Mechanically jointed beams and columns

STEP lecture B11
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Objectives
To explain the computation and design of mechanically jointed beams and columns, to provide analytical solutions, and to illustrate the use of computer programs.

Prerequisites
B2 Tension and compression
B3 Bending
B6 Columns
C1 Joints

Summary
An example of a beam made of two parts is illustrated, for which analytical solutions for computing stresses and deformations are derived. The possibility of using a computer program for the design of such beams is indicated. A design example is provided.

Introduction
Cross-sections of beams or columns may be composed of several components, connected by mechanical joints. Longitudinally the cross-sections are not jointed. In the junction between the individual composites, the mechanical joints mainly carry shear forces.

Thus a wide variety of cross-sections (see Figures 1 and 2) may be built. The dowelled beam is known from ancient timber constructions. Adding additional cross-section parts is a suitable way of strengthening an existing profile. These parts may be of solid timber, glued laminated timber or wood-based materials.

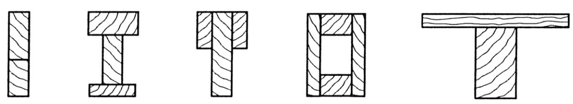

Figure 1 Typical cross-sections.

For columns, cross-sectional parts are often separated by gussets at a given distance. Especially for beams the cross-section with two flanges connected by a web, which carries the shear, is very common. The flanges may be of solid timber or glued laminated timber, the web may be of planks, wood-based panels or lately steel. It is also possible to build a composite structure from a concrete plate and a timber tension flange.

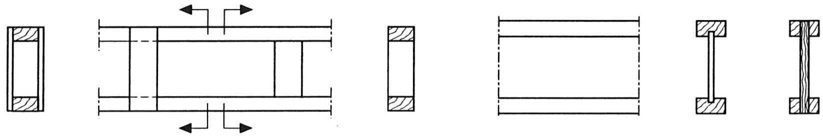

Figure 2 Cross-sections with two flanges and discrete or continuous connection.

Semi-rigid joint

The connection of a number of cross-sections is made by mechanical fasteners such as nails, bolts, dowels or nail plates (glued joints are regarded as rigid connections). Each joint is stressed by shear forces causing a displacement. The relation between the displacement of the cross-section parts u and the force is specified by the slip modulus K. Figure 3 shows some common patterns of joints, the displacement u and the shear force v.

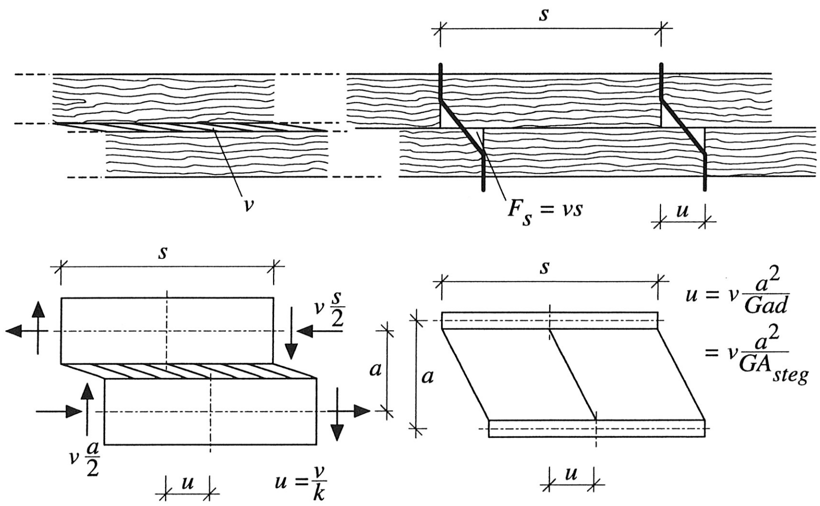

Figure 3 Displacement and shear force between the parts.

For the computation, and in order to develop application equations, it is necessary to distribute the joints continuously along the beam. The effect of this is a continuously acting shear force v, such that:

$$v = \frac{F_s}{s} \qquad k = \frac{K}{s} \qquad v = k\,u \qquad (1)$$

If the distance between the fasteners is considerable or if the joints are concentrated at very few points, the computational model of a continuous joint is no longer valid, and a different mechanical model is required, for instance a frame model.

Computation methods

Beams

For beam design the following parameters are required: stresses σ and τ in all parts, forces in the joints and deflections. For mechanically jointed beams, the bending-theory for beams is no longer applicable because of the slip in the joints. However, the theory is applicable to individual components.

Analytical solutions are developed by use of differential equations of equilibrium (Möhler, 1956; Heimeshoff, 1987) or energy considerations and specially developed design programs are available, see for example Kneidl (1991). The development of the differential equations is conveniently shown in a T-cross-section made of two parts (Figure 4).

The solutions require that for every part simple bending-theory is valid and shear displacement is omitted. The connection is regarded as continuous and the profile and the joint stiffness are constant in the direction of the beam's axis.

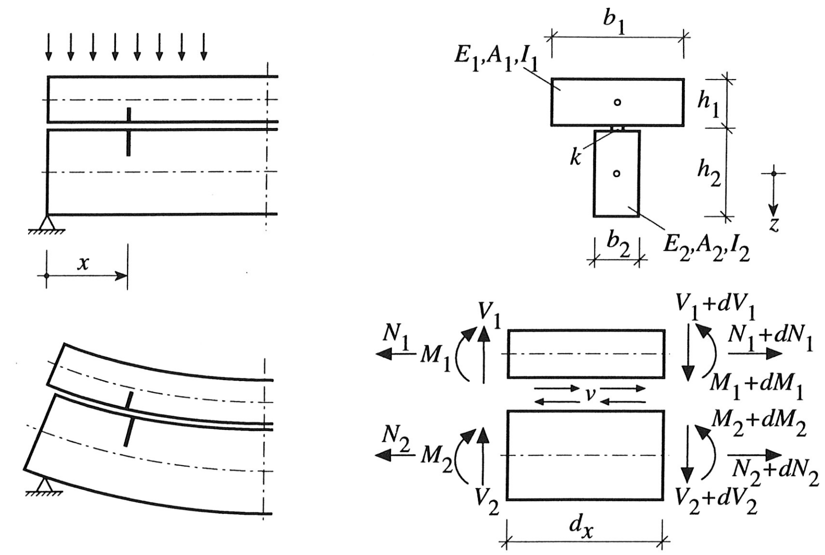

Figure 4 *Beam details and equilibrium of an incremental element. System, cross-section, deformations, element dx.*

The deformations are (see Figure 5):

u_1, u_2 are the longitudinal displacement of the axis of cross-section 1 and 2,
w is the common bending deflection and
u is the relative displacement of the cross-section parts at the location of the joints.

$$u = u_2 - u_1 + w'\left(\frac{h_1}{2} + \frac{h_2}{2}\right) = u_2 - u_1 + w'a \tag{2}$$

u is independent of the position of the joints. The critical dimension is the distance a of the axes of the cross-sectional parts. The derived equations are not only valid for cross-sectional parts located one upon another, as shown in the T-profile, they also apply to cross-sectional parts located side by side. This is only true if shear deformation is neglected.

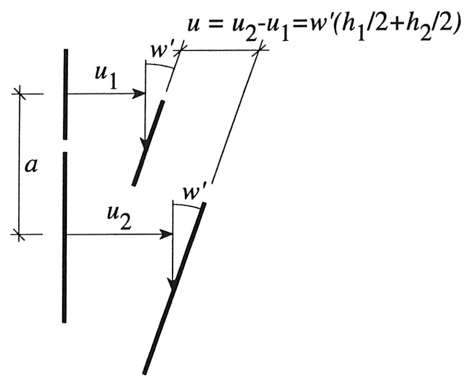

Figure 5 *Deformations.*

Elasticity principles matching the simple bending theory:

$$N_1 = E_1\, A_1\, u'_1 \qquad N_2 = E_2\, A_2\, u'_2 \tag{3}$$
$$M_1 = -E_1\, I_1\, w'' \qquad M_2 = -E_2\, I_2\, w'' \tag{4}$$
$$V_1 = -E_1\, I_1\, w''' \qquad V_2 = -E_2\, I_2\, w''' \tag{5}$$
$$v = k\, u_t = k\,(u_2 - u_1 + w'a) \tag{6}$$

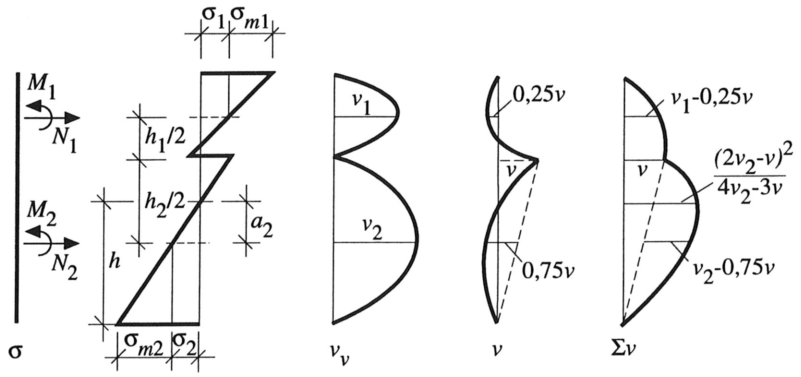

Figure 6 *Stress distribution.*

Equilibrium of the two elements in x and z direction: $[p_x = 0, (N_1 + N_2)' = 0]$

$$N_1' + v = 0 \qquad (7) \qquad\qquad M_1' = V_1 - v\frac{h_1}{2} \qquad (9a)$$

$$N_2' + v = 0 \qquad (8) \qquad\qquad M_2' = V_2 - v\frac{h_2}{2} \qquad (9b)$$

$$V_1' + V_2' = -p = V' \qquad (9c)$$

The sum of (9a) and (9b) is differentiated once with respect to x and V' is replaced by the term $-p$:

$$M_1'' + M_2'' + v'\, a + p = 0 \qquad (10)$$

If the internal forces and moments are replaced using elasticity principles, the following system of differential equations results:

$$E_1 A_1 u_1'' + k\,(u_2 - u_1 + w'\, a) = 0 \qquad (11)$$
$$E_2 A_2 u_2'' - k\,(u_2 - u_1 + w'\, a) = 0 \qquad (12)$$
$$(E_1 I_1 + E_2 I_2)\, w'''' - k\,(u'_2 - u'_1 + w''\, a)\, a = p \qquad (13)$$

In this way three equations of equilibrium (7), (8) and (10) are formulated for the three deformations u_1, u_2 and w.

The variation of the elastic energy is also determined from these equations:

$$\Pi = \frac{1}{2}\int\Big[E_1 A_1 u'^2_1 + E_2 A_2 u'^2_2 + (E_1 I_1 + E_2 I_2)\, w''^2 + k\,(u_2 - u_1 + w' a)^2 - 2pw\Big]dx \qquad (14)$$

Elastic foundation effect k_w, and the influence of second order theory effects could be taken into account by adding the term $k_w\, w - N_0\, w''$ to Equation (13).

For single span beams with a sinusoidal load distribution, a simple, analytical solution can be given because the shape of the deformations in the direction of the axes corresponds to cos- or sin-functions. Although the derivation is based on the synusoidal load distribution, the solution is also applicable to most other load distributions.

$$p = p_0 \sin\left(\frac{\pi}{1}\, x\right) \qquad (15)$$

$$u_1 = u_{10} \cos\left(\frac{\pi}{1} x\right); \quad u_2 = u_{20} \cos\left(\frac{\pi}{1} x\right); \quad w = w_0 \sin\left(\frac{\pi}{1} x\right) \tag{16a,b,c}$$

These terms, when placed in Equations (11), (12) and (13), give a system of equations for the constants u_{10}, u_{20} and w_0:

u_{10}	u_{20}	w_0	$= p_0$
$-\frac{\pi^2}{l^2} E_1 A_1 - k$	k	$k \frac{\pi}{l} a$	$= 0$
k	$-\frac{\pi^2}{l^2} E_2 A_2 - k$	$-k \frac{\pi}{l} a$	$= 0$
$k \frac{\pi}{l} a$	$-k \frac{\pi}{l} a$	$\frac{\pi^4}{l^4}(E_1 I_1 + E_2 I_2) - \frac{k\pi^2}{l^2} a_2$	$= -1$

The solution is:

$$w_0 = p \frac{l^4}{\pi^4} \frac{1}{E_1 I_1 + E_2 I_2 + \dfrac{E_1 A_1 \gamma_1 a^2}{1 + \gamma_1 \dfrac{E_1 A_1}{E_2 A_2}}} = p_0 \frac{l^4}{\pi^4} \frac{1}{(EI)_{ef}} \tag{17a}$$

$$u_{10} = w_0 \frac{\pi}{l} \frac{a \gamma_1 E_2 A_2}{\gamma_1 E_1 A_1 + E_2 A_2} \qquad u_{20} = -w_0 \frac{\pi}{l} \frac{a \gamma_1 E_1 A_1}{\gamma_1 E_1 A_1 + E_2 A_2} \tag{17b,c}$$

$$k_1 = \frac{\pi^2}{l^2} \frac{E_1 A_1}{k} \text{ and } \gamma_1 = \frac{1}{(1 + k_1)} \tag{18a,b}$$

With these deformations and applying elastic principles, the stresses can be computed. The stress in the axis of part 1 of the cross-section is (Figure 6):

$$\sigma_1 = E_1 u'_1 (x = l/2) = -E_1 u_{10} \frac{\pi}{l} \tag{19}$$

Using the following terms

$$w_0 = p_0 \frac{l^4}{\pi^4} \frac{1}{(EI)_{ef}}; \qquad M_0 = p_0 \frac{l^2}{\pi^2};$$

$$a_2 = \frac{\gamma_1 E_1 A_1 a}{\gamma_1 E_1 A_1 + E_2 A_2}; \qquad a_1 = a - a_2; \tag{20a,b,c,d}$$

the stress is

$$\sigma_1 = \frac{\gamma_1 E_1 a_1 M_0}{(EI)_{ef}} \tag{21}$$

This type of the equation is equivalent to the equation for the stress in a simple beam. In EC5, Annex B, further equations are given.

EC5: Part 1-1: 5.1.9, 5.1.10

The bending stresses and the stresses in the axes of the members must verify the condition of combined bending with axial tension or axial compression. If necessary the stability condition must also be satisfied such that:

EC5: Part 1-1: 5.2.2

$$\sigma_{m,d} \leq k_{crit} f_{m,d} \tag{22}$$

k_{crit} takes account of the bending stress according to the lateral deformation resulting from 2nd order theory effects. For this purpose the critical bending stress is necessary. The bending stiffness of the beam about the weak axis and the torsion stiffness are required.

Columns

The computation of mechanically jointed columns has to allow for buckling, and the influence of 2nd order theory. It is clear that the effective bending stiffness $(EI)_{ef}$ is the dominant factor for buckling. If the expression $N_0 \cdot w''$ is included in the Equation of equilibrium (13) and if the determinant of the equilibrium equations is set to zero, the buckling load is also obtained.

$$F_{ki} = \frac{\pi^2}{l^2}(EI)_{ef} \tag{23}$$

The axial stiffness of a composite column is

$$(EA)_{ef} = (EA)_{tot} = \sum E_i A_i \tag{24}$$

since the joints are not considered to be stressed by the axial forces.

The slenderness of a mechanically jointed column can be computed in a similar manner to a simple column.

$$i_{ef} = \sqrt{\frac{(EI)_{ef}}{(EA)_{ef}}}; \qquad \lambda_{ef} = \frac{l}{i_{ef}} \tag{25a,b}$$

Each member of a composite column corresponds to the simple column, and for each member of the cross-section the relative slenderness and the buckling factor can be computed.

$$\lambda_{rel} = \lambda_{ef}\sqrt{\frac{f_{c,0,k,i}}{\pi^2 E_{0,05,i}}} \qquad \sigma_i = \frac{F_{c,d}}{(EA)_{ef}} E_i = \sigma_{c,0,d,i} \tag{26, 27}$$

If, at the same time the column is stressed by bending, the bending stress must be superimposed. Normally the design will be governed by compression in a single member such that

$$\frac{\sigma_{c,0,d,i}}{k_{c,i} f_{c,0,d,i}} + \frac{\sigma_{m,i}}{f_{m,d,i}} \leq 1 \tag{28}$$

The compression force and its corresponding deformation results in a transverse force V_d, which is dependent on the slenderness. To this, any transverse force due to direct loading must be added.

In EC5, Annex C, equations are given to compute the effective slenderness of columns with different cross-sections. For spaced columns with packs or gussets, and for lattice columns, the effective bending stiffness can be computed using frame programs. Here, the deflection w_0 in relation to a sinusoidal transverse load p_0 and taking into account the yielding of the connection, results in an effective bending-stiffness given by:

$$(EI)_{ef} = \frac{p_0}{w_0}\frac{l^4}{\pi^4} \tag{29}$$

Design example

Figure 7 shows a beam made up of a single plywood flange and a timber web jointed by nails. The design stresses and moduli are also given.

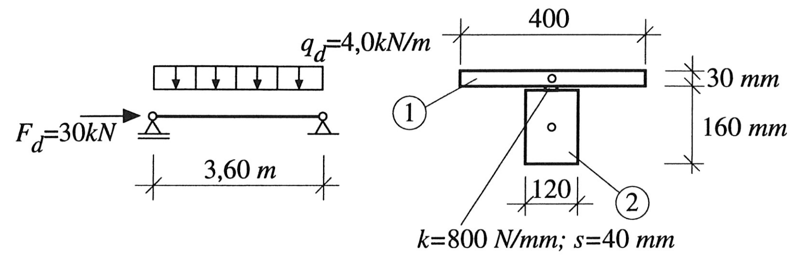

Figure 7 System, cross-section, design values.

		$E_{0,mean}$	$f_{c,o,k}$	$f_{c,o,d}$	$f_{t,o,d}$	$f_{m,d}$	(N/mm^2)
Design values	1	4500	19,5	12,0	8,0	12,0	
	2	11000	21,0	12,9	8,6	14,7	

For both ultimate and serviceability limit states $E_{0,mean}$ values are to be used. For calculation of deflections the slip modulus K_{ser} will be used, for ultimate limit state $K_u = 2/3\ K_{ser}$.

Computation

EC5: Part 1-1: Annex B

Values of cross-section:

(B2b) $A_1 = 400 \cdot 30 = 12000\ mm^2$ $\qquad A_2 = 19200\ mm^2$

(B2c) $I_1 = 400 \cdot 30^3/12 = 0,9 \cdot 10^6\ mm^4$ $\qquad I_2 = 41,0 \cdot 10^6\ mm^4$

(B2d) $\gamma_2 = 1$

(B2e) $\gamma_1 = (1 + \pi^2 \cdot 4500 \cdot 12000 \cdot 40 / 800 \cdot 3600^2)^{-1} = 0,33$

$$a_2 = \frac{0,33 \cdot 4500 \cdot 12000\ (30 + 160)}{2\ (0,33 \cdot 4500 \cdot 12000 + 11000 \cdot 19200)} = 7,33\ mm$$

$$a_1 = \frac{(30 + 160)}{2} - 7,33 = 87,7\ mm$$

(B2f)

(B2a)
$$(EJ)_{ef} = 4500 \cdot 0,9 \cdot 10^6 + 11000 \cdot 41,0 \cdot 10^6 + 0,33 \cdot 4500 \cdot 12000 \cdot 87,7^2 + 11000 \cdot 19200 \cdot 7,33^2$$
$$= 602 \cdot 10^9\ Nmm^2$$

Stresses in the middle of the span caused by a bending moment $M_d = 6,48\ kNm$

(B3a)
$$\sigma_1 = 0,33 \cdot 4500 \cdot 87,7 \cdot 6,48 \cdot 10^6 / 602 \cdot 10^9 = 1,40\ N/mm^2$$
$$\sigma_2 = 1 \cdot 11000 \cdot 7,33 \cdot 6,48 \cdot 10^6 / 602 \cdot 10^9 = 0,87\ N/mm^2$$
$$\sigma_{m1} = 0,5 \cdot 4500 \cdot 30 \cdot 6,48 \cdot 10^6 / 602 \cdot 10^9 = 0,73\ N/mm^2$$
$$\sigma_{m2} = 0,5 \cdot 11000 \cdot 160 \cdot 6,48 \cdot 10^6 / 602 \cdot 10^9 = 9,47\ N/mm^2$$

Stresses caused by a compression force $F_d = 30\ kN$

$$\sigma_1 = 30000 \cdot 4500 / (4500 \cdot 12000 + 11000 \cdot 19200) = 0,51\ N/mm^2$$
$$\sigma_2 = 30000 \cdot 11000 / (4500 \cdot 12000 + 11000 \cdot 19200) = 1,25\ N/mm^2$$

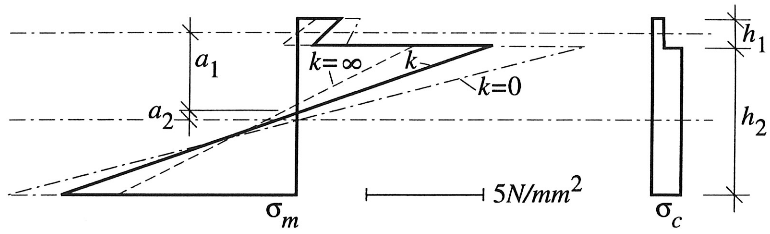

Figure 8 Stresses (rigid jointed k = ∞, not jointed k = 0).

Maximum force F_1 in the joints for a shear force $V_d = 4 \cdot 3{,}6/2 = 7{,}2\ kN$.

(B3b) $F_1 = 0{,}33 \cdot 4500 \cdot 12000 \cdot 87{,}7 \cdot 40 \cdot 7200 / 601{,}8 \cdot 10^8 = 741\ N$

Concluding summary

- The basis for the computation of mechanically jointed beams and columns is shown and the analytical solutions given in EC5, Annex B and C, are shown for simply supported beams and columns with a span length l.

- For more complicated systems such as frames or beams and columns with varying cross-sections, along the main axis, it is necessary to use numerical solutions offered by computer programs. The members must then be modelled as bars and the joints as either bars or springs.

References

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