Buckling lengths

STEP lecture B7
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Objective
To describe the concept of buckling (effective) length and its application in design to practical columns and frames.

Prerequisite
B6 Columns

Summary
The concept of the effective or buckling length is described. The principal factors influencing the buckling lengths of columns and frames as well as simple approximations for practical cases are given. An example of a three-hinged frame with semi-rigid corner connections complements the lecture.

Introduction
Buckling curves for the design of timber columns are generally based on the load bearing capacity of columns where both ends are simply supported (see Figure 1). The support conditions of compression members in actual timber structures often differ from those shown in Figure 1. In order to be able to employ the buckling curves in EC5 for these more practical cases, the concept of an effective length is used.

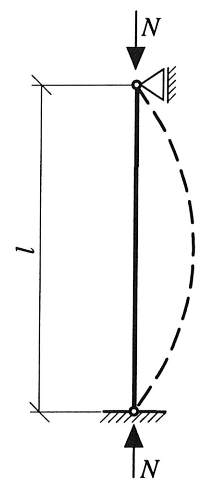

Figure 1 Buckling of a two-hinged column.

One example of the difference between real length and buckling length can be found in the internal member of a truss. In practice, the external members (chords) are often braced at the outer edges of the top and bottom chord, respectively. In this case, the buckling lengths of the internal members can be assumed to correspond to the distance between the braces and hence are larger than the distance between the member nodes.

The effective or buckling length of a compression member is defined as the length of a hypothetical two-hinged column with the same elastic critical buckling load as the member in question. The effective length can be visualised as the distance between two consecutive points of contraflexure of the actual compression member (see Figure 2). In practice, an effective length factor β is used which denotes the ratio of the effective length to the real length of the member. Figure 3 shows the four Euler cases where the buckling length is given for different idealised support conditions of the column.

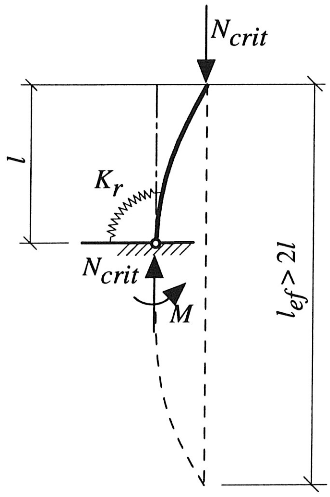

Figure 2 Effective length of a clamped column with a semi-rigid base connection.

In this lecture, approximate solutions for the buckling lengths of different systems are given. Where the approximate solutions do not apply, a second order analysis should be carried out, calculating the equilibrium of moments and forces and considering the deformed shape of the respective member or structure.

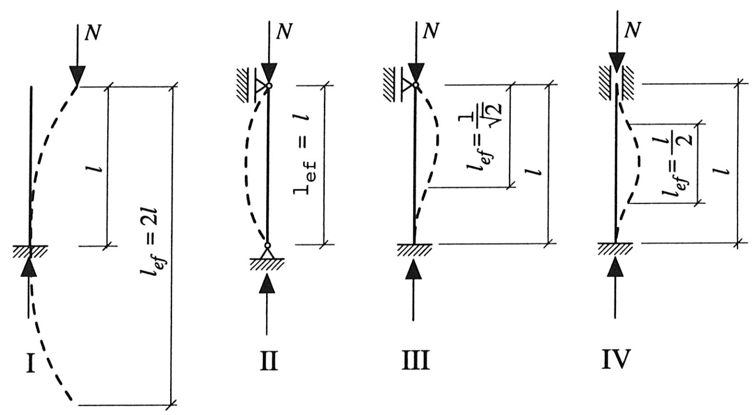

Figure 3 Buckling lengths for idealised support conditions (Euler cases I to IV).

Influence of rotations in semi-rigid connections

Since completely rigid connections are almost impossible in timber structures, the rotations in semi-rigid joints should be taken into account when determining buckling lengths. The rotational stiffness K_r of a semi-rigid connection is defined as the moment necessary to cause an angle of rotation of one radian. With the slip modulus K_u of the fastener, the rotational stiffness of a semi-rigid connection is calculated as:

EC5: Part 1-1: 6.1

$$K_r = \sum_{i=1}^{n} K_u r_i^2 \qquad (1)$$

where r_i denotes the distance between the single fastener and the centre of the connection. As an example, the buckling length of the column in Figure 2 is derived considering the influence of the rotation in the semi-rigid joint at the base of the column.

The approximate solutions for buckling lengths, taking into account the influence of the rotation in semi-rigid joints, are valid in those cases where this influence decreases the critical elastic buckling load by not more than about 20%.

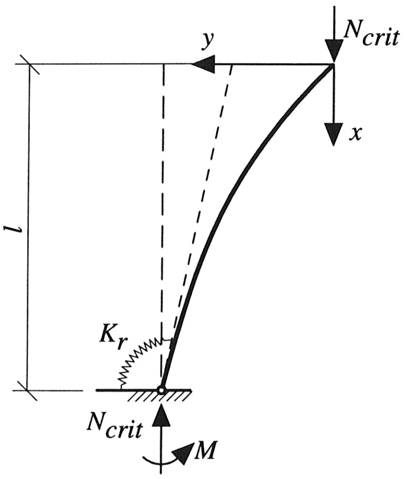

Figure 4 Deformed shape of the clamped column with a semi-rigid base connection.

Using the notation of Figure 4, the bending moment M is:

$$M(x) = N\ y(x) \tag{2}$$

where

$$y'' = -\frac{M(x)}{EI} \tag{3}$$

This results in the following differential equation

$$y'' + \frac{N}{EI}\ y = 0 \tag{4}$$

with the solution

$$y = A\ \sin(\mu\ x) \tag{5}$$

where

$$\mu = \sqrt{N\ /\ EI} \tag{6}$$

Using the condition

$$M(x{=}l) = N\ y(x{=}l) = K_r\ y'(x{=}l) \tag{7}$$

yields

$$(\mu\ l)\ \frac{EI}{l\ K_r}\ \tan(\mu\ l) = 1 \tag{8}$$

An analytical solution of equation (8) does not exist. However, for

$$-\frac{\pi}{2} < \mu\, l < \frac{\pi}{2} \tag{9}$$

$$\tan(\mu\, l) \approx \frac{\mu\, l}{1 - \dfrac{4\, \mu^2\, l^2}{\pi^2}} \tag{10}$$

Substituting for tan (μ l) using the approximation in equation (10), the critical elastic buckling load becomes:

$$N_{crit} = \frac{1}{\dfrac{4\, l^2}{\pi^2\, EI} + \dfrac{l}{K_r}} \tag{11}$$

Compared to the critical elastic buckling load of a two-hinged column (Euler case II)

$$N_{crit} = \frac{\pi^2\, EI}{l_{ef}^2} \tag{12}$$

the effective length factor β is given by

$$\beta = \frac{l_{ef}}{l} = \sqrt{4 + \frac{\pi^2\, EI}{l\, K_r}} \tag{13}$$

Interconnected columns

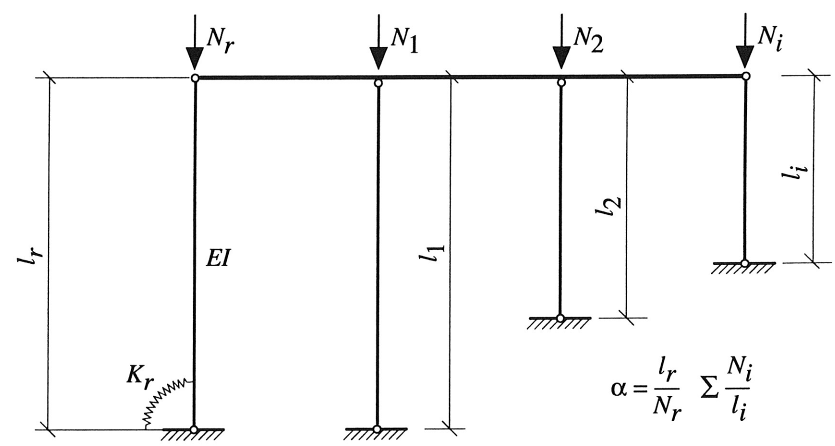

Figure 5　Interconnected columns.

If two-hinged columns are braced by a column clamped at its base (see Figure 5), the critical buckling load of the clamped column decreases due to the additional forces N_i which cause a horizontal force in the deformed system. Considering the effect of the rotation in the semi-rigid joint at the column base, the effective length factor for the system shown in Figure 5 (buckling in the system plane) is approximately:

$$\beta = \pi \sqrt{\frac{5 + 4\,\alpha}{12} + \frac{(1 + \alpha)\, EI}{l_r\, K_r}} \tag{14}$$

with α as defined as in Figure 5. The two-hinged columns braced by the clamped column are of course to be designed with a buckling length corresponding to their real length.

Arches

For three- and two-hinged arches (see Figure 6) with a ratio h/l between 0,15 and 0,5 and essentially uniform cross-section the effective length for buckling in the arch plane may be assumed to be

$$l_{ef} = 1{,}25\ s \tag{15}$$

where s equals half the arch length. The normal force at the quarter point should be used in the buckling design.

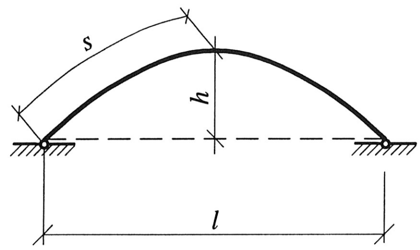

Figure 6 Two-hinged arch.

Two- and three-hinged frames

For two- and three-hinged frames with an angle of inclination of the columns of less than about 15° (see Figure 7), the following equation for the buckling length of the column applies:

$$l_{ef} = h \sqrt{4 + 3{,}2\ \frac{I\ s}{I_o\ h} + 10\ \frac{EI}{h\ K_r}} \tag{16}$$

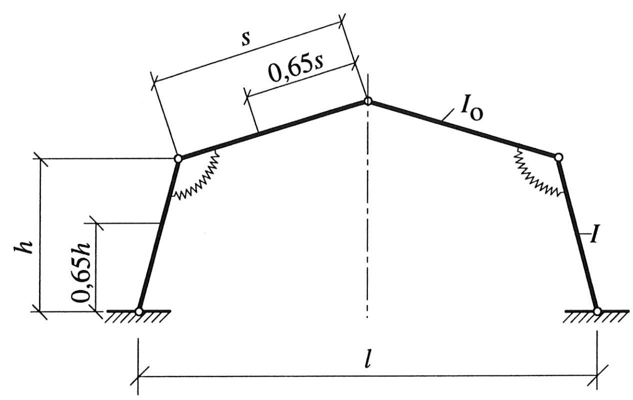

Figure 7 Three-hinged frame.

The respective buckling length of the rafter is:

$$l_{ef} = h \sqrt{4 + 3{,}2\ \frac{I\ s}{I_o\ h} + 10\ \frac{EI}{h\ K_r}} \sqrt{\frac{I_0\ N}{I\ N_0}} \tag{17}$$

where N and N_0 denote the axial forces in the column and the rafter, respectively. For tapered rafters or columns equations (16) and (17) may be used provided the second moments of area of the rafter and the column are taken at $0{,}65s$ and $0{,}65h$, respectively (see Figure 7). These second moments of area are also used to determine the slenderness ratios.

Columns or rafters with knee bracing

The buckling lengths of the columns of portal frames as shown in Figure 8 (left) and the buckling lengths of the rafters in Figure 8 (right) for buckling in the frame plane can be estimated as:

$$l_{ef} = 2\,s_l + 0{,}7\,s_o \qquad (18)$$

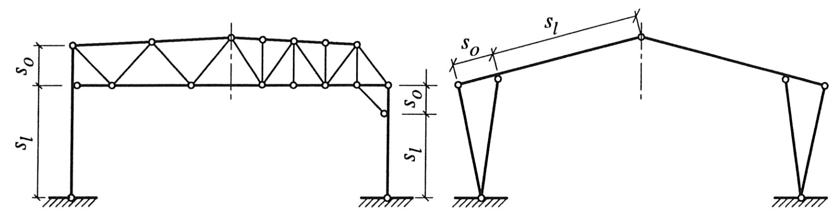

Figure 8 Portal frame (left) and three-hinged frame with V-shaped columns (right).

Torsional buckling of spatial frames

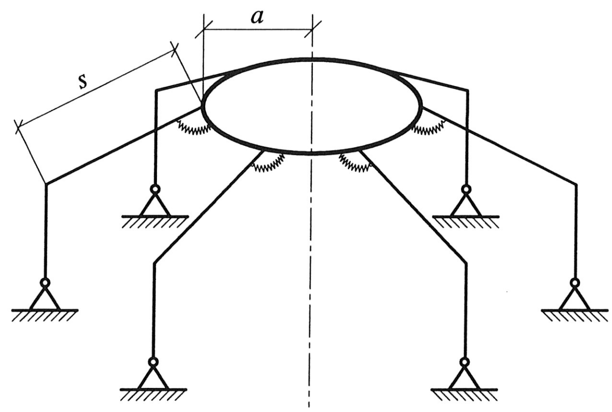

Figure 9 Rotationally symmetrical spatial frame.

For axi-symmetrical structures two types of buckling may occur: the first is the buckling within the plane of the half-frame, the second is the rotational buckling of the spatial structure (see Figure 9). The latter is characterised by a rotation of the compression ring about the vertical axis of symmetry. For $1 < \beta < 2$ and $a/s < 0{,}2$ the following approximate solution for the effective length factor β for the rafter of the half-frame exists:

$$\beta = \sqrt{1 + \frac{2\,a}{s} + \frac{3\,\pi^2\,a\,EI}{4\,s^2\,(1 + a/s)\,K_r}} \qquad (19)$$

Here, EI is the bending stiffness of the rafter for bending about the vertical axis and K_r is the rotational stiffness of the connection between the rafter and the compression ring, also for bending about the vertical axis. For tapered rafters, the bending stiffness is taken at a distance of 0,65 s from the semi-rigid rafter-compression-ring connection similar to the procedure for two- and three-hinged frames.

Example

In the following example, the buckling lengths of the column and rafter of the three-hinged frame shown in Figure 10 are calculated. The influence of the semi-rigid connections in the frame corners is taken into account.

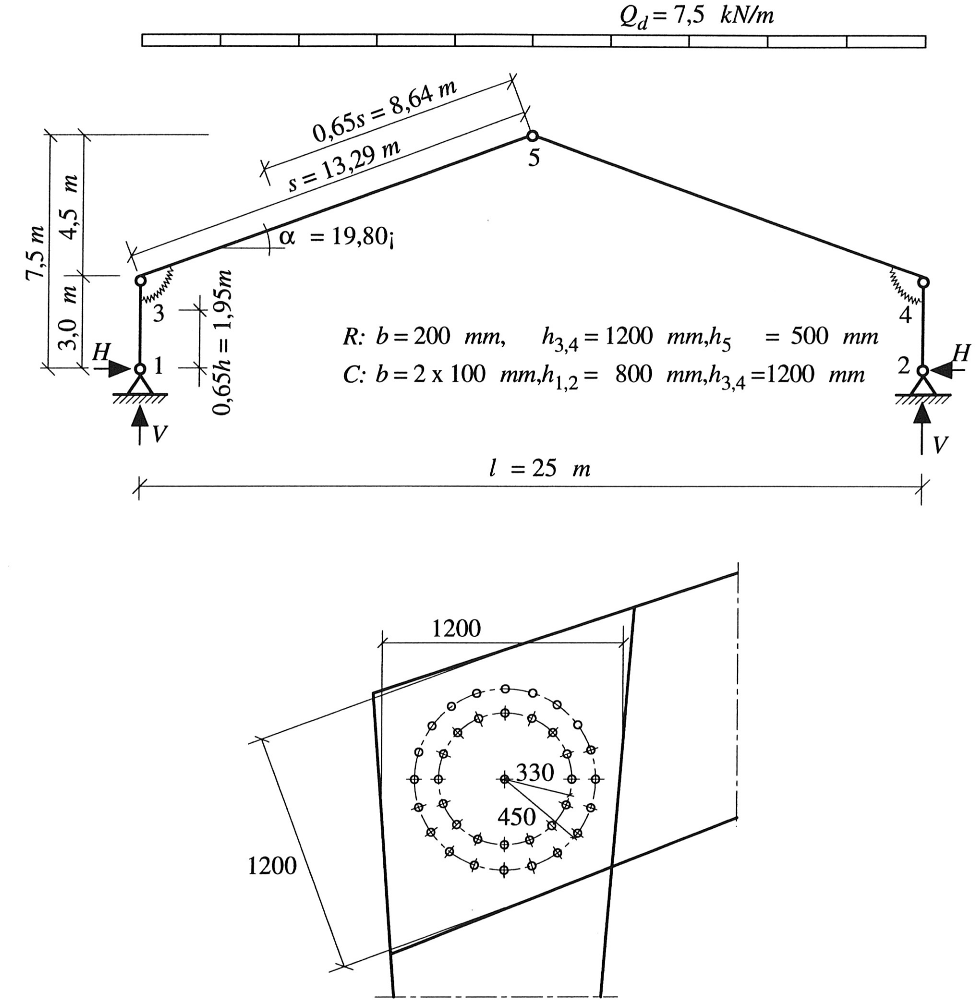

Figure 10 *Three-hinged frame with semi-rigid frame corners. R: Rafter, C: Column. Outer circle: 20 dowels ø 24 mm, inner circle: 16 dowels ø 24 mm.*

prEN 1194: 1993

Glued laminated timber strength class GL28.

$E_{05} = 9600\ N/mm^2$

$\rho_k = 410\ kg/m^3$

EC5: Part 1-1: 4.2 — $K_{ser} = \rho_k^{1,5}\ d\ /\ 20 = 410^{1,5} \cdot 24\ /\ 20 = 9960\ N/mm$

EC5: Part 1-1: 6.1 — $K_u = 2\ K_{ser}\ /\ 3 = 6640\ N/mm$

Equation (1) — $K_r = 2 \cdot 6640 \cdot (20 \cdot 450^2 + 16 \cdot 330^2) = 76,9 \cdot 10^9\ Nmm$

$I = 2 \cdot 100 \cdot 1060^3\ /\ 12 = 19,9 \cdot 10^9\ mm^4$

$I_0 = 200 \cdot 955^3\ /\ 12 = 14,5 \cdot 10^9\ mm^4$

$N = 93,8\ kN$

$N_0 = 105,3\ kN$

Equation (16) — Column:

$$l_{ef} = 3000 \sqrt{4 + 3{,}2 \frac{1{,}99 \cdot 13290}{1{,}45 \cdot 3000} + 10 \frac{9600 \cdot 1{,}99}{3000 \cdot 7{,}69}}$$

$$l_{ef} = 3000 \cdot 5{,}63 = 16900\ mm$$

Equation (17)

Rafter:

$$l_{ef} = 3000 \cdot 5{,}63 \sqrt{\frac{1{,}45 \cdot 93{,}8}{1{,}99 \cdot 105{,}3}}$$

$$l_{ef} = 3000 \cdot 5{,}63 \cdot 0{,}81 = 13600\ mm$$

Concluding summary

- The concept of effective length enables buckling curves for two-hinged columns to be used for the practical design of compression members with different support conditions.
- Rotations in semi-rigid connections generally decrease the elastic critical buckling load of timber compression members.
- Where the approximate solutions given here do not apply, a second order analysis should be carried out, calculating the equilibrium of moments and forces considering the deformed shape of the respective member or structure.

References

Brüninghoff, H. et al. (1989). Holzbauwerke - eine ausführliche Erläuterung zu DIN 1052 Teil 1 bis Teil 3. Beuth, Berlin Köln, Germany, 238 pp.

Heimeshoff, B. (1979). Bemessung von Holzstützen mit nachgiebigem Fußanschluß. Holzbau Statik Aktuell No. 3, Arbeitsgemeinschaft Holz, Düsseldorf, Germany.

Additional Notation

K_r Rotational stiffness of a semi-rigid connection

r_i Distance between a single fastener and the centre of a connection

β Effective length factor