Volume and stress distribution effects

STEP lecture B1
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Objectives

To explain volume and stress distribution effects as a consequence of the weakest link theory for brittle materials. To describe the options of Eurocode 5 and CEN supporting standards for deriving characteristic values and evaluating design stresses.

Prerequisites

- A7 Solid timber - Strength classes
- A8 Glued laminated timber - Production and strength classes
- B2 Tension and compression
- B3 Bending Members
- B4 Shear and torsion

Summary

The lecture begins with a presentation of the weakest link theory, for tension in brittle materials, and explains volume effects. This theory is expanded to other stress fields, with attention to bending, tension, shear and tension perpendicular to grain. Research results are summarised. The options of EC5 for bending and tension perpendicular to grain are explained. Some examples of calculations are given.

Theory

The weakest link theory has been developed by Pierce (1926), Tucker (1927) and Weibull (1939) who studied brittle materials, including concrete. This theory says that "when subjected to tension, a chain is as strong as its weakest link". To explain this theory, consider a reference volume subjected to tension. The probability of failure P_f of this volume is defined by:

$$P_f = F(\sigma) = \textit{Probability}\ (\textit{Strength} \leq \sigma) \qquad (1)$$

where F is the cumulative distribution of the strength, as illustrated in Figure 1.

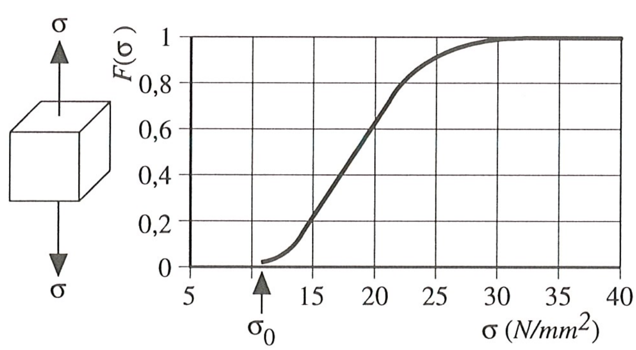


Figure 1 Cumulative probability of failure for a reference volume.

Now consider a series assembly of N reference volumes. This system survives if each of the members survives, i.e.:

$$P_S = P_S(1)\,P_S(2)\,...\,P_S(N) = [1-P_f(1)]\,[1-P_f(2)]\,...\,[1-P_f(N)] \quad (2)$$

where P_S is the probability of survival of the system and $P_S(i)$ is the probability of survival of an individual element i. From Equation (2) and *assuming that reference volumes have the same probability of failure and that the events of failure are independent in all reference volumes*, the probability of failure of the system can be evaluated:

$$P_f = 1-P_S = 1-[1-F(\sigma)]^N = 1-e^{N\,Log(1-F(\sigma))} \propto 1-e^{-N\,F(\sigma)} \quad (3)$$

Now, assume that the lower tail of F has been fitted by a power model, i.e.

$$F(\sigma) = a\,(\sigma-\sigma_0)^k \quad (4)$$

The probability of failure is then expressed by:

$$P_f(\sigma) = 1-e^{-Na\,(\sigma-\sigma_0)^k} = 1-e^{-V\,(\frac{\sigma-\sigma_0}{m})^k} \quad (5)$$

This model is known as the 3 parameters Weibull model. It is also well known as the 2 parameters Weibull model when $\sigma_0 = 0$. The parameters m and k can be estimated from the mean of σ *($E(\sigma)$)* and the coefficient of variation of σ *($CV(\sigma)$)* by solving the following equations:

$$(CV(\sigma))^2 = \frac{\Gamma(1+\frac{2}{k})}{\Gamma^2(1+\frac{1}{k})} - 1 \quad (6)$$

$$m = \frac{E(\sigma)}{\Gamma(1+\frac{1}{k})}\,V^{\frac{1}{k}} \quad (7)$$

where $\Gamma(x)$ is the Gamma function.

The theory can be used to explain the size effect in tension. Consider a volume V_1 which has a given probability of failure $P_f(\sigma_1)$ at level σ_1 and a volume V_2 which has a given probability of failure $P_f(\sigma_2)$ at level σ_2. If the characteristic strengths of these two volumes are compared, the following is obtained:

$$P_f(\sigma_1) = P_f(\sigma_2) \Rightarrow V_1\,(\frac{\sigma_1}{m})^k = V_2\,(\frac{\sigma_2}{m})^k \Rightarrow \frac{\sigma_2}{\sigma_1} = (\frac{V_1}{V_2})^{\frac{1}{k}} \quad (8)$$

This equation is the basic explanation of size effect. In the case of stress fields other than tension, these equations are modified to take into account the stress variations:

$$\sigma(x,y,z) = \sigma\,w(x,y,z) \quad (9)$$

where σ is the maximum stress in volume V
$w(x,y,z)$ is a spatial distribution function (dimension-free).

The Weibull model is then written:

$$P_f(\sigma) = 1-e^{-V^*(\frac{\sigma}{m})^k} \quad (10)$$

where V^* is defined by:

$$V^* = \int_V w\,(x,y,z)^k\, dV \tag{11}$$

For example, a simply supported beam with rectangular cross-section and loaded at the midpoint by a concentrated force gives the following value for V^*:

$$V^* = \frac{V}{2(k+1)^2} \tag{12}$$

This method of calculating the stress distribution effect has been used by Larsen (1986) and Colling (1986) to evaluate the volume and stress distribution effects on the shear strength and tension perpendicular to grain for curved, tapered and cambered beams. In Larsen's paper, the term "distribution factor" (k_{dis}) is used, where:

$$k_{dis} = \left(\frac{V}{V^*}\right)^{\frac{1}{k}} \tag{13}$$

The k_{dis} factor is used to calculate the design tension perpendicular to grain strength for different load configurations:

$$f_{t,90,d} = k_{vol}\, k_{dis}\, f^*_{t,90,d} \quad with \quad k_{vol} = \left(\frac{V_0}{V}\right)^{1/k} \tag{14}$$

where $f^*_{t,90,d}$ refers to a reference volume V_0 under uniform stress.

In Colling (1986), the following notation is used:

$$V^* = V \int_L w^k dL \int_D w^k dD = V\, \lambda_L^k\, \lambda_D^k \tag{15}$$

where λ_L and λ_D are called "fullness parameters".

Research results

A vast amount of data has been published to explain size effect for structural size timber. These results are sometimes conflicting (Barrett and Lam, 1992; Madsen, 1992), and might be due to the following reasons:

- The size effect is justified by a brittle failure theory, which is applicable to tension parallel and perpendicular to grain (Barrett, 1974; Colling, 1986), and to shear (Foschi and Barrett, 1976; Foschi, 1985; Colling, 1986). But in the case of compression, and particularly in bending which is a mixed mode of failure between tension and compression, the use of this theory is debatable.
- The size effect is based on an equal probability of failure of the "reference volumes". This assumption is not always verified for all the species, especially for pines in which knots are not randomly located.
- For visually graded lumber, defect sizes increase with the size of the member. This means that the material changes with the size, which can mask a pure size effect. In particular, when size effect is investigated in a mixture of grades, the effect of grading will have an influence on the size effect.
- When tests are conducted for constant span to depth ratios in bending, the size effect is a combination of a depth effect and a length effect (Barrett and Fewell, 1990). These effects cannot be identified separately.

The following tables summarize these results. They show some discrepancies, which have been explained by Barrett and Lam (1992).

In Table 1, different factors for bending size effects are recorded:

- a length factor S_L (for beams tested at constant depths) which is calculated from:

$$\frac{\sigma_2}{\sigma_1} = \left(\frac{L_1}{L_2}\right)^{\frac{1}{k_L}} = \left(\frac{L_1}{L_2}\right)^{S_L} \tag{16}$$

- a depth factor S_h (for beams tested at constant spans) which is calculated from:

$$\frac{\sigma_2}{\sigma_1} = \left(\frac{h_1}{h_2}\right)^{\frac{1}{k_h}} = \left(\frac{h_1}{h_2}\right)^{S_h} \tag{17}$$

- a "size factor" S_R (for beams tested at constant span to depth ratio, i.e. $L_i = k\ h_i$), which, according to the combination of equations (16) and (17), is calculated from:

$$\frac{\sigma_2}{\sigma_1} = \left(\frac{L_1}{L_2}\right)^{S_L}\left(\frac{h_1}{h_2}\right)^{S_h} = \left(\frac{h_1}{h_2}\right)^{S_L}\left(\frac{h_1}{h_2}\right)^{S_h} = \left(\frac{h_1}{h_2}\right)^{S_h+S_L} = \left(\frac{h_1}{h_2}\right)^{S_R} \tag{18}$$

Author	S_L	S_h	S_R
Barrett and Larsen, 1992	0,17	0,23	0,40
Madsen, 1992	0,20	0,0	0,20
Ehlbeck and Colling, 1990	0,15	0,15	0,30

Table 1 Size factors for bending.

Additional results are reported for glulam (Ehlbeck and Colling, 1990), but are based on a sample size which was much smaller. The size effects for glulam are lower than for solid timber, probably due to a lamination effect which increases the strength.

In Table 3, load configuration factors for different bending cases are reported according to Johnson (1953). These load configuration factors are derived according to Equations (9), (10) and (11), and normalized to the reference four points bending case.

Tension results are slightly different from those for bending. This might be due to a pure brittle failure mechanism (see Table 2).

Author	S_L	S_h	S_R
Barrett and Larsen, 1992	0,17	0,23	0,40
Madsen, 1992	0,20	0,10	0,30

Table 2 Size factors for tension.

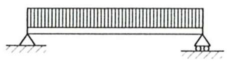
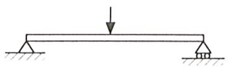
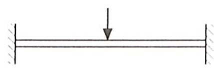
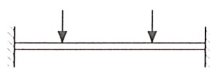
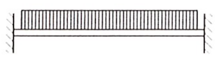
Load Case	*Load Configu-ration Factor*
	0,87
	1,00
	1,04
	1,22
	1,22
	1,35
	1,43

Table 3 Load configuration factors.

For compression, the results of the different studies are in general agreement:

$S_L = 0{,}10$ $S_h = 0{,}11$ $S_R = 0{,}21$

For tension perpendicular to grain and for shear, a volume factor (S_V) has been derived by Colling (1986), who also derived load configuration factors for tension perpendicular to grain

$S_V = 0{,}20$

These results are subject to different opinions but show an evidence of size effects for many stresses, together with a stress distribution effect which can be as significant as the size effect itself. For code purposes, the approach has been simplified, especially in the case of stress distribution effects.

Size and stress distribution effects related to EC5

prEN 338: 1991

The first application of size effects concerns the modification of characteristic strengths given in prEN338 "Structural Timber - Strength classes". The characteristic strengths in bending and in tension are given for a reference depth of 150 *mm* for solid timber and 600 *mm* for glulam. For depths less than these reference values, these strengths are multiplied by a size factor, which has a fixed upper limit. This means that size effect is only applied in one direction, as shown in Figure 2.

EC5: Part 1-1: 3.2.2

For solid timber :

$$k_h = \min.\begin{cases} \left(\dfrac{150}{h}\right)^{0,2} \\ 1,3 \end{cases} \tag{19}$$

where h is the beam depth in *mm*.

ECEC5: Part 1-1: 3.3.2

For glulam :

$$k_h = \min.\begin{cases} \left(\dfrac{600}{h}\right)^{0,2} \\ 1,15 \end{cases} \tag{20}$$

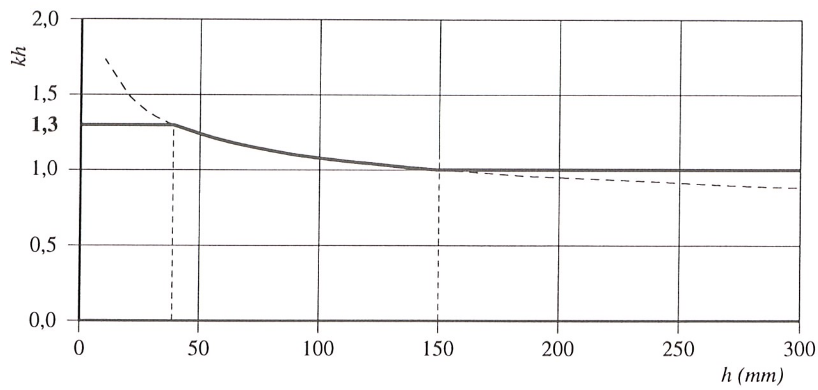


Figure 2 *EC5 size factor for solid timber in bending or tension (solid line), related to theory (dashed line).*

EC5: Part 1-1: 5.1.3

For tension perpendicular to grain and for shear, characteristic strengths are also given for a reference volume. But, for simplicity, a size factor is only proposed for tension perpendicular in glulam. The designer is then required to verify the following equation:

$$\sigma_{t,90,d} \le f_{t,90,d} \left(\frac{V_0}{V}\right)^{0,2} \tag{21}$$

where V_0 is a reference volume of 0,01 m^3.

EC5: Part 1-1: 5.2.4

For double tapered, curved and pitched cambered beams, an additional requirement is included to account of the stress distribution effects. The designer must verify the following equation in the apex zone:

$$\sigma_{t,90,d} \le k_{dis} \left(\frac{V_0}{V}\right)^{0,2} f_{t,90,d} \tag{22}$$

where

k_{dis} is a stress distribution factor which has been fixed for special cases:
k_{dis} = 1,4 for double tapered and curved beams
k_{dis} = 1,7 for pitched cambered beams.

For simplicity, other aspects of size and stress distribution effects like compression size effect and load configuration factors have not been taken into account.

Calculation examples

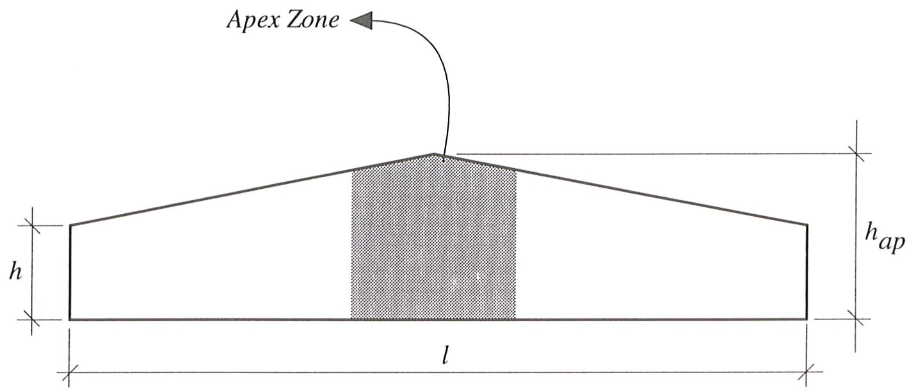
Example 1: Bending Strength of a solid timber beam of cross-section 40x100 *mm*, strength class C24.

C24 strength class provides $f_{m,k} = 24\ N/mm^2$

$k_h = (150\ /\ 100)^{0,2} = 1{,}08 < 1{,}3$

$f_{m,k}$ (modified) $= 26\ N/mm^2$

Example 2: Design of a double tapered beam. Verification of tensile stresses perpendicular to grain.



Span: $L = 20\ m$ $h = 1\ m$ $h_{ap} = 1{,}20\ m$ $b = 150\ mm$

prEN 1194 : 1993

Glulam Strength Class: GL 36

GL 36 strength class provides $f_{t,90,k} =\ 0{,}45\ N/mm^2$

To calculate a design strength, take $k_{mod} = 0{,}8$ and $\gamma_M = 1{,}3$

This implies $f_{t,90,d} = 0{,}277\ N/mm^2$

EC5: Part 1-1: 5.2.4

The volume of the apex zone is equal to: $V = 0{,}2097\ m^3$

Thus $(V_0/V)^{0,2} = 0{,}544$

For a double tapered beam, $k_{dis} = 1{,}4$

The maximum design stress perpendicular to the grain is equal to :

$\sigma_{t,90,d}$ (max) $= 0{,}277\ .\ 0{,}544\ .\ 1{,}4 = 0{,}21\ N/mm^2$

Concluding summary

- Size and stress distribution effects are explained by Weibull theory.
- Research results show discrepancy, especially for depth effect in bending.
- EC5 provides a simplistic approach to size and stress distribution effects to aid the designer.

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