Tapered, curved and pitched cambered beams

STEP lecture B8
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Objectives

To describe the special aspects of tapered, curved and pitched cambered beams and to present the design methods of EC5.

Prerequisites

A8 Glued laminated timber - Production and strength classes
B1 Volume and stress distribution effects

Summary

The lecture starts with basic information related to stress calculations for tapered, curved and pitched cambered beams and explains the parameters influencing the bending strength. EC5 equations for calculation and design are given. Two practical examples, one for a curved beam and the other for a pitched cambered beam, complete the lecture.

Introduction

Glued laminated beams are often tapered and/or curved in order to meet architectural requirements, to provide pitched roofs, to obtain maximum interior clearance, and to reduce wall height requirements at the end supports. The most commonly used types are the single tapered beam, the curved beam with constant cross-section, the double tapered beam and the pitched cambered beam (see Figure 1).

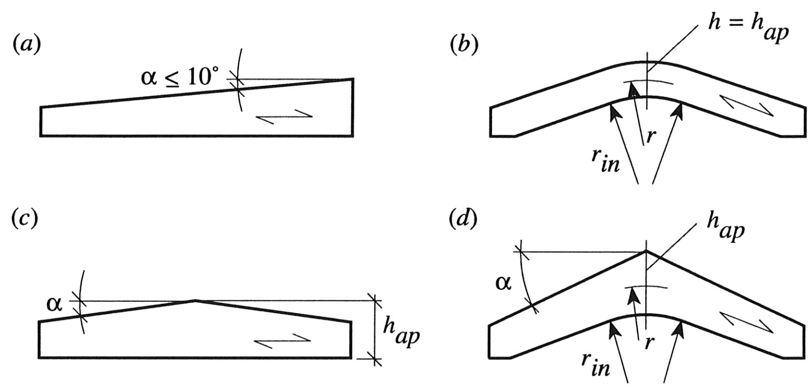

Figure 1 *(a) Single tapered beam, (b) curved beam with constant cross-section, (c) double tapered beam, (d) pitched cambered beam.*

As a result of their shape and the manufacturing procedure, these beams usually have parts with sawn taper cuts and apex zones with or without curved laminations. It is recommended that the laminations should be parallel to the tension edge of the beam with the tapered edges located on the compression edge.

The distribution of bending stresses in tapered beams is non-linear and therefore should be calculated using the theory of thin anisotropic plates, taking into account the ratios of E_0/E_{90} and E_0/G and Poisson's ratio. For design purposes the maximum bending stresses at the tapered edge can be calculated approximately (Riberholt, 1979) according to simple bending theory modified by a factor depending on the slope of the top face (see Equation (4)).

In the apex zone of curved and tapered beams the distribution of the bending stresses is also non-linear. In the apex zone of curved and tapered beams the distribution of the bending stresses is also nonlinear. Additionally, radial stresses perpendicular to the grain are caused by bending moments. Figure 2 shows an incremental section of a curved beam to illustrate the distribution of the bending stresses. The fibres on the inner side of the beam are shorter than those on the outer side. Based on Navier's theory and assuming the neutral axis at mid-depth the strains at the edges are as follows:

$$\varepsilon_i = \frac{\Delta dl_i}{dl_i} > \frac{\Delta dl_o}{dl_o} = \varepsilon_o \qquad (1)$$

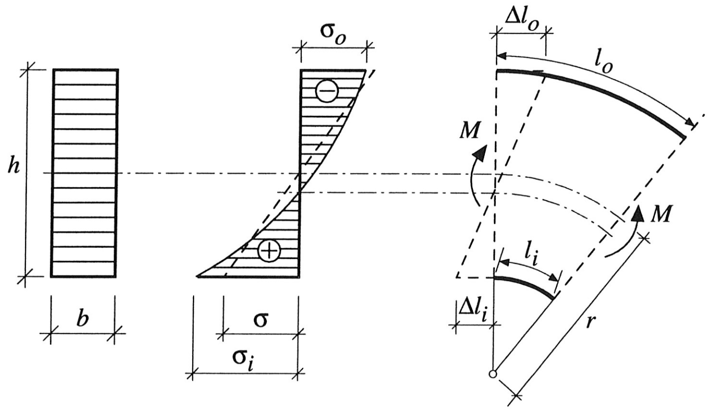

Figure 2 Distribution of bending stresses in a curved beam.

Thus, in accordance with Hooke's law, the maximum bending stress $|\sigma_i|$ is greater than $|\sigma_o|$. Equilibrium of the internal forces over the cross-section is only possible if the neutral axis is closer towards the inner edge. The distribution of the bending stresses is therefore non-linear and hyperbolic with the maximum stress at the inner fibre. For design purposes the maximum bending stresses can be calculated approximately (Blumer, 1975, 1979) by modifying *M/W* with a shape factor k_l (k_l > 1, see Figure 7) which depends on the ratio of the cross-section depth at the apex, h_{ap}, to the radius of curvature of the centerline of the member, *r*, as well as for tapered beams on the slope of the top face, α. For curved beams of constant depth, $\alpha = 0$.

Bending moments in curved members cause radial stresses perpendicular to the grain. Figure 3 shows the apex section of a curved beam under a constant moment. Assuming, for simplification, a linear stress distribution, it can easily be shown that the resulting tensile and compressive forces, F_t and F_c, lead to the force *U* in the radial direction. If the moment increases the radius of curvature, the radial stresses are in tension. The maximum tension stress, max $\sigma_{t,90}$, at the apex can be calculated approximately by modifying *M/W* with a shape factor k_p ($k_p < 1$, see Figure 7).

In addition to the bending stresses in glued laminated curved beams, consideration must be given to the bending of the laminations during glulam production, especially in beams with a small radius of curvature. The bending stress in a curved lamination with thickness *t*, ratio of curvature $r_1 / t = 240$ and $E_0 = 10\,000$ N/mm^2, is theoretically:

$$\sigma_m = \frac{E_0\, t}{2\, r_1} = \frac{10000}{2 \cdot 240} = 20{,}8\ N/mm^2 \qquad (2)$$

These stresses are reduced due to plastic deformations and relaxation, but they have to be taken into account in cases of large curvature. Therefore, the design bending strength of the beam has to be modified by a curvature factor k_r.

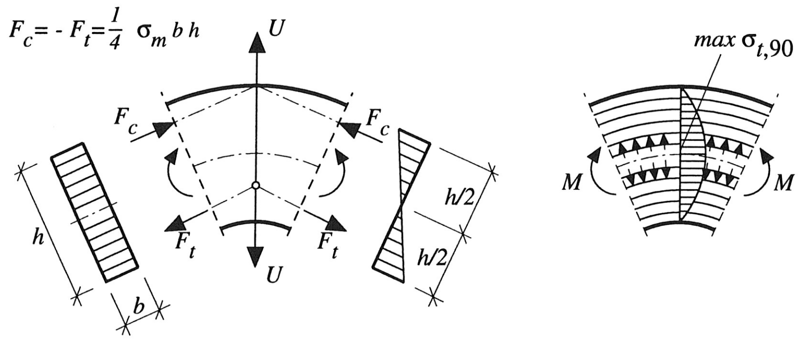

Figure 3 *Stresses perpendicular to grain under constant moment.*

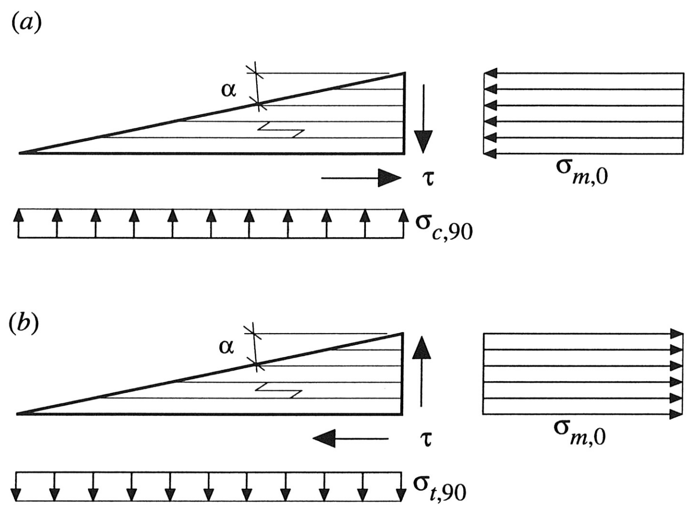

Figure 4 *Stresses parallel and perpendicular to grain and shear stresses at a tapered edge under, (a) compressive bending stress, (b) tensile bending stress.*

At the edges of tapered beams with sawn tapered cuts, stresses perpendicular-to-grain and shear stresses coexist with bending stresses (see Figure 4). The perpendicular-to-grain stresses are in compression or in tension, depending on compressive or tensile bending stresses, respectively.

This stress combination can be taken into account in the design procedures by using a reduced design bending strength, $f_{m,\alpha,d}$ as demonstrated in Equation (7).

Design Procedures

Single tapered beams

Where the grain is parallel to one of the surfaces, and the slope $\alpha \leq 10°$, the design bending stress in the outermost fibre, where the grain is parallel to the surface, should be calculated as (see Figure 5):

$$\sigma_{m,0,d} = (1 + 4 \tan^2\alpha) \frac{6M_d}{bh^2} \tag{3}$$

and on the tapered side as

$$\sigma_{m,\alpha,d} = (1 - 4\tan^2\alpha)\frac{6M_d}{bh^2} \tag{4}$$

The maximum stress condition occurs at the point x, where $\partial\sigma/\partial x = 0$. In the case of uniformly distributed load x results in:

$$x = l / (1 + h_{ap} / h_s) \tag{5}$$

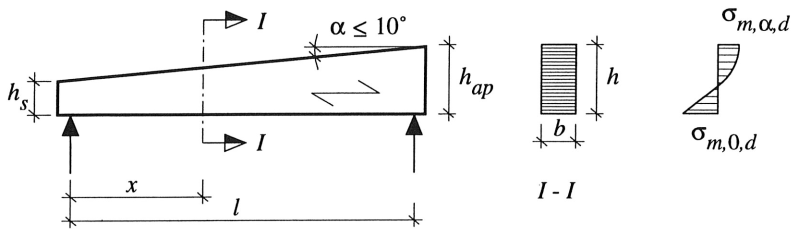

Figure 5   Single tapered beam showing critical cross-section I-I.

In the outermost fibre at the tapered edge the stresses should satisfy the following condition:

$$\sigma_{m,\alpha,d} \le f_{m,\alpha,d} \tag{6}$$

where

$$f_{m,\alpha,d} = \frac{f_{m,d}}{\dfrac{f_{m,d}}{f_{c,90,d}}\sin^2\alpha + \cos^2\alpha} \tag{7}$$

in the case of compressive stresses parallel to the tapered edge (in the case of tensile stresses, $f_{c,90,d}$ in Equation (7) is replaced by $f_{t,d}$).

Double tapered, curved and pitched cambered beams

The apex design bending stress should be calculated as follows:

$$\sigma_{m,d} = k_l \frac{6M_{ap,d}}{bh_{ap}^2} \tag{8}$$

where h_{ap} is defined in Figure 6 and

$$k_l = k_1 + k_2\left(\frac{h_{ap}}{r}\right) + k_3\left(\frac{h_{ap}}{r}\right)^2 + k_4\left(\frac{h_{ap}}{r}\right)^3 \tag{9}$$

with

$$k_1 = 1 + 1{,}4\tan\alpha + 5{,}4\tan^2\alpha \tag{10}$$

$$k_2 = 0{,}35 - 8\tan\alpha \tag{11}$$

$$k_3 = 0{,}6 + 8{,}3\tan\alpha - 7{,}8\tan^2\alpha \tag{12}$$

$$k_4 = 6\tan^2\alpha \tag{13}$$

The slope angle α is defined in Figure 6. For curved beams with constant cross-sections, the slope angle α should be assumed as α = 0°.

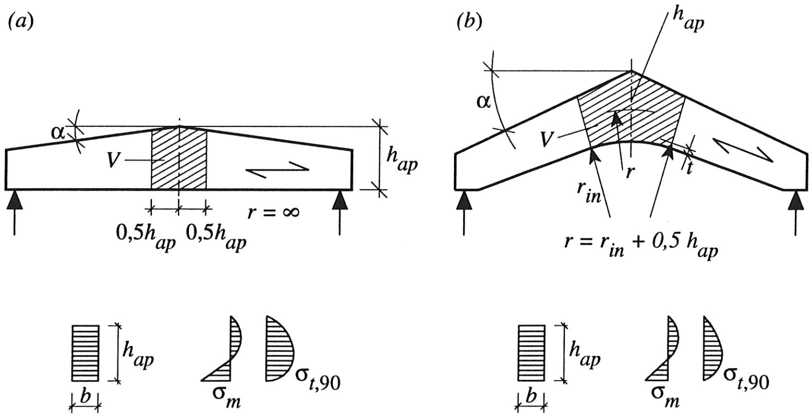

Figure 6 *Elevation and stress distribution at apex for (a) double tapered beam, (b) pitched cambered beam.*

The design tensile stress perpendicular to the grain due to the bending moment should be calculated as follows:

$$\sigma_{t,90,d} = k_p \frac{6M_{ap,d}}{bh_{ap}^2} \tag{14}$$

where

$$k_p = k_5 + k_6\left(\frac{h_{ap}}{r}\right) + k_7\left(\frac{h_{ap}}{r}\right)^2 \tag{15}$$

with

$$k_5 = 0{,}2\ \tan\alpha \tag{16}$$

$$k_6 = 0{,}25 - 1{,}5\ \tan\alpha + 2{,}6\ \tan^2\alpha \tag{17}$$

$$k_7 = 2{,}1\ \tan\alpha - 4\ \tan^2\alpha \tag{18}$$

$M_{ap,d}$ is the design bending moment at the apex. In the apex zone, the design bending stresses shall satisfy the following condition:

$$\sigma_{m,d} \le k_r\, f_{m,d} \tag{19}$$

where

$$k_r = \begin{cases} 1 & \text{for} \quad r_{in}/t \ge 240 \\ 0{,}76 + 0{,}001\ r_{in}/t & \text{for} \quad r_{in}/t < 240 \end{cases} \tag{20}$$

In the apex zone the design tensile stress perpendicular to the grain should satisfy the following condition:

$$\sigma_{t,90,d} \le k_{dis}\,(V_0/V)^{0,2}\, f_{t,90,d} \tag{21}$$

where k_{dis} is a factor which takes into account the stress distribution. The ratio of the reference volume $V_0 = 0{,}01\ m^3$ to the stressed volume V considers the influence of the volume on the perpendicular-to-grain tensile strength (see STEP lecture B1). V should as a maximum be taken as 2/3 of the whole beam volume V_b (see Table 1).

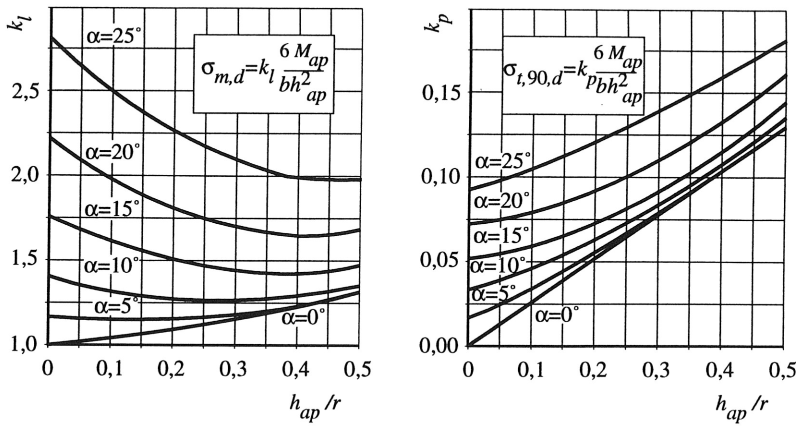

Figure 7 Factors k_l and k_p for different radius of curvature, r, and slope angles, α.

Curved beam with constant cross-section	$k_{dis} = 1{,}4$	$V = \frac{\beta\pi}{180}\, b\,(h_{ap}^2 + 2r_{in}h_{ap}) \leq \frac{2}{3}V_b$
Double tapered beam	$k_{dis} = 1{,}4$	$V = bh_{ap}^2\,(1 - \frac{\tan\alpha}{4}) \leq \frac{2}{3}\,V_b$
Pitched cambered beam	$k_{dis} = 1{,}7$	$V = b\left(\sin\alpha\cos\alpha\,(r_{in} + h_{ap})^2 - r_{in}^2\,\frac{\pi\,\alpha}{180}\right) \leq \frac{2}{3}\,V_b$

Table 1 Factor k_{dis} and volume V for different types of beams.

Design examples

Curved beam with constant cross-section

Material: Glued laminated timber made of spruce. Strength class GL28 according to prEN 1194 "Glued laminated timber - Strength classes and determination of characteristic values"

l	$= 20\ m$	q_d	$= 7\ kN/m$	h	$= 1{,}0\ m$
β	$= 10°$	$M_{ap,d}$	$= 350\ kNm$	b	$= 0{,}2\ m$
r	$= 20\ m$	k_{mod}	$= 0{,}8$	t	$= 30\ mm$

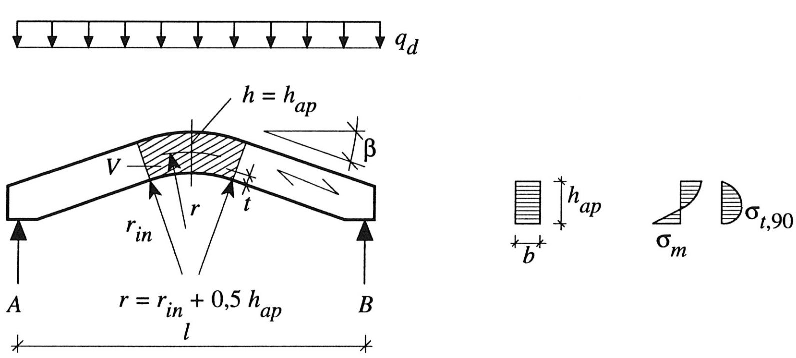

Figure 8 Elevation and stress distribution at apex for curved beam of constant cross-section under uniform load, q_d.

Design bending stress and design tensile stress at apex:

$$V = 1{,}40 m^3 \qquad k_{dis} = 1{,}4 \qquad k_r = 1{,}0 \qquad k_{Vol} = \left(\frac{0{,}01}{1{,}40}\right)^{0{,}2} = 0{,}37$$

$$k_p = 0{,}25 \frac{h_{ap}}{r} = 0{,}013 \qquad k_l = \left(1 + 0{,}35 \frac{1}{20} + 0{,}6 \left(\frac{1}{20}\right)^2\right) = 1{,}02$$

$$\sigma_{t,90,d} = 0{,}013 \cdot \frac{6 \cdot 350 \cdot 10^6}{200 \cdot 1000^2} = 0{,}131 \ N/mm^2$$

$$\sigma_{m,d} = 1{,}02 \cdot \frac{6 \cdot 350 \cdot 10^6}{200 \cdot 1000^2} = 10{,}7 \ N/mm^2$$

Characteristic material properties (GL28, prEN 1194):
$f_{m,g,k} = 28\ N/mm^2$ $\qquad f_{t,90,g,k} = 0{,}45\ N/mm^2$

The design bending and tension perpendicular to the grain strengths are:

$$f_{t,90,g,d} = \frac{k_{mod}\, f_{t,90,g,k}}{\gamma_M} = \frac{0{,}8 \cdot 0{,}45}{1{,}3} = 0{,}277 \ N/mm^2$$

$$f_{m,g,d} = \frac{k_{mod}\, f_{m,g,k}}{\gamma_M} = \frac{0{,}8 \cdot 28}{1{,}3} = 17{,}2 \ N/mm^2$$

Verification of failure condition:

$$\sigma_{t,90,d} = 0{,}131 \ N/mm^2 \leq k_{Vol}\, k_{dis}\, f_{t,90,g,d} = 0{,}37 \cdot 1{,}4 \cdot 0{,}277 = 0{,}143 \ N/mm^2$$

$$\sigma_{m,d} = 10{,}7 \ N/mm^2 \leq k_r\, f_{m,g,d} = 1{,}0 \cdot 17{,}2 = 17{,}2 \ N/mm^2$$

Pitched cambered beam
System and loading as before; apex with glued haunch (see Figure 6b).
Depth at apex: $h_{ap} = 1{,}32\ m$ $\qquad$ Radius: $r = 19{,}50 + 1{,}32/2 = 20{,}16\ m$

Design stress perpendicular to the grain at apex ($\alpha = \beta = 10°$):
with: $V = 1{,}55\ m^3$, $\quad k_{dis} = 1{,}7$, $\quad k_r = 1{,}0$

$$k_{vol} = \left(\frac{V_0}{V}\right)^{0{,}2} = 0{,}36 \qquad \left(\frac{h_{ap}}{r}\right) = 0{,}065$$

$$k_p = 0{,}2 \tan 10° + (0{,}25 - 1{,}5 \tan 10° + 2{,}6 \tan^2 10°) \cdot 0{,}065$$

$$+ (2{,}1 \tan 10° - 4 \tan^2 10°) \cdot 0{,}065^2 = 0{,}041$$

$$\sigma_{t,90,d} = 0{,}041 \cdot \frac{6 \cdot 350 \cdot 10^6}{200 \cdot 1320^2} = 0{,}249 \ N/mm^2$$

Verification of failure condition:

$$\sigma_{t,90,d} = 0{,}249 \ N/mm^2 > k_{Vol}\, k_{dis}\, f_{t,90,g,d} = 0{,}36 \cdot 1{,}7 \cdot 0{,}277 = 0{,}172 \ N/mm^2$$

The failure condition of the perpendicular-to-grain tensile stress is not satisfied. This is surprising compared to the curved beam. One reason is the fact that in Equations

(14)-(18) a constant moment is assumed to act in the curved part of the beam. A more accurate calculation shows that the stresses in a pitched cambered beam under uniformly distributed loads are 20 % less, whereas the stresses in the curved beam remain almost unchanged (Ehlbeck, Kürth, 1990). Nevertheless, differences exist between the results from the design methods of EC5 and test results showing that both beam types have similar failure loads.

Concluding summary

- In single tapered, curved, double tapered and pitched cambered beams the tensile bending stresses at the inner edge are greater than in straight beams.

- In curved zones with a bending moment increasing the radius of curvature, tension stresses perpendicular to the grain occur.

- Tapered edges reduce the bending strength because of the combined effects of bending, compression, tension and shear parallel and perpendicular to the grain.

- Bending of the laminates in curved beams reduces the bending strength when the radius of curvature is small.

References

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