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Editör: Doç.Dr. Ahmet KAZAN

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İÇİNDEKİLER

ON SOME PERFECT ARF NUMERICAL SEMIGROUPS

Sedat İLHAN[1]

1. INTRODUCTION

Numerical semigroups, which emerged towards the end of the 19th century, are an important subject within Algebra and Number Theory, which has an important place in mathematics. The numerical semigroups are used in many areas of mathematics. Recently, the expression numerical semigroups; Algebra has facilitated wider applications in fields such as Algebraic Geometry, Topology and Differential Geometry. We can see that Local, Noteherian Local, Gorenstein and Arf rings can be characterized in terms of numerical semigroups, especially under certain conditions (For details see [1, 17,18,19,20,21]).

The topic of numerical semigroups emerged when the problem known as the "Frobenius Problem" was put forward by Sylvester ([14]). Later, Brauer derived the Frobenius formula for a class of numerical semigroups ([16]).

Arf Numerical semigroups, a class of numeric semigroups were formed after Cahit Arf's study on multiple points in algebraic curves ([15]). Lipman, on the other hand, reintroduced the Arf rings in his studies, based on Arf's studies, and saw that the numerical semigroups formed based on the values of the rings also have the Arf property ([22]).

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However, recently many researchers have carried out various studies on Arf numerical semigroups. (For details see [5,10,23,25,28]).

The gaps and isolated gaps of numerical semigroups are also important and some studies have been done on this subject ([2,13,24,30]). Many studies have been carried out on perfect numerical semigroups, especially those obtained from discrete spaces. For example, Frais and his colleagues studied the basic properties of perfect numerical semigroups and investigated their relationships with other concepts ([11,12]). Various properties are also given about the parts of perfect numerical semigroups ([13,26]). On the other hand, mathematical relations of perfect numerical semigroups with Frobenius number was also studied by Moreno and his colleagues ([27]).

P. A. Garcia-Sanchez et al. ([25]) gave all Arf numerical semigroups with fixed Frobenius number and multiplicity up to five in Theorem 3.1. We note that if Arf numerical semigroups with multiplicity one then the only numerical semigroup with multiplicity one is $\mathbb{N}$, which is trivially Arf. Also, Let us state here that $\theta \equiv 1(m(\delta))$. On the contrary, if $\theta \equiv 1(m(\delta))$ then $\tau(\delta) \in \delta$ contradiction arises, where θ is the conductor of δ.

2. NOTATION

Let $\mathbb{N} = \{u \in \mathbb{Z} : u \geq 0\}$ and $\mathbb{Z}$ be integer set. If *(i)* $p_1 + p_2 \in \delta$, for $p_1, p_2 \in \delta$ *(ii)* $\mathbb{N} \backslash \delta$ is finite *(iii)* $0 \in \delta$ then $\delta \subseteq \mathbb{N}$ is a numerical semigroup.

For δ numerical semigroup we write $\delta = < q_1, q_2, ..., q_s > = \left\{ \sum_{j=1}^{s} u_j q_j : u_j \in \mathbb{N} \right\}$.

$L \subset \mathbb{N}$ is minimal system of generators of S if $< L > = S$ and there isn't any subset $T \subset L$ such that $< T > = S$. Also, $\mu(S) = \min\{k \in S : k > 0\}$ and $e(S) = s$ are called multiplicity and embedding dimension of S, respectively. It is know that $e(S) \le \mu(S)$. If $e(S) = \mu(S)$ then S is called maksimal embedding dimension (MED) (See [6,8,29]).

Let S be a numerical semigroup, then $\tau(S) = \max(\mathbb{Z} \backslash S)$ is called as Frobenius number of S. $\sigma(S) = Card(\{x \in \mathbb{N} : 0 \le x \le \tau(S)\} \cap S)$ and $\theta = \tau(S) + 1$ are called as the determine number and the conductor of S, respectively. . Here, we will indicate the number of elements of the set S by $Card(S)$.

S be numerical semigroup such that $S = < \theta_1, \theta_2, ..., \theta_s >$, then we write

$$S = < \theta_1, \theta_2, ..., \theta_s > = \{r_0 = 0, r_1, r_2, ..., r_{s-1}, r_s = \tau(S) + 1, \rightarrow ...\}$$

, where $r_j < r_{j+1}$, $s = \sigma(S)$ and the arrow means that every integer greater than $\tau(S) + 1 \in S$ for $1 \le j \le s = \sigma(S)$.

Let S be a numerical semigroup. If $\theta \in \mathbb{N}$ and $\theta \notin S$ then θ is called gap of S. We denote the set of gaps of S, by $\Delta(S)$, i.e, $\Delta(S) = \mathbb{N} \backslash S = \{\theta \in \mathbb{N} : \theta \notin S\}$.

The number $\Omega(S) = Card(\Delta(S))$ is called the genus of S, and it known that $\Omega(S) + \sigma(S) = \tau(S) + 1$ (see [5,8]) .

S is called symmetric numerical semigroup if $\tau(S) - w \in S$, for $w \in \mathbb{Z} \backslash S$. It is know the numerical semigroup $S = < \lambda_1, \lambda_2 >$ is symmetric and $\tau(S) = \lambda_1 \lambda_2 - \lambda_1 - \lambda_2$. In this case, we write $\sigma(S) = \Omega(S) = \frac{\tau(S) + 1}{2}$. S is called pseudo symmetric numerical semigroup if $t(d)$ is even and $t(d)-p \notin d,\ p=\frac{t(d)}{2}$, for $p \in \mathbb{Z} \setminus d$. (For details see [2,3,4,9]).

Let d be a numerical semigroup. We define the set $\frac{d}{z}=\{u \in \mathbb{N} : zu \in d\}$, for $z \in \mathbb{N}$. If $z=2$ then the set $\frac{d}{2}$ is called the half of d ([7,8]).We note that while d is symmetric (pseudo symmetric) numerical semigroup, $\frac{d}{2}$ need not be symmetric (pseudo symmetric). $d=<5,9>=\{0,5,9,10,14,15,18,19,20,23,24,25,27,28,29,30,32,\to \ldots\}$ is symmetric numerical semigroup but $\frac{d}{2}=\{u \in \mathbb{N} : 2u \in d\}=\{0,5,7,9,10,12,14,\to \ldots\}$ is not symmetric

($d=<3,8,13>=\{0,3,6,8,9,11,\to \ldots\}$ is pseudo symmetric numerical semigroup but $\frac{d}{2}=\{u \in \mathbb{N} : 2u \in d\}=\{0,3,4,6,\to \ldots\}=<3,4>$ is not pseudo symmetric) .

A numerical semigroup d is Arf if $z_1+z_2-z_3 \in d$, for all $z_1,z_2,z_3 \in d$ such that $z_1 \geq z_2 \geq z_3$. The intersection of any family of Arf numerical semigroups is again an Arf numerical semigroup. We note that $\mathbb{N}$ is an Arf numerical semigroup and every Arf numerical semigroup is MED ([10,23,25]).

Let d be a numerical semigroup and $r \in \Delta(d)$. If $r-1, r+1 \in d$ then is r is called isolated gap of d. We denote the set of isole gaps of d, by $Y(d)$, i.e, $Y(d)=\{r \in \Delta(d) : r-1, r+1 \in d\}$. A numerical semigroup d is perfect if $Y(d) = f$ (see [12,13,26,27]). We note that Arf and perfect numerical semigroups do not require each other. For example, $d = < 3,11,13 > = \{0,3,6,9,11,® \ ...\}$ is an Arf numerical semigroup but d is not perfect since $Y(d) = \{10\}$ ¹ f On the other hand, $V = < 5,8,9 > = \{0,5,8,9,10,13,® \ ...\}$ is perfect numerical semigroup since $Y(V) = f$. But V is not Arf because $9 + 8 - 5 = 12$ Ï V .

In this study, Arf numerical semigroups with multiplicity 5 and less than 5 are examined and whether half of them are perfect or not.

3. PERFECT ARF NUMERICAL SEMIGROUPS

THEOREM 3.1. ([25]) Let d be an Arf numerical semigroup. q and $m(d)$ are conductor and multiplicity of d, recpectively. Then the following is true

(a) If q ° $0 \pmod 2$ and $m(d) = 2$ then $d = < 2, q + 1 >$.

(b) (i) if q ° $0 \pmod 3$ and $m(d) = 3$ then $d = < 3, q + 1, q + 2 >$

(ii) if q ° $2 \pmod 3$ and $m(d) = 3$ then $d = < 3, q, q + 2 >$.

(c) $m(d) = 4$ and conductor q, where q ° $0, 2$ *or* $3 \pmod 4$. Then the all Arf numerical semigroups as follows:

(i) If $q \equiv 0 \pmod 4$, then $d = < 4, 4v+2, q+1, q+3 >$ for some $v \in \left\{1, 2, ..., \frac{q}{4}\right\}$. (ii) If $q \equiv 2 \pmod 4$, then $d = < 4, 4v+2, q+1, q+3 >$ for some $v \in \left\{1, 2, ..., \frac{q-2}{4}\right\}$

(iii) If $q \equiv 3 \pmod 4$, then $d = < 4, q, q+2, q+3 >$.

(d) $m(d) = 5$ and conductor q , where $q \equiv 0, 2, 3 \; or \; 4 \pmod 5$. Then the all Arf numerical semigroups as follows:

(i) If $q \equiv 0 \pmod 5$, then $d = < 5, q-2, q+1, q+2, q+4 >$ or $d = < 5, q+1, q+2, q+3, q+4 >$

(ii) If $q \equiv 2 \pmod 5$, then $d = < 5, q, q+1, q+2, q+4 >$

(iii) If $q \equiv 3 \pmod 5$, then $d = < 5, q, q+1, q+3, q+4 >$

(iv) If $q \equiv 4 \pmod 5$, then $d = < 5, q-2, q, q+2, q+4 >$ or $d = < 5, q, q+2, q+3, q+4 >$.

PROPOSITION 3.2. Let d be an Arf numerical semigroup. q and $m(d)$ are conductor and multiplicity of d, recpectively. Then the following is true

(a) If $m(d) = 1$ then $d = \infty$ is perfect.

(b) If $m(d) = 2$ and $q \equiv 0 \pmod 2$ then $d = < 2, q+1 >$ is not perfect.

PROOF. Let d be an Arf numerical semigroup.with q and $m(d)$ are conductor and multiplicity of d, recpectively.

(a If $m(d) = 1$ then $d = \infty$ is perfect, since $Y(d) = Y(\infty) = f$.

(b) If $m(\delta)=2$ and $\theta \equiv 0 \pmod 2$ then $\delta=<2,\theta+1>$ $=\{0,2,4,6,\ldots,\theta,\rightarrow \ldots\}$. Thus, we write that $\Psi(\delta)=\{p \in \Delta(\delta): p-1, p+1 \in \delta\}=\{1,3,5,\ldots,\theta-1\} \neq \phi$. So, δ is not perfect.

PROPOSITION 3.3. Let δ be an Arf numerical semigroup with multiplicity $m(\delta)=3$ and conductor θ. Then we have

(i) if $\theta \equiv 0 \pmod 3$ then $\delta=<3,\theta+1,\theta+2>$ is perfect.

(ii) if $\theta \equiv 2 \pmod 3$ then $\delta=<3,\theta,\theta+2>$ is not perfect.

PROOF. Let δ be an Arf numerical semigroup with multiplicity $m(\delta)=3$ and conductor θ. Then we have

(i) if $\theta \equiv 0 \pmod 3$ then $\delta=<3,\theta+1,\theta+2>=\{0,3,6,9,\ldots,\theta-3,\theta,\rightarrow \ldots\}$ is perfect since $\Psi(\delta)=\{p \in \Delta(\delta): p-1, p+1 \in \delta\}=\phi$.

(ii) if $\theta \equiv 2 \pmod 3$ then $\delta=<3,\theta,\theta+2>=\{0,3,6,9,\ldots,\theta-2,\theta,\rightarrow \ldots\}$ is not perfect since $\tau(\delta)=\theta-1 \in \Psi(\delta)$.

PROPOSITION 3.4. Let δ be an Arf numerical semigroup with multiplicity $m(\delta)=4$ and conductor θ. In this case,

(i) if $\theta \equiv 0 \pmod 4$ and $\delta=<4,4v+2,\theta+1,\theta+3>$ for some $v \in \left\{1,2,\ldots,\frac{\theta}{4}\right\}$, then δ is not perfect.

(ii) If $q \equiv 2 \pmod 4$ and $d = <4, 4v+2, q+1, q+3>$ for some $v \in \left\{1, 2, \ldots, \frac{q-2}{4}\right\}$, then d is not perfect.

(iii) if $q \equiv 3 \pmod 4$ then $d = <4, q, q+2, q+3>$ is perfect.

PROOF. Let d be an Arf numerical semigroup with multiplicity $m(d) = 4$ and conductor q in Theorem 3.1/ (c).

(i) If $q \equiv 0 \pmod 4$ then $d = <4, 4v+2, q+1, q+3>$ for some $v \in \left\{1, 2, \ldots, \frac{q}{4}\right\}$. Then we write that $d = <4, 4v+2, q+1, q+3> = \{0, 4, 8, 12, \ldots, 4v, 4v+2, 4v+4, 4v+6, \ldots, q-2, q, \rightarrow \ldots\}$ In this case, it is trivial $Y(d) \neq f$ since $q-1 \in Y(d)$. Thus, d is not perfect numerical semigroup.

(ii) If $q \equiv 2 \pmod 4$, then $d = <4, 4v+2, q+1, q+3>$ for some $v \in \left\{1, 2, \ldots, \frac{q-2}{4}\right\}$. Thus, d is not perfect numerical semigroup. Because $d = <4, 4v+2, q+1, q+3> = \{0, 4, 8, 12, \ldots, 4v, 4v+2, 4v+4, 4v+6, \ldots, q-2, q, \rightarrow \ldots\}$ and, also $Y(d) = \{p \in D(d) : p-1, p+1 \in d\} = \{4v+1, 4v+3, \ldots, q-1\} \neq f$

(iii) If $q \equiv 3 \pmod 4$ then $d = <4, q, q+2, q+3> = \{0, 4, 8, 12, \ldots, q-3, q, \rightarrow \ldots\}$ is perfect since $Y(d) = f$.

PROPOSITION 3.5. Let d be an Arf numerical semigroup with multiplicity $m(d) = 5$ and conductor q in Theorem 3.1/(d) .

(a) If $q \equiv 0 \pmod 5$, then $d = <5, q-2, q+1, q+2, q+4>$ is not perfect.

(b) If $q^{\circ}\ 0(\mathrm{mod}\,5)$, then $d = <5, q+1, q+2, q+3, q+4>$ is perfect.

(c) If $q^{\circ}\ 2(\mathrm{mod}\,5)$, then $d = <5, q, q+1, q+2, q+4>$ is not perfect.

(d) If $q^{\circ}\ 3(\mathrm{mod}\,5)$, then $d = <5, q, q+1, q+3, q+4>$ is perfect.

(e) If $q^{\circ}\ 4(\mathrm{mod}\,5)$, then $d = <5, q-2, q, q+2, q+4>$ is not perfect.

(f) If $q^{\circ}\ 4(\mathrm{mod}\,5)$, then $d = <5, q, q+2, q+3, q+4>$ is perfect.

PROOF. Let d be an Arf numerical semigroup with multiplicity $m(d) = 5$ and conductor q in Theorem 3.1/(d) .

(a) If $q^{\circ}\ 0(\mathrm{mod}\,5)$ then $d = <5, q-2, q+1, q+2, q+4> = \{0, 5, 10, 15, ..., q-5, q-2, q, ®\}$ is not perfect since $Y(d) = \{t(d) = q-1\} ¹\ f$.

(b) If $q^{\circ}\ 0(\mathrm{mod}\,5)$ then $d = <5, q+1, q+2, q+3, q+4> = \{0, 5, 10, 15, ..., q-5, q, ®\ ...\}$ is perfect since $Y(d) = f$.

(c) If $q^{\circ}\ 2(\mathrm{mod}\,5)$ then $d = <5, q, q+1, q+2, q+4> = \{0, 5, 10, 15, ..., q-2, q, ®\ ...\}$ is not perfect since $Y(d) = \{t(d) = q-1\} ¹\ f$.

(d) If $q^{\circ}\ 3(\mathrm{mod}\,5)$ then $d = <5, q, q+1, q+3, q+4> = \{0, 5, 10, 15, ..., q-3, q, ®\ ...\}$ is perfect. Because $Y(d) = f$.

(e) If $q \equiv 4(\bmod 5)$, then $d = <5, q-2, q, q+2, q+4>$ is not perfect. Because $d = <5, q-2, q, q+2, q+4> = \{0,5,10,15,...,q-4,q-2,q,\rightarrow ...\}$ and $Y(d) = \{p \in D(d) : p-1, p+1 \in d\} = \{q-3, q-1\} \neq f$.

(f) If $q \equiv 4(\bmod 5)$ then $d = <5, q, q+2, q+3, q+4> = \{0,5,10,15,...,q-4,q,\rightarrow ...\}$ is perfect since $Y(d) = f$.

4. THE HALF OF PERFECT ARF NUMERICAL SEMIGROUPS

Let d be numerical semigroup. We define the set $\frac{d}{z} = \{y \in \mathbb{N} : zy \in d\}$. If $z = 2$ then the set $\frac{d}{2}$ is called the half of d. We note that If d is MED numerical semigroup then $\frac{d}{2}$ may be not a MED numerical semigroup. For example $d = <3,8,13> = \{0,3,6,8,9,11,\rightarrow ...\}$ is a MED numerical semigroup but $\frac{d}{2} = \{y \in \mathbb{N} : 2y \in d\} = \{0,3,4,6,\rightarrow ...\} = <3,4>$ is not a MED numerical semigroup.

LEMMA 4.1. If d is Arf numerical semigroup then $\frac{d}{2}$ is Arf.

PROOF. Let d be an Arf numerical semigroup. Then we have $u+v-w \in d$ for all $u,v,w \in d$ where $u \geq v \geq w$. If $u+v-w \in d$ then $2(u+v-w) \in d \Rightarrow u+v-w \in \frac{d}{2}$ for all

$u,v,w \in \frac{d}{2}$ where $u \geq v \geq w$ since $d \subseteq \frac{d}{2}$. Thus, we obtain that $\frac{d}{2}$ is Arf.

LEMMA 4.2. Let d be a perfect Arf numerical semigroup.with q and $m(d)$ is conductor and multiplicity of d, recpectively. If S given by Propotıon 3.2 then $\frac{d}{2} = \mathbb{N}$ is perfect.

PROOF. Let d be a perfect Arf numerical semigroup given by Proposition 3.2.

(a) If $m(d) = 1$ then $d = \mathbb{N}$. Thus $\frac{d}{2} = \mathbb{N}$ since $2 \in d$. So, $\frac{d}{2} = \mathbb{N}$ is perfect.

(b) If $m(d) = 2$ and $q \equiv 0 \pmod 2$ then $d = \langle 2, q+1 \rangle = \{0, 2, 4, 6, ..., q, \to ...\}$. So, it is clear that $\frac{d}{2} = \mathbb{N}$ is perfect.

THEOREM 4.3. Let d be an Arf numerical semigroup with multiplicity $m(d) = 3$ and conductor d such that $q \equiv 0 \pmod 3$ and $d = \langle 3, q+1, q+2 \rangle$. In this case,

(a) if q is odd then $\frac{d}{2} = \langle 3, \frac{q+1}{2}, \frac{q+5}{2} \rangle = \left\{0, 3, 6, 9, ..., \frac{q+1}{2} - 2, \frac{q+1}{2}, \to ...\right\}$ is not perfect;

(b) if q is even then $\frac{d}{2} = \langle 3, \frac{q+2}{2}, \frac{q+4}{2} \rangle = \left\{0, 3, 6, 9, ..., \frac{q}{2} - 3, \frac{q}{2}, \to ...\right\}$ is perfect.

PROOF. Let d be an Arf numerical semigroup with multiplicity $m(d)=3$ and conductor d such that $q \equiv 0\,(\bmod 3)$ and $d=<3,q+1,q+2>$. We write that $d=<3,q+1,q+2>=\{0,3,6,9,\ldots,q-3,q,\rightarrow\ldots\}$. In this case,

(a) if q is odd then $\frac{q+1}{2}$ and $\frac{q+5}{2}$ are even. So,

$$
\begin{aligned}
x\in \frac{d}{2} &\Leftrightarrow x=3a_1+(\frac{q+1}{2})a_2+(\frac{q+5}{2})a_3,\ for\ a_1,a_2,a_3\in \mathbb{N}\\
&\Leftrightarrow 2x=3(2a_1)+(q+1)a_2+(q+5)a_3\\
&\Leftrightarrow 2x=3(2a_1)+(q+1)a_2+(q+2)a_3+3a_3\\
&\Leftrightarrow 2x=3(2a_1+a_3)+(q+1)a_2+(q+2)a_3\\
&\Leftrightarrow 2x\in <3,q+1,q+2>=d.
\end{aligned}
$$

Thus, we find that $\frac{d}{2}=<3,\frac{q+1}{2},\frac{q+5}{2}>$.

Let's $T=\left\{0,3,6,9,\ldots,\frac{q+1}{2}-2,\frac{q+1}{2},\rightarrow\ldots\right\}$. It is evidence $T\subseteq \frac{d}{2}$. If

$y\in \frac{d}{2}=<3,\frac{q+1}{2},\frac{q+5}{2}> \Rightarrow y=3u_1+(\frac{q+1}{2})u_2+(\frac{q+5}{2})u_3\ ;u_1,u_2,u_3\in \mathbb{N}$

$\Rightarrow y=3u_1+(\frac{q+1}{2})u_2+(\frac{q+1}{2}+2)u_3\in T$ since the conductor of T is $\frac{q+1}{2}$. Thus we obtain $\frac{d}{2}=T$ when q is odd. In this case, $\frac{d}{2}$ is not perfect since $\frac{q+1}{2}-1\in Y(\frac{d}{2})$.

(b) if we make similar operations for the even condition of q, we can easily find that

$$\frac{d}{2} = <3, \frac{q+2}{2}, \frac{q+4}{2}> = <3, \frac{q}{2}+1, \frac{q}{2}+2> = \left\{0, 3, 6, 9, \ldots, \frac{q}{2}-3, \frac{q}{2}, \rightarrow \ldots\right\}$$

. Thus, it is clear that $\frac{d}{2} = \mathbb{N}$ is perfect since $Y(\frac{d}{2}) = \varphi$.

THEOREM 4.4. Let d be an Arf numerical semigroup with multiplicity $m(d) = 3$ and conductor q such that $q \equiv 2 \pmod 3$ and $d = <3, q, q+2>$. In this case,

(a) if q is odd then $\frac{d}{2} = <3, \frac{q+3}{2}, \frac{q+5}{2}> = \left\{0, 3, 6, 9, \ldots, \frac{q+1}{2}-3, \frac{q+1}{2}, \rightarrow \ldots\right\}$ is perfect;

(b) if q is even then $\frac{d}{2} = <3, \frac{q}{2}, \frac{q}{2}+1> = \left\{0, 3, 6, 9, \ldots, \frac{q-2}{2}-3, \frac{q-2}{2}, \rightarrow \ldots\right\}$ is perfect.

PROOF. Let d be an Arf numerical semigroup with multiplicity $m(d) = 3$ and conductor q such that $q \equiv 2 \pmod 3$ and $d = <3, q, q+2>$. We write that $d = <3, q, q+2> = \{0, 3, 6, 9, \ldots, q-2, q, \rightarrow \ldots\}$. In this case,

(a) if q is odd then $\frac{q+3}{2}$ and $\frac{q+5}{2}$ are even. So,

$$\begin{aligned} x \in \frac{d}{2} &\Leftrightarrow x = 3a_1 + (\frac{q+3}{2})a_2 + (\frac{q+5}{2})a_3, \ for\ a_1, a_2, a_3 \in \mathbb{N} \\ &\Leftrightarrow 2x = 3(2a_1) + (q+3)a_2 + (q+5)a_3 \\ &\Leftrightarrow 2x = 3(2a_1) + (q+3)a_2 + (q+2)a_3 + 3a_3 \\ &\Leftrightarrow 2x = 3(2a_1 + a_2 + a_3) + qa_2 + (q+2)a_3 \\ &\Leftrightarrow 2x \in <3, q, q+2> = d. \end{aligned}$$

Thus, we find that $\frac{d}{2} = <3, \frac{q+3}{2}, \frac{q+5}{2}>$

Let's $K = \left\{0, 3, 6, 9, \ldots, \frac{\theta+1}{2} - 3, \frac{\theta+1}{2}, \rightarrow \ldots\right\}$. It is evidence $K \subseteq \frac{\delta}{2}$. If $y \in \frac{\delta}{2} = < 3, \frac{\theta+3}{2}, \frac{\theta+5}{2} > \Rightarrow y = 3u_1 + (\frac{\theta+3}{2})u_2 + (\frac{\theta+5}{2})u_3 \; ; u_1, u_2, u_3 \in \mathbb{N}$ $\Rightarrow y = 3u_1 + (\frac{\theta+1}{2})u_2 + (\frac{\theta+1}{2} + 2)u_3 \in K$ since the conductor of K is $\frac{\theta+1}{2}$. Thus we obtain $\frac{\delta}{2} = < 3, \frac{\theta+3}{2}, \frac{\theta+5}{2} > = \left\{0, 3, 6, 9, \ldots, \frac{\theta+1}{2} - 3, \frac{\theta+1}{2}, \rightarrow \ldots\right\}$ when θ is odd.

(b) if we make similar operations for the even condition of θ, we can easily obtain that $\frac{\delta}{2} = < 3, \frac{\theta}{2}, \frac{\theta}{2} + 1 > = \left\{0, 3, 6, 9, \ldots, \frac{\theta-2}{2} - 3, \frac{\theta-2}{2}, \rightarrow \ldots\right\}$.

Thus, we find that $\frac{\delta}{2}$ is perfect since $\Psi(\frac{\delta}{2}) = \phi$ in both cases.

THEOREM 4.5. Let δ be an Arf numerical semigroup with multiplicity $\mu(\delta) = 4$ and conductor θ such that $\theta \equiv 0 \,(\mathrm{mod}\, 4)$ and $\delta = < 4, 4v+2, \theta+1, \theta+3 >$ for some $v \in \left\{1, 2, \ldots, \frac{\theta}{4}\right\}$. In this case, $\frac{\delta}{2} = < 2, 2v+1 > = \{0, 2, 4, 6, \ldots, 2v, \rightarrow \ldots\}$ is not perfect.

PROOF. It is clear.

THEOREM 4.6. Let δ be an Arf numerical semigroup with multiplicity $\mu(\theta) = 4$ and conductor θ such that

$\theta \equiv 2 \pmod 4$ and $\delta = <4, 4v+2, \theta+1, \theta+3>$ for some $v \in \left\{1, 2, \ldots, \frac{\theta-2}{4}\right\}$. In this case, $\frac{\delta}{2} = <2, 2v+1> = \{0, 2, 4, 6, \ldots, 2v, \rightarrow \ldots\}$ is not perfect.

PROOF. It is trivial.

THEOREM 4.7. Let δ be a Arf numerical semigroup with multiplicity $\mu(\delta) = 4$ and conductor θ such that $\theta \equiv 3 \pmod 4$, i.e $\theta = 4m+3,\ for\ m \in \mathbb{Z}^+$ and $\delta = <4, \theta, \theta+2, \theta+3>$. Then $\frac{\delta}{2} = <2, 2m+3> = \{0, 2, 4, 6, \ldots, 2m, 2m+2, \rightarrow \ldots\}$ is not perfect.

PROOF. Let δ be an Arf numerical semigroup with multiplicity $\mu(\delta) = 4$ and conductor θ such that $\theta \equiv 3 \pmod 4$, i.e. $\theta = 4m+3,\ for\ m \in \mathbb{Z}^+$ and $\delta = <4, \theta, \theta+2, \theta+3>$.

In this case,

$$
\begin{aligned}
x \in \frac{\delta}{2} &\Leftrightarrow x = 2a_1 + (2m+3)a_2,\ for\ a_1, a_2 \in \mathbb{N} \\
&\Leftrightarrow 2x = 2(2a_1) + (4m+6)a_2 \\
&\Leftrightarrow 2x = 4a_1 + (\theta+3)a_2 \\
&\Leftrightarrow 2x = 4a_1 + 0.\theta + 0.(\theta+2) + (\theta+3)a_3 \\
&\Leftrightarrow 2x \in <4, \theta, \theta+2, \theta+3> = \delta.
\end{aligned}
$$

Let's $D = \{0, 2, 4, 6, \ldots, 2m, 2m+2, \rightarrow \ldots\}$. It is clear that $D \subseteq \frac{\delta}{2}$. If

$y \in \frac{\delta}{2} = <2, 2m+3> \Rightarrow y = 2u_1 + (2m+3)u_2 \in D\ ; u_1, u_2 \in \mathbb{N}$

since the conductor of D is $2m+2$. Thus we obtain $\frac{\delta}{2} = <2, 2m+3> = \{0, 2, 4, 6, \ldots, 2m, 2m+2, \rightarrow \ldots\}$. And it is clear that $\frac{\delta}{2}$ is not perfect since $I(\frac{\delta}{2}) = \{u \in \Delta(\delta) : u-1, u+1 \in \delta\} = \{1, 3, 5, \ldots, 2m+1\} \neq \varphi$.

THEOREM 4.8. Let δ be an Arf numerical semigroup with multiplicity $\mu(\delta) = 5$ and conductor θ such that $\theta \equiv 0 \pmod 5$, $\theta > 5$.

(a) Let's $\delta = <5, \theta-2, \theta+1, \theta+2, \theta+4>$ .In this case,

(1) if θ is even then;

(i) if $\theta = 10$ then $\frac{\delta}{2} = <4, 5, 6, 7> = \{0, 4, \rightarrow \ldots\}$ is perfect;

(ii) if $\theta > 10$ then $\frac{\delta}{2} = <5, \frac{\theta}{2}-1, \frac{\theta}{2}+1, \frac{\theta}{2}+2, \frac{\theta}{2}+3> = \left\{0, 5, 10, 15, \ldots, \frac{\theta}{2}-5, \frac{\theta}{2}-1, \rightarrow \ldots\right\}$ is perfect.

(2) if θ is odd then $\frac{\delta}{2} = <5, \frac{\theta+1}{2}, \frac{\theta+1}{2}+1, \frac{\theta+1}{2}+3, \frac{\theta+1}{2}+4> = \left\{0, 5, 10, 15, \ldots, \frac{\theta+1}{2}-3, \frac{\theta+1}{2}, \rightarrow \ldots\right\}$ is perfect.

(b) Let's $\delta = <5, \theta+1, \theta+2, \theta+3, \theta+4>$. So,

(1) if θ is even then $\frac{\delta}{2} = <5, \frac{\theta}{2}+1, \frac{\theta}{2}+2, \frac{\theta}{2}+3, \frac{\theta}{2}+4> = \left\{0, 5, 10, 15, \ldots, \frac{\theta}{2}-5, \frac{\theta}{2}, \rightarrow \ldots\right\}$ is perfect.

(2) if q is odd then $\frac{d}{2} = < 5, \frac{q+1}{2}, \frac{q+1}{2} + 1, \frac{q+1}{2} + 3, \frac{q+1}{2} + 4 > = \left\{ 0, 5, 10, 15, ..., \frac{q+1}{2} - 3, \frac{q+1}{2}, ® \; ... \right\}$ is perfect.

PROOF. Let d be an Arf numerical semigroup with multiplicity $m(d) = 5$ and conductor q such that $q ° \; 0 (\bmod 5)$.

(a) Let's $d = < 5, q - 2, q + 1, q + 2, q + 4 >$ .In this case,

(1) if q is even then;

(i) if $q = 10$ then it is clear that $\frac{d}{2} = < 4, 5, 6, 7 >$. Also, $\frac{d}{2} = < 4, 5, 6, 7 > = \{0, 4, ® \; ...\}$ is perfect since $Y(\frac{d}{2}) = f$.

If we follow the path in the proof of the above theorems, we see that there are the following equations.

(ii) if $q > 10$ then $\frac{d}{2} = < 5, \frac{q}{2} - 1, \frac{q}{2} + 1, \frac{q}{2} + 2, \frac{q}{2} + 3 > = \left\{ 0, 5, 10, 15, ..., \frac{q}{2} - 5, \frac{q}{2} - 1, ® \; ... \right\}$ is perfect.

(2) if q is odd then $\frac{d}{2} = < 5, \frac{q+1}{2}, \frac{q+1}{2} + 1, \frac{q+1}{2} + 3, \frac{q+1}{2} + 4 > = \left\{ 0, 5, 10, 15, ..., \frac{q+1}{2} - 3, \frac{q+1}{2}, ® \; ... \right\}$ is perfect.

We will give the following theorems without proof. Because the proofs can be easily done in a similar way to the above.

THEOREM 4.9. Let d be an Arf numerical semigroup with multiplicity $m(d) = 5$ and conductor q such that $q ° \; 2 (\bmod 5)$ and $d = < 5, q, q + 1, q + 2, q + 4 >$. In this case,

(1) if θ is even then

$$\frac{\delta}{2}=<5,\frac{\theta}{2},\frac{\theta}{2}+1,\frac{\theta}{2}+2,\frac{\theta}{2}+3>=\left\{0,5,10,15,\ldots,\frac{\theta}{2}-6,\frac{\theta}{2}-1,\rightarrow\ldots\right\}$$

is perfect,

(2) if θ is odd then

$$\frac{\delta}{2}=<5,\frac{\theta+1}{2},\frac{\theta+1}{2}+2,\frac{\theta+1}{2}+3,\frac{\theta+1}{2}+4>=\left\{0,5,10,15,\ldots,\frac{\theta+1}{2}-4,\frac{\theta+1}{2},\rightarrow\right\}$$

is perfect.

THEOREM 4.10. Let δ be an Arf numerical semigroup with multiplicity $\mu(\delta)=5$ and conductor θ such that $\theta\equiv 3(\bmod 5)$ and $\delta=<5,\theta,\theta+1,\theta+3,\theta+4>$. In this case,

(1) if θ is even then

$$\frac{\delta}{2}=<5,\frac{\theta}{2},\frac{\theta}{2}+2,\frac{\theta}{2}+3,\frac{\theta}{2}+4>=\left\{0,5,10,15,\ldots,\frac{\theta}{2}-4,\frac{\theta}{2},\rightarrow\ldots\right\}$$

is perfect,

(2) if θ is odd then

$$\frac{\delta}{2}=<5,\frac{\theta+1}{2},\frac{\theta+1}{2}+1,\frac{\theta+1}{2}+2,\frac{\theta+1}{2}+4>=\left\{0,5,10,15,\ldots,\frac{\theta+1}{2}-2,\frac{\theta+1}{2},\rightarrow\ldots\right\}$$

is not perfect.

THEOREM 4.11. Let δ be an Arf numerical semigroup with multiplicity $\mu(\delta)=5$ and conductor θ such that $\theta\equiv 4(\bmod 5)$. Thus,

(i) let's $\delta=<5,\theta-2,\theta,\theta+2,\theta+4>$. In this case,

(1) if θ is even then

$$\frac{\delta}{2}=<5,\frac{\theta}{2}-1,\frac{\theta}{2},\frac{\theta}{2}+1,\frac{\theta}{2}+2>=\left\{0,5,10,15,\ldots,\frac{\theta-4}{2}-5,\frac{\theta-4}{2},\rightarrow\ldots\right\}$$

is perfect,

(2) if θ is odd then $\frac{\delta}{2} = <5, \frac{\theta+3}{2}, \frac{\theta+3}{2}+1, \frac{\theta+3}{2}+2, \frac{\theta+3}{2}+3> = \left\{0,5,10,15,\ldots,\frac{\theta+1}{2}-5, \frac{\theta+1}{2}, \rightarrow \ldots\right\}$ is perfect.

(ii) let's $\delta = <5,\theta,\theta+2,\theta+3,\theta+4>$. So,

(1) if θ is even then $\frac{\delta}{2} = <5, \frac{\theta}{2}, \frac{\theta}{2}+1, \frac{\theta}{2}+2, \frac{\theta}{2}+4> = \left\{0,5,10,15,\ldots,\frac{\theta}{2}-2, \frac{\theta}{2}, \rightarrow \ldots\right\}$ is not perfect,

(2) if θ is odd then $\frac{\delta}{2} = <5, \frac{\theta+3}{2}, \frac{\theta+3}{2}+1, \frac{\theta+3}{2}+2, \frac{\theta+2}{2}+3> = \left\{0,5,10,15,\ldots,\frac{\theta+1}{2}-5, \frac{\theta+1}{2}, \rightarrow \ldots\right\}$ is perfect.

EXAMPLE 4. 12. Let's $\theta = 23$ in Theorem 4.10/(2). Thus, we write $\delta = <5,23,24,26,27> = \{0,5,10,15,20,23,\rightarrow \ldots\}$ Arf numerical semigroup. Here, $\tau(\delta) = 22$, $\mu(\delta) = 5$, $\sigma(\delta) = 5$, $e(\delta) = 5$, $\Delta(\delta) = \{1,2,3,4,6,7,8,9,11,12,13,14,16,17,18,19,21,22\}$ and $\Omega(\delta) = Card(\Delta(\delta)) = 18$. In this case, δ is perfect since $Y(\delta) = \{p \in \Delta(\delta) : p-1, p+1 \in \delta\} = \phi$. Also, δ is MED since $\mu(\delta) = e(\delta) = 5$. So we obtain that $\frac{\delta}{2} = \{a \in \mathbb{N} : 2a \in \delta\} = \{0,5,10,12,\rightarrow \ldots\} = <5,12,13,14,16>$ is not perfect since the set of isolated gaps of $\frac{\delta}{2}$ is $Y(\frac{\delta}{2}) = \{11\} \neq \phi$.

EXAMPLE 4. 13. Let's $\theta = 14$ in Proposition 3.3./(ii). Thus, we write $\delta = <3,14,16> = \{0,3,6,9,12,14,\rightarrow \ldots\}$ Arf numerical semigroup. Here, $t(d) = 13$, $m(d) = 3 = e(d)$, $s(d) = 5$, $\mathrm{D}(d) = \{1, 2, 4, 5, 7, 8, 10, 11, 13\}$ and $\mathrm{W}(d) = Card(\mathrm{D}(d)) = 9$. In this case, d is not perfect since $Y(d) = \{13\} \text{¹} f$. Also, d is MED since $m(d) = e(d) = 3$. So we obtain that $\frac{d}{2} = \{a \text{ Î } \text{¥} : 2a \text{ Î } d\} = \{0, 3, 6, \text{®} \ldots\} = < 3, 7, 8 >$ is perfect since $Y(\frac{d}{2}) = f$.

EXAMPLE 4.14. Let's $q = 12$ in Proposition 3.3./(i). Thus, we write $d = < 3, 13, 14 > = \{0, 3, 6, 9, 12, \text{®} \ldots\}$ Arf numerical semigroup. Here, $t(d) = 11$ and d is MED since $m(d) = e(d) = 3$. Also, $s(d) = 4$, $\mathrm{D}(d) = \{1, 2, 4, 5, 7, 8, 10, 11\}$ and $\mathrm{W}(d) = Card(\mathrm{D}(d)) = 8$. In this case, d is perfect since $Y(d) = f$. So, we obtain that $\frac{d}{2} = \{a \text{ Î } \text{¥} : 2a \text{ Î } d\} = \{0, 3, 6, \text{®} \ldots\} = < 3, 7, 8 >$ is perfect since the set of isolated gaps of $\frac{d}{2}$ is f.

EXAMPLE 4.15. Let's $q = 16$ and $v = 3$ in Proposition 3.4./(i). Thus, the Arf numerical semigroup $d = < 4, 14, 17, 19 > = \{0, 4, 8, 12, 14, 16, \text{®} \ldots\}$ is not perfect since $Y(d) = \{13, 15\} \text{¹} f$. Here, $t(d) = 15$ and d is MED since $m(d) = e(d) = 4$. Also, $s(d) = 5$, $\mathrm{D}(d) = \{1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 15\}$ and $\mathrm{W}(d) = Card(\mathrm{D}(d)) = 11$. So, we obtain that

$\frac{d}{2} = \{a \in \mathbb{N} : 2a \in d\} = \{0,2,4,6,\rightarrow \ldots\} = \langle 2,7 \rangle$ is not perfect since the set of isolated gaps of $\frac{d}{2}$ is $Y(\frac{d}{2}) = \{1,3,5\} \neq \emptyset$.

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4-BOYUTLU ÖKLİD UZAYINDA HOMOTHETİCAL HİPERYÜZEYLERİN LB^{IV} OPERATÖRÜ

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1. GİRİŞ VE ÖN BİLGİLER

3-boyutlu uzayların genişletilmesi olarak düşünülebilen 4-boyutlu Öklid uzayında ve daha yüksek boyutlu Öklid uzaylarında homothetical (hiper)yüzeylere ait değişik karakterizasyonlar içeren pekçok çalışma matematikçiler tarafından yapılmıştır. Örneğin, (Büyükkütük ve Öztürk, 2019) kodlu çalışmada, 4-boyutlu Öklid uzayında homothetical (factorable) yüzeylerin Gaussian eğriliği, Gaussian torsiyonu ve ortalama eğriliği elde edilerek bu kavramlarla ilgili çeşitli karakterizasyonlar elde edilmiş ve (Jiu ve Sun, 2007) kodlu çalışmada ise, $n \geq 3$ olmak üzere $(n+1)$-boyutlu Öklid uzayında homothetical hiperyüzeylerin minimalliği ele alınmıştır.

Ayrıca, Laplacian'ın bir genelleştirilmesi olarak düşünülebilen Laplace-Beltrami (LB) operatörü de araştırmacıların ilgisini fazlasıyla çekmektedir. Biz bu çalışmada, 4-boyutlu Öklid uzayı E^4'te homothetical hiperyüzeylerin dördüncü Laplace-Beltrami operatörlerini ile ilgileneceğimizden dolayı, öncelikle E^4 uzayı ile alakalı bazı temel bilgiler verelim.

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E^4 uzayında $\vec{u} = (u_1, u_2, u_3, u_4)$, $\vec{v} = (v_1, v_2, v_3, v_4)$ ve $\vec{w} = (w_1, w_2, w_3, w_4)$ üç vektör olmak üzere, bu uzayda $\vec{u}$ ve $\vec{v}$ vektörlerinin iç çarpımı

$$\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 \tag{1}$$

şeklinde, bir $\vec{u}$ vektörünün normu

$$\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle} \tag{2}$$

ile ve $\vec{u}$, $\vec{v}$, $\vec{w}$ vektörlerinin vektörel çarpımı ise

$$\vec{u} \times \vec{v} \times \vec{w} = \det \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix} \tag{3}$$

olarak tanımlanır. Ayrıca,

$$\Lambda: E^3 \longrightarrow E^4 \tag{4}$$

$$(x_1, x_2, x_3) \longrightarrow \Lambda(x_1, x_2, x_3)$$
$$= (\Lambda_1(x_1, x_2, x_3), \Lambda_2(x_1, x_2, x_3), \Lambda_3(x_1, x_2, x_3), \Lambda_4(x_1, x_2, x_3))$$

E^4'te bir hiperyüzey olmak üzere, bu hiperyüzeyin birim normal vektörü ile birinci temel formunun, ikinci temel formunun ve üçüncü temel formunun matrissel gösterimleri, sırasıyla

$$N_\Lambda = \frac{\Lambda_{x_1} \times \Lambda_{x_2} \times \Lambda_{x_3}}{\|\Lambda_{x_1} \times \Lambda_{x_2} \times \Lambda_{x_3}\|}, \tag{5}$$

$$I = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}, \tag{6}$$

$$II = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \tag{7}$$

ve

$$III = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \tag{8}$$

ile bulunur. Burada $g_{ij} = \left\langle \Lambda_{x_i}, \Lambda_{x_j} \right\rangle$, $h_{ij} = \left\langle \Lambda_{x_i x_j}, N_\Lambda \right\rangle$, $m_{ij} = \left\langle (N_\Lambda)_{x_i}, (N_\Lambda)_{x_j} \right\rangle$, $\Lambda_{x_i} = \frac{\partial \Lambda(x_1,x_2,x_3)}{\partial x_i}$, $\Lambda_{x_i x_j} = \frac{\partial^2 \Lambda(x_1,x_2,x_3)}{\partial x_i x_j}$, $(N_\Lambda)_{x_i} = \frac{\partial N_\Lambda(x_1,x_2,x_3)}{\partial x_i}$, $i,j \in \{1,2,3\}$'dir.

I matrisinin tersi I^{-1} ile gösterilirse, (4) eşitliği ile verilen hiperyüzeyin şekil operatörü

$$S = I^{-1}.II \tag{9}$$

formülü ile elde edilir.

(6), (7) ve (9) eşitlikleri yardımıyla E^4'te (4) eşitliği ile verilen hiperyüzeyin Gaussian ve ortalama eğrilikleri, sırasıyla,

$$K = \det(S) = \frac{\det[II]}{\det[I]} \tag{10}$$

ve

$$3H = iz(S) \tag{11}$$

formülleri ile elde edilir.

E^4'te bir hiperyüzeyin dördüncü temel formu IV ile gösterilirse,

$$\left.\begin{array}{l} II = I.S, \\ III = II.S = I.S.S, \\ IV = III.S = II.S.S = I.S.S.S \end{array}\right\} \tag{12}$$

eşitlikleri geçerlidir.

Ayrıca

$$[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{13}$$

şeklindeki bir matrisin determinantı ve tersi ise, sırasıyla

$$\det[a_{ij}] = -a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{11}a_{22}a_{33} \tag{14}$$

ve

$$[a_{ij}]^{-1} = \frac{1}{\det[a_{ij}]}\begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ a_{23}a_{31} - a_{21}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{13}a_{21} - a_{11}a_{23} \\ a_{21}a_{32} - a_{22}a_{31} & a_{12}a_{31} - a_{11}a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix} \quad (15)$$

şeklindedir. 4-boyutlu Öklid uzayında hiperyüzeylerle ilgili daha detaylı bilgi için (Altın vd., 2020), (Altın vd., 2021), (Güler, 2020a), (Güler, 2020b), (Aydın ve Ergüt, 2015) kaynaklarına bakılabilir.

Diğer taraftan 3-boyutlu Öklid uzayı E^3'te bir homothetical yüzey (bu yüzey factorable yüzey olarak da bilinir), f ve g diferensiyellenebilir fonksiyonlar olmak üzere, parametrik olarak $X(u,v) = (u, v, f(u)g(v))$ şeklinde tanımlanmıştır (Woestyne, 1993). 4-boyutlu Öklid uzayı E^4'te ise bir homothetical yüzey, $f_1(u)$, $f_2(u)$, $g_1(v)$ ve $g_2(v)$ diferensiyellenebilir fonksiyonları için $X(u,v) = (u, v, f_1(u)g_1(v), f_2(u)g_2(v))$ şeklinde parametrik olarak ifade edilmiştir (Büyükkütük ve Öztürk, 2019). 3-boyutlu Öklid uzayında homothetical yüzeylerin minimalliği, flatlığı gibi farklı karakterizasyonları elde edilmişken; dört-boyutlu Öklid uzayında bu yüzeylerin Gaussian eğrilikleri ve ortalama eğrilik fonksiyonları yardımıyla önemli karakterizasyonları elde edilmiştir. Benzer şekilde, $(n+1)$-boyutlu Öklid uzayı E^{n+1}'de ise $f_1, f_2, \ldots, f_n$ diferensiyellenebilir fonksiyonları için bir homothetical hiperyüzey, $X(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, f_1(x_1)f_2(x_2)\ldots f_n(x_n))$ şeklinde tanımlanmıştır. Literatürde, $n \geq 3$ olmak üzere, $(n+1)$-boyutlu Öklid uzayında homothetical hiperyüzeylerin başta minimalliği ve flatlığı olmak üzere, çeşitli geometrik karakterizasyonları ile ilgili birçok çalışma yapılmıştır.

Biz de bu çalışmamızda, 4-boyutlu Öklid uzayında $\Lambda(\mathrm{x},\mathrm{y},\mathrm{z}) = (x, y, z, f(x).g(y).h(z))$ şeklinde parametrelendirilen homothetical hiperyüzeyinde $f(x), g(y)$ ve

$h(z)$ fonksiyonlarının özel durumlarına göre oluşturulan 1. tip $(g(y) = y, h(z) = z)$, 2. tip $(f(x) = x, h(z) = z)$ ve 3. tip $(f(x) = x, g(y) = y)$ homothetical hiperyüzeyleri inceledik. Öncelikle bu hiperyüzeylerin 1. tipini detaylı bir şekilde ele aldık. Bu bağlamda, ilk olarak 1. tip homothetical hiperyüzeyin birim normal vektör alanını, birinci, ikinci ve üçüncü temel formlarının katsayılarını ve şekil operatörünü verdik. Ayrıca, bu temel formlar ve şekil operatörü yardımıyla 1. tip homothetical hiperyüzeyin dördüncü temel formunun katsayılarını hesapladık ve bu temel formu kullanarak da dördüncü Laplace-Beltrami (LB^{IV}) operatörünü ayrıntılı bir şekilde elde ettik. Ardından, 2. tip ve 3. tip homothetical hiperyüzeyler için dördüncü Laplace-Beltrami operatörlerini içeren teoremleri ifade ettik. Son olarak da, 1. tip homothetical hiperyüzey için özel bir örnek oluşturarak, bu hiperyüzeyin dördüncü Laplace-Beltrami operatörünü elde ettik.

2. E^4'TE HOMOTHETICAL HİPERYÜZEYLERİN LB^{IV} OPERATÖRÜ

Bu bölümde, ilk olarak 4-boyutlu Öklid uzayında 1. tip homothetical hiperyüzey olarak adlandırdığımız hiperyüzeyin LB^{IV} operatörünü ayrıntılı bir şekilde elde edeceğiz. Ardından 2. ve 3. tip homothetical hiperyüzeyler olarak adlandırdığımız hiperyüzeylerin LB^{IV} operatörlerini vererek, bu hiperyüzeylerin LB^{IV} operatörleri ile ilgili sonuçlar vereceğiz.

$f(x)$ diferensiyellenebilir bir fonksiyon olmak üzere,

$$\Lambda^1 : E^3 \longrightarrow E^4$$

$$(x, y, z) \longrightarrow (x, y, z, f(x).y.z) \quad (16)$$

ile verilen 1. tip homothetical hiperyüzeyini inceleyelim.

E^4'te 1. tip homothetical hiperyüzeyi (16)'nın birim normal vektörü, (5) numaralı eşitlikten,

$$N_{\Lambda^1} = \frac{1}{W}(yzf'(x), zf(x), yf(x), -1) \tag{17}$$

şeklinde bulunur. Burada

$W = \sqrt{1 + (y^2 + z^2)f^2(x) + y^2z^2f'^2(x)}$ ve $f'(x) = \frac{df}{dx}$ dir.

Ayrıca, (6)-(8) ve (17)'den, 1. tip homothetical hiperyüzey (16)'nın birinci temel formu, ikinci temel formu ve üçüncü temel formu, sırasıyla,

$$I = \begin{bmatrix} 1 + y^2z^2f'^2(x) & yz^2f(x)f'(x) & y^2zf(x)f'(x) \\ yz^2f(x)f'(x) & 1 + z^2f^2(x) & yzf^2(x) \\ y^2zf(x)f'(x) & yzf^2(x) & 1 + y^2f^2(x) \end{bmatrix}, \tag{18}$$

$$II = -\frac{1}{W}\begin{bmatrix} yzf''(x) & zf'(x) & yf'(x) \\ zf'(x) & 0 & f(x) \\ yf'(x) & f(x) & 0 \end{bmatrix} \tag{19}$$

ve

$$III = \frac{1}{W^4}\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \tag{20}$$

olarak bulunur. Burada

$$m_{11} = y^2z^2(y^2 + z^2)f'^4(x) + y^2z^2\big(1 + (y^2 + z^2)f^2(x)\big)f''^2(x) - (y^2 + z^2)f'^2(x)\big(-1 + 2y^2z^2f(x)f''(x)\big),$$

$$m_{12} = m_{21} = yf'(x)\big(f(x) - z^4f(x)f'^2(x) + z^2f''(x) + z^4f^2(x)f''(x)\big),$$

$$m_{13} = m_{31} = zf'(x)\big(f(x) - y^4f(x)f'^2(x) + y^2f''(x) + y^4f^2(x)f''(x)\big),$$

$$m_{22} = \big(1 + z^2f^2(x)\big)\big(f^2(x) + z^2f'^2(x)\big),$$

$$m_{23} = m_{32} = yz\big(-f^4(x) + f'^2(x)\big),$$

$$m_{33} = (1 + y^2f^2(x))(f^2(x) + y^2f'^2(x))$$

dir. Bu temel formların determinantları ise (14) yardımıyla,

$$\det[I] = W^2,$$

$$\det[II] = \frac{yzf(x)(-2f'^2(x)+f(x)f''(x))}{W^3}, \tag{21}$$

$$\det[III] = \frac{y^2z^2f^2(x)(-2f'^2(x)+f(x)f''(x))^2}{W^8}$$

şeklindedir.

(15) ve (18)'den, 1. tip homothetical hiperyüzey (16)'nın birinci temel formunun tersi

$$I^{-1} = \begin{bmatrix} \frac{1+(y^2+z^2)f^2}{W^2} & -\frac{yz^2ff'}{W^2} & -\frac{y^2zff'}{W^2} \\ -\frac{yz^2ff'}{W^2} & 1-\frac{z^2f^2}{W^2} & -\frac{yzf^2}{W^2} \\ -\frac{y^2zff'}{W^2} & -\frac{yzf^2}{W^2} & 1-\frac{y^2f^2}{W^2} \end{bmatrix} \tag{22}$$

şeklinde elde edilir. Burada $f = f(x), f' = f'(x)$'dir.

Dolayısıyla, (9) numaralı eşitlikte (19) ve (22) kullanılırsa, E^4'te 1. tip homothetical hiperyüzey (16)'nın şekil operatörünün

$$S = \frac{1}{W^3}\begin{bmatrix} yz\left(-f'' + (y^2+z^2)f\left(f'^2 - ff''\right)\right) & -z(1+z^2f^2)f' & -y(1+y^2f^2)f' \\ zf'\left(-1+y^2z^2\left(-f'^2+ff''\right)\right) & yzf\left(f^2+z^2f'^2\right) & -f(1+y^2f^2) \\ yf'\left(-1+y^2z^2\left(-f'^2+ff''\right)\right) & -f(1+z^2f^2) & yzf(f^2+y^2f'^2) \end{bmatrix} \tag{23}$$

şeklinde olduğu görülür.

Şimdi de (16) eşitliği ile verilen 1. tip homothetical hiperyüzeyin dördüncü temel formunu hesaplayalım.

(12) numaralı eşitliklerde, (18)-(20) ve (23) kullanılırsa, 1. tip homothetical hiperyüzey (16)'nın dördüncü temel formu

$$n_{11} = \frac{1}{W^7}(yz\big(2ff'^2\big(-1 + (y^4 + z^4)f'^2 + y^2z^2(y^4 + y^2z^2 + z^4)f'^4\big)$$
$$- f'^2\big(2(y^2 + z^2 + (y^4 + z^4)f^2)$$
$$+ y^2z^2(2(y^2 + z^2) + (5y^4 + 6y^2z^2 + 5z^4)f^2)f'^2\big)f''$$
$$+ 2y^2z^2f(2(y^2 + z^2) + (2y^4 + 3y^2z^2 + 2z^4)f^2)f'^2f''^2$$
$$- y^2z^2(1 + (y^2 + z^2)f^2)^2f''^3\big)),$$

$$n_{12} = n_{21} = \frac{1}{W^7}(zf'\big(f^2(-1 + (y - z)(y + z)f^2)$$
$$- \big(y^2 + z^2 - (y - z)(y + z)f^2(y^2 + z^2 + y^2z^2f^2)\big)f'^2$$
$$- y^2z^2(y^2 + z^2 + z^2(y^2 + 2z^2)f^2)f'^4$$
$$+ y^2f(1 + z^2f^2)\big(-1 + (-y^2 + z^2)f^2$$
$$+ (2y^2z^2 + 3z^4)f'^2\big)f''$$
$$- y^2z^2(1 + z^2f^2)(1 + (y^2 + z^2)f^2)f''^2\big)),$$

$$n_{13} = n_{31} = \frac{1}{W^7}(yf'\big(f^2(-1 + (-y^2 + z^2)f^2)$$
$$- \big(y^2 + z^2 + (y - z)(y + z)f^2(y^2 + z^2 + y^2z^2f^2)\big)f'^2$$
$$- y^2z^2(y^2 + z^2 + y^2(2y^2 + z^2)f^2)f'^4$$
$$+ z^2f(1 + y^2f^2)\big(-1 + (y - z)(y + z)f^2$$
$$+ (3y^4 + 2y^2z^2)f'^2\big)f''$$
$$- y^2z^2(1 + y^2f^2)(1 + (y^2 + z^2)f^2)f''^2\big)),$$

$$n_{22} = \frac{1}{W^7}(yz(1 + z^2f^2)\big(2f\big(f^4 + (-1 + z^2f^2)f'^2 + z^4f'^4\big)$$
$$- z^2(1 + z^2f^2)f'^2f''\big)),$$

$$n_{23} = n_{32} = \frac{1}{W^7}(-2y^2z^2f^7 + (y^2 + z^2)ff'^2\big(-1 + y^2z^2f'^2\big)$$
$$- (y^2 + z^2)f^5\big(1 + y^2z^2f'^2\big)$$
$$+ f^3\big(-1 - (y^4 + z^4)f'^2 + y^4z^4f'^4\big)$$
$$- y^2z^2(1 + y^2f^2)(1 + z^2f^2)f'^2f''),$$

$$n_{33} = \frac{1}{W^7}(yz(1 + y^2f^2)\big(2f\big(f^4 + (-1 + y^2f^2)f'^2 + y^4f'^4\big)$$
$$- y^2(1 + y^2f^2)f'^2f''\big))$$

olmak üzere

$$IV = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix} \tag{24}$$

dir. Dördüncü temel formun determinantı ise

$$\det[IV] = \frac{y^3 z^3 f^3 (-2f'^2 + ff'')^3}{W^{13}} \qquad (25)$$

olarak bulunur.

Şimdi de (16) ile verilen 1. tip homothetical hiperyüzeyin LB^{IV} operatörünü elde edelim.

$\Lambda = \Lambda(x_1, x_2, x_3)|_D$ $(D \subset \mathbb{R}^3)$, bir hiperyüzey üzerinde C^3-sınıfından diferensiyellenebilir bir fonksiyon ve $(IV)^{-1}$, dördüncü temel forma karşılık gelen IV matrisinin tersi olmak üzere, bu fonksiyonun LB^{IV} operatörü

$$\Delta^{IV}\Lambda = \frac{1}{\sqrt{\det[IV]}} \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(\sqrt{\det[IV]}(IV)^{-1}\frac{\partial \Lambda}{\partial x_j}\right) \qquad (26)$$

şeklinde tanımlanır.

$\Lambda = \Lambda(x, y, z)$ şeklindeki fonksiyonun LB^{IV} operatörü; (15), (24) ve (26) eşitlikleri yardımıyla açık bir şekilde,

$$\Delta^{IV}\Lambda = \frac{1}{\sqrt{\det[IV]}}\left\{\begin{array}{l} \frac{\partial}{\partial x}\left(\frac{(n_{22}n_{33}-n_{23}^2)\Lambda_x+(n_{13}n_{23}-n_{12}n_{33})\Lambda_y+(n_{12}n_{23}-n_{13}n_{22})\Lambda_z}{\sqrt{\det[IV]}}\right) \\ +\frac{\partial}{\partial y}\left(\frac{(n_{13}n_{23}-n_{12}n_{33})\Lambda_x+(n_{11}n_{33}-n_{13}^2)\Lambda_y+(n_{12}n_{13}-n_{11}n_{23})\Lambda_z}{\sqrt{\det[IV]}}\right) \\ +\frac{\partial}{\partial z}\left(\frac{(n_{12}n_{23}-n_{13}n_{22})\Lambda_x+(n_{12}n_{13}-n_{11}n_{23})\Lambda_y+(n_{11}n_{22}-n_{12}^2)\Lambda_z}{\sqrt{\det[IV]}}\right) \end{array}\right\} \qquad (27)$$

olarak yazılabilir.

E^4'te 1. tip homothetical hiperyüzey (16)'nın LB^{IV} operatörünü, $\Delta^{IV}\Lambda^1$ ile gösterelim. Bu durumda, (24), (25) ve (27) yardımıyla

$$\left.\begin{aligned}
U_i &= \sqrt{\frac{W^{13}}{y^3z^3f^3(-2f'^2+ff'')^3}}\left\{(n_{22}n_{33}-n_{23}^2)(\Lambda^1{}_i)_x+(n_{13}n_{23}-n_{12}n_{33})(\Lambda^1{}_i)_y+(n_{12}n_{23}-n_{13}n_{22})(\Lambda^1{}_i)_z\right\},\\
V_i &= \sqrt{\frac{W^{13}}{y^3z^3f^3(-2f'^2+ff'')^3}}\left\{(n_{13}n_{23}-n_{12}n_{33})(\Lambda^1{}_i)_x+(n_{11}n_{33}-n_{13}^2)(\Lambda^1{}_i)_y+(n_{12}n_{13}-n_{11}n_{23})(\Lambda^1{}_i)_z\right\},\\
W_i &= \sqrt{\frac{W^{13}}{y^3z^3f^3(-2f'^2+ff'')^3}}\left\{(n_{12}n_{23}-n_{13}n_{22})(\Lambda^1{}_i)_x+(n_{12}n_{13}-n_{11}n_{23})(\Lambda^1{}_i)_y+(n_{11}n_{22}-n_{12}^2)(\Lambda^1{}_i)_z\right\}.
\end{aligned}\right\} \tag{28}$$

olmak üzere

$$\Delta^{IV}\Lambda^1 = ((\Delta^{IV}\Lambda^1)_1,(\Delta^{IV}\Lambda^1)_2,(\Delta^{IV}\Lambda^1)_3,(\Delta^{IV}\Lambda^1)_4) =$$
$$\sqrt{\frac{W^{13}}{y^3z^3f^3(-2f'^2+ff'')^3}}\begin{pmatrix}(U_1)_x+(V_1)_y+(W_1)_z,(U_2)_x+(V_2)_y+(W_2)_z,\\ (U_3)_x+(V_3)_y+(W_3)_z,(U_4)_x+(V_4)_y+(W_4)_z\end{pmatrix} \tag{29}$$

şeklinde yazılabilir.

(28)'de, (16) ve (24) ifadeleri kullanılırsa,

$$\left.\begin{aligned}
U_1 &= \frac{-(\sqrt{f}((y^2-z^2)^2f'^4-2(y^2+z^2)f^2f'^2(-1+y^2z^2f'^2)+f^4(1+2y^2z^2f'^2-3y^4z^4f'^4)+2y^2z^2ff'^2f''+2y^2z^2(y^2+z^2)f^3f'^2f''+2y^4z^4f^5f'^2f'')}{\sqrt{y^3z^3W^{7/2}(-2f'^2+ff'')^3}},\\
U_2 &= \frac{f'(f^4+2y^2f^2f'^2+y^4f'^4-z^4f'^4+2y^2z^4f^2f'^4+3y^4z^4f^4f'^4-z^2f(1+y^2f^2)(-f^2+(y^2-z^2+5y^2z^2f^2)f'^2)f''+y^2z^2f^2(1+y^2f^2)(1+2z^2f^2)f''^2)}{\sqrt{yz^3fW^{7/2}(-2f'^2+ff'')^3}},\\
U_3 &= \frac{f'(f^4+2z^2f^2f'^2-y^4f'^4+z^4f'^4+2y^4z^2f^2f'^4+3y^4z^4f^4f'^4-y^2f(1+z^2f^2)(-f^2+(-y^2+z^2+5y^2z^2f^2)f'^2)f''+y^2z^2f^2(1+2y^2f^2)(1+z^2f^2)f''^2)}{\sqrt{zy^3fW^{7/2}(-2f'^2+ff'')^3}},\\
U_4 &= \frac{\begin{pmatrix}\sqrt{f}f'(-(y^2-z^2)^2f'^4+(y^4-4y^2z^2+z^4)ff'^2f''+2y^2z^2f^5(1-6y^2z^2f'^2)f''-(y^2+z^2)f^3(-1+7y^2z^2f'^2)f''+4y^4z^4f^6f''^2\\ +2y^2z^2f^2(2(y^2+z^2)f'^4+f''^2)+f^4(1+y^2z^2(-2f'^2+9y^2z^2f'^4+3(y^2+z^2)f''^2)))\end{pmatrix}}{\sqrt{yzW^{7/2}(-2f'^2+ff'')^3}};
\end{aligned}\right\} \tag{30}$$

$$\left.\begin{aligned}
V_1 &= \frac{f'(f^4+2y^2f^2f'^2+y^4f'^4-z^4f'^4+2y^2z^4f^2f'^4+3y^4z^4f^4f'^4-z^2f(1+y^2f^2)(-f^2+(y^2-z^2+5y^2z^2f^2)f'^2)f''+y^2z^2f^2(1+y^2f^2)(1+2z^2f^2)f''^2)}{\sqrt{yz^3fW^{7/2}(-2f'^2+ff'')^3}},\\
V_2 &= \frac{\begin{pmatrix}\sqrt{y}(-(y^2+z^2)^2f'^6-2(y-z)(y+z)f^2f'^4(1+y^2z^2f'^2)+f^4f'^2(-1+2y^2z^2f'^2+3y^4z^4f'^4)-2z^2f(1+y^2f^2)f'^2(f^2+(-2y^2-z^2+4y^2z^2f^2)f'^2)f''\\ +z^2f^2(1+y^2f^2)(-2y^2-z^2+7y^2z^2f^2)f'^2f''^2-2y^2z^4f^5(1+y^2f^2)f''^3)\end{pmatrix}}{\sqrt{z^3f^3W^{7/2}(-2f'^2+ff'')^3}},\\
V_3 &= \frac{\begin{pmatrix}(y^2+z^2)^2f'^6+f^4f'^2\left(-1+2y^2z^2f'^2+3y^4z^4f'^4\right)-ff'^2\left(f^2(y^2+z^2+2y^2z^2f^2)+(y^4+4y^2z^2+z^4+y^2z^2(y^2+z^2)f^2+8y^4z^4f^4)f'^2\right)f''\\ +y^2z^2f^2(3+2(y^2+z^2)f^2+7y^2z^2f^4)f'^2f''^2-y^2z^2f^3(1+(y^2+z^2)f^2+2y^2z^2f^4)f''^3\end{pmatrix}}{\sqrt{yzf^3W^{7/2}(-2f'^2+ff'')^3}},\\
V_4 &= \frac{\begin{pmatrix}\sqrt{y}(-f^4f'^2+2z^2f^2(1+2y^2f^2)f'^4+(y^4-z^4+(-2y^4z^2+4y^2z^4)f^2+9y^4z^4f^4)f'^6-ff'^2(f^2(y^2+2z^2+3y^2z^2f^2)+(y^4+y^2z^2-2z^4\\ +(-2y^4z^2+11y^2z^4)f^2+21y^4z^4f^4)f'^2)f''+z^2f^2(2y^2-z^2+(y^4+10y^2z^2)f^2+16y^4z^2f^4)f'^2f''^2-y^2z^2f^3(1+(y^2+3z^2)f^2+4y^2z^2f^4)f''^3)\end{pmatrix}}{\sqrt{zfW^{7/2}(-2f'^2+ff'')^3}};
\end{aligned}\right\} \tag{31}$$

$$W_1=\frac{f'(f^4+2z^2f^2f'^2-y^4f'^4+z^4f'^4+2y^4z^2f^2f'^4+3y^4z^4f^4f'^4-y^2f(1+z^2f^2)(-f^2+(-y^2+z^2+5y^2z^2f^2)f'^2)f''+y^2z^2f^2(1+2y^2f^2)(1+z^2f^2)f''^2)}{\sqrt{y^3zfW^{7/2}(-2f'^2+ff'')^3}},$$

$$W_2=\frac{\begin{pmatrix}(y^2+z^2)^2f'^6+f^4f'^2\left(-1+2y^2z^2f'^2+3y^4z^4f'^4\right)-ff'^2\left(f^2(y^2+z^2+2y^2z^2f^2)+(y^4+4y^2z^2+z^4+y^2z^2(y^2+z^2)f^2+8y^4z^4f^4)f'^2\right)f''\\+y^2z^2f^2(3+2(y^2+z^2)f^2+7y^2z^2f^4)f'^2f''^2-y^2z^2f^3(1+(y^2+z^2)f^2+2y^2z^2f^4)f''^3\end{pmatrix}}{\sqrt{yzf^3W^{7/2}(-2f'^2+ff'')^3}},$$

$$W_3=\frac{\begin{pmatrix}\sqrt{z}(-(y^2+z^2)^2f'^6+2(y-z)(y+z)f^2f'^4(1+y^2z^2f'^2)+f^4f'^2(-1+2y^2z^2f'^2+3y^4z^4f'^4)-2y^2f(1+z^2f^2)f'^2(f^2+(-y^2-2z^2+4y^2z^2f^2)f'^2)f''\\+y^2f^2(1+z^2f^2)(-y^2-2z^2+7y^2z^2f^2)f'^2f''^2-2y^4z^2f^5(1+z^2f^2)f''^3)\end{pmatrix}}{\sqrt{y^3f^3W^{7/2}(-2f'^2+ff'')^3}},$$

$$W_4=\frac{\begin{pmatrix}\sqrt{z}(-f^4f'^2+2y^2f^2(1+2z^2f^2)f'^4+(-y^4+z^4+(4y^4z^2-2y^2z^4)f^2+9y^4z^4f^4)f'^6-ff'^2(f^2(2y^2+z^2+3y^2z^2f^2)\\+(-2y^4+y^2z^2+z^4+(11y^4z^2-2y^2z^4)f^2+21y^4z^4f^4)f'^2)f''+y^2f^2(-y^2+2z^2+(10y^2z^2+z^4)f^2+16y^2z^4f^4)f'^2f''^2-y^2z^2f^3(1+(3y^2+z^2)f^2+4y^2z^2f^4)f''^3)\end{pmatrix}}{\sqrt{yfW^{7/2}(-2f'^2+ff'')^3}}$$

(32)

olarak bulunur.

O halde, (29) ifadesinde (30)-(32) eşitlikleri kullanılırsa, 1. tip homothetical hiperyüzey (16)'nın LB^{IV} operatörü şu şekilde elde edilir:

Teorem. E^4'te 1. tip homothetical hiperyüzey (16)'nın dördüncü Laplace-Beltrami (LB^{IV}) operatörü,

$$(\Delta^{IV}\Lambda^1)_1=\frac{W^3}{2y^3z^3f^2(-2f'^2+ff'')^4}\big(-2(7y^4+2y^2z^2+7z^4)f'^7+4(4y^4+3y^2z^2\\+4z^4)ff'^5f''+2y^4z^4f^7f'f''(3f''^2+f'f^{(3)})+2f^3f'f''(-2(y^2\\+z^2)f'^2+6y^2z^2(y^2+z^2)f'^4-y^2z^2f''^2+y^2z^2f'f^{(3)})+f^2f'^3(8(y^2\\+z^2)f'^2-4y^2z^2(y^2+z^2)f'^4-(9y^4+10y^2z^2+9z^4)f''^2+(3y^4\\+2y^2z^2+3z^4)f'f^{(3)})+f^4f'(8y^2z^2f'^4-54y^4z^4f'^6-9(y^2\\+z^2)f''^2+2f'^2(11-5y^2z^2(y^2+z^2)f''^2)+6(y^2+z^2)f'f^{(3)}\\+2y^2z^2(y^2+z^2)f'^3f^{(3)})+2f^5f'f''(-10+y^2z^2(-2f'^2\\+46y^2z^2f'^4-(y^2+z^2)f''^2+(y^2+z^2)f'f^{(3)}))+f^6(3f^{(3)}\\-y^2z^2f'((9+49y^2z^2f'^2)f''^2+f'(-6+y^2z^2f'^2)f^{(3)}))\big),$$

$$(\Delta^{IV}\Lambda^1)_2 = \frac{W^3}{2y^2z^3f^3(-2f'^2+ff'')^4}\big(2(11y^2-7z^2)(y^2+z^2)f'^8 - 2(12y^4+9y^2z^2 - 13z^4)ff'^6f'' + 2y^4z^4f^8f''^2(-7f''^2+f'f^{(3)}) + f^3f'^2f''(2(-6y^2 + z^2)f'^2 + 6y^2z^2(-3y^2+4z^2)f'^4 + z^2(7y^2+5z^2)f''^2 - z^2(7y^2 + z^2)f'f^{(3)}) + f^2f'^4((8y^2-4z^2)f'^2 + (8y^4z^2-4y^2z^4)f'^4 + (11y^4 + 6y^2z^2 - 15z^4)f''^2 - (3y^4-4y^2z^2+z^4)f'f^{(3)}) + f^5f''(2y^2z^2f'^4 + 158y^4z^4f'^6 + 2z^2f''^2 + f'^2(12+y^2z^2(7y^2+36z^2)f''^2) - z^2f'f^{(3)} - y^2z^2(7y^2+12z^2)f'^3f^{(3)}) + y^2z^2f^7f''((2+87y^2z^2f'^2)f''^2 - f'(1 + 11y^2z^2f'^2)f^{(3)}) + f^4(-4y^2z^2f'^6 - 54y^4z^4f'^8 + (13y^2 + 5z^2)f'^2f''^2 - y^2z^2f''^4 + 2f'^4(-7+y^2z^2(3y^2-25z^2)f''^2) - 2(3y^2+2z^2)f'^3f^{(3)} + 2y^2z^2(2y^2+5z^2)f'^5f^{(3)} + y^2z^2f'f''^2f^{(3)}) + f^6(f''^2(2-y^2z^2(-5f'^2+179y^2z^2f'^4+(y^2+6z^2)f''^2)) + f'(-3 + y^2z^2(-4f'^2+11y^2z^2f'^4+(y^2+2z^2)f''^2))f^{(3)})\big),$$

$$(\Delta^{IV}\Lambda^1)_3 = \frac{W^3}{2y^3z^2f^3(-2f'^2+ff'')^4}\big(2(-7y^4+4y^2z^2+11z^4)f'^8 + 2(13y^4-9y^2z^2 - 12z^4)ff'^6f'' + 2y^4z^4f^8f''^2(-7f''^2+f'f^{(3)}) + f^3f'^2f''(2(y^2 - 6z^2)f'^2 + 6(4y^4z^2-3y^2z^4)f'^4 + y^2(5y^2+7z^2)f''^2 - y^2(y^2 + 7z^2)f'f^{(3)}) + f^2f'^4(-4(y^2-2z^2)f'^2 + (-4y^4z^2+8y^2z^4)f'^4 + (-15y^4+6y^2z^2+11z^4)f''^2 - (y^4-4y^2z^2+3z^4)f'f^{(3)}) + f^5f''(2y^2z^2f'^4 + 158y^4z^4f'^6 + 2y^2f''^2 + f'^2(12+y^2z^2(36y^2 + 7z^2)f''^2) - y^2f'f^{(3)} - y^2z^2(12y^2+7z^2)f'^3f^{(3)}) + y^2z^2f^7f''((2 + 87y^2z^2f'^2)f''^2 - f'(1+11y^2z^2f'^2)f^{(3)}) + f^4(-4y^2z^2f'^6 - 54y^4z^4f'^8 + (5y^2+13z^2)f'^2f''^2 - y^2z^2f''^4 - 2f'^4(7+y^2z^2(25y^2 - 3z^2)f''^2) - 2(2y^2+3z^2)f'^3f^{(3)} + 2y^2z^2(5y^2+2z^2)f'^5f^{(3)} + y^2z^2f'f''^2f^{(3)}) + f^6(f''^2(2-y^2z^2(-5f'^2+179y^2z^2f'^4+(6y^2 + z^2)f''^2)) + f'(-3+y^2z^2(-4f'^2+11y^2z^2f'^4+(2y^2 + z^2)f''^2))f^{(3)})\big)$$

$$(\Delta^{IV}\Lambda^1)_4 = \frac{W^3}{2y^2z^2f^2(-2f'^2+ff'')^4}\big(-2(7y^4+2y^2z^2+7z^4)f'^8 + 2(13y^4+12y^2z^2 + 13z^4)ff'^6f'' + 4y^4z^4f^8f''^2(-9f''^2+f'f^{(3)}) + f^2f'^4(-4(y^2 + z^2)f'^2 - 16y^2z^2(y^2+z^2)f'^4 - 5(3y^2+z^2)(y^2+3z^2)f''^2 - (y^4 - 10y^2z^2+z^4)f'f^{(3)}) + f^3f'^2f''(2(y^2+z^2)f'^2 + 66y^2z^2(y^2 + z^2)f'^4 + (5y^4+36y^2z^2+5z^4)f''^2 - (y^4+12y^2z^2+z^4)f'f^{(3)}) + f^5f''(16y^2z^2f'^4 + 576y^4z^4f'^6 + 2(y^2+z^2)f''^2 + f'^2(12 + 65y^2z^2(y^2+z^2)f''^2) - (y^2+z^2)f'f^{(3)} - 17y^2z^2(y^2+z^2)f'^3f^{(3)}) + 2y^2z^2f^7f''(2(1+59y^2z^2f'^2)f''^2 - f'(1+10y^2z^2f'^2)f^{(3)}) + f^4(-16y^2z^2f'^6 - 234y^4z^4f'^8 + 5(y^2+z^2)f'^2f''^2 - 6y^2z^2f''^4 - 2f'^4(7+53y^2z^2(y^2+z^2)f''^2) - 4(y^2+z^2)f'^3f^{(3)} + 16y^2z^2(y^2 + z^2)f'^5f^{(3)} + 2y^2z^2f'f''^2f^{(3)}) + f^6(f''^2(2+y^2z^2(-3f'^2 - 553y^2z^2f'^4 - 11(y^2+z^2)f''^2)) + f'(-3+y^2z^2(-2f'^2 + 21y^2z^2f'^4 + 3(y^2+z^2)f''^2))f^{(3)})\big)$$

olmak üzere,

$$\Delta^{IV}\Lambda^{1}=((\Delta^{IV}\Lambda^{1})_{1},(\Delta^{IV}\Lambda^{1})_{2},(\Delta^{IV}\Lambda^{1})_{3},(\Delta^{IV}\Lambda^{1})_{4}) \qquad (33)$$

dir.

Sonuç. E^4'te 1. tip homothetical hiperyüzey (16)'nın dördüncü Laplace-Beltrami (LB^{IV}) operatörü, $f(x)=\frac{c_1}{x+c_2}$, $(c_1, c_2 \in \mathbb{R}, c_1 \neq 0)$, için hesaplanamaz.

Şimdi de 2. tip ve 3. tip homothetical hiperyüzeylerin LB^{IV} operatörlerini verelim.

$g(y)$ diferensiyellenebilir bir fonksiyon olmak üzere,

$$\Lambda^{2}: \mathrm{E}^{3} \longrightarrow \mathrm{E}^{4}$$

$$(x,y,z) \longrightarrow (x,y,z,x.g(y).z) \qquad (34)$$

ile verilen 2. tip homothetical hiperyüzeyin dördüncü Laplace-Beltrami operatörünü, yukarıdakine benzer yöntemle, uzun işlemlerden sonra şu şekilde elde ederiz:

Teorem. E^4'te 2. tip homothetical hiperyüzey (34)'ün dördüncü Laplace-Beltrami (LB^{IV}) operatörü,

$$Q=\sqrt{1+(x^{2}+z^{2})g^{2}(y)+x^{2}z^{2}g'^{2}(y)} \text{ için}$$

$$\begin{aligned}(\Delta^{IV}\Lambda^{2})_{1}=\frac{Q^{3}}{2x^{2}z^{3}g^{3}(-2g'^{2}+gg'')^{4}}\Big(&2(11x^{2}-7z^{2})(x^{2}+z^{2})g'^{8}-2(12x^{4}+9x^{2}z^{2}\\ &-13z^{4})gg'^{6}g''+2x^{4}z^{4}g^{8}g''^{2}(-7g''^{2}+g'g^{(3)})+g^{3}g'^{2}g''(2(-6x^{2}\\ &+z^{2})g'^{2}+6x^{2}z^{2}(-3x^{2}+4z^{2})g'^{4}+z^{2}(7x^{2}+5z^{2})g''^{2}-z^{2}(7x^{2}\\ &+z^{2})g'g^{(3)})+g^{2}g'^{4}((8x^{2}-4z^{2})g'^{2}+(8x^{4}z^{2}-4x^{2}z^{4})g'^{4}+(11x^{4}\\ &+6x^{2}z^{2}-15z^{4})g''^{2}-(3x^{4}-4x^{2}z^{2}+z^{4})g'g^{(3)})+g^{5}g''(2x^{2}z^{2}g'^{4}\\ &+158x^{4}z^{4}g'^{6}+2z^{2}g''^{2}+g'^{2}(12+x^{2}z^{2}(7x^{2}+36z^{2})g''^{2})-z^{2}g'g^{(3)}\\ &-x^{2}z^{2}(7x^{2}+12z^{2})g'^{3}g^{(3)})+x^{2}z^{2}g^{7}g''((2+87x^{2}z^{2}g'^{2})g''^{2}-g'(1\\ &+11x^{2}z^{2}g'^{2})g^{(3)})+g^{4}(-4x^{2}z^{2}g'^{6}-54x^{4}z^{4}g'^{8}+(13x^{2}\\ &+5z^{2})g'^{2}g''^{2}-x^{2}z^{2}g''^{4}+2g'^{4}(-7+x^{2}z^{2}(3x^{2}-25z^{2})g''^{2})\\ &-2(3x^{2}+2z^{2})g'^{3}g^{(3)}+2x^{2}z^{2}(2x^{2}+5z^{2})g'^{5}g^{(3)}+x^{2}z^{2}g'g''^{2}g^{(3)})\\ &+g^{6}(g''^{2}(2-x^{2}z^{2}(-5g'^{2}+179x^{2}z^{2}g'^{4}+(x^{2}+6z^{2})g''^{2}))+g'(-3\\ &+x^{2}z^{2}(-4g'^{2}+11x^{2}z^{2}g'^{4}+(x^{2}+2z^{2})g''^{2}))g^{(3)})\Big),\end{aligned}$$

$$
\begin{aligned}
(\Delta^{IV}\Lambda^2)_2 = {} & \frac{Q^3}{2x^3z^3g^2(-2g'^2+gg'')^4}\big(-2(7x^4+2x^2z^2+7z^4)g'^7+4(4x^4+3x^2z^2 \\
& +4z^4)gg'^5g''+2x^4z^4g^7g'g''(3g''^2+g'g^{(3)})+2g^3g'g''(-2(x^2 \\
& +z^2)g'^2+6x^2z^2(x^2+z^2)g'^4-x^2z^2g''^2+x^2z^2g'g^{(3)})+g^2g'^3(8(x^2 \\
& +z^2)g'^2-4x^2z^2(x^2+z^2)g'^4-(9x^4+10x^2z^2+9z^4)g''^2+(3x^4 \\
& +2x^2z^2+3z^4)g'g^{(3)})+g^4g'(8x^2z^2g'^4-54x^4z^4g'^6-9(x^2 \\
& +z^2)g''^2+2g'^2(11-5x^2z^2(x^2+z^2)g''^2)+6(x^2+z^2)g'g^{(3)} \\
& +2x^2z^2(x^2+z^2)g'^3g^{(3)})+2g^5g'g''(-10+x^2z^2(-2g'^2 \\
& +46x^2z^2g'^4-(x^2+z^2)g''^2+(x^2+z^2)g'g^{(3)}))+g^6(3g^{(3)} \\
& -x^2z^2g'((9+49x^2z^2g'^2)g''^2+g'(-6+x^2z^2g'^2)g^{(3)}))\big),
\end{aligned}
$$

$$
\begin{aligned}
(\Delta^{IV}\Lambda^2)_3 = {} & \frac{Q^3}{2x^3z^2g^3(-2g'^2+gg'')^4}\big(2(-7x^4+4x^2z^2+11z^4)g'^8+2(13x^4-9x^2z^2 \\
& -12z^4)gg'^6g''+2x^4z^4g^8g''^2(-7g''^2+g'g^{(3)})+g^3g'^2g''(2(x^2 \\
& -6z^2)g'^2+6(4x^4z^2-3x^2z^4)g'^4+x^2(5x^2+7z^2)g''^2-x^2(x^2 \\
& +7z^2)g'g^{(3)})+g^2g'^4(-4(x^2-2z^2)g'^2+(-4x^4z^2+8x^2z^4)g'^4 \\
& +(-15x^4+6x^2z^2+11z^4)g''^2-(x^4-4x^2z^2+3z^4)g'g^{(3)}) \\
& +g^5g''(2x^2z^2g'^4+158x^4z^4g'^6+2x^2g''^2+g'^2(12+x^2z^2(36x^2 \\
& +7z^2)g''^2)-x^2g'g^{(3)}-x^2z^2(12x^2+7z^2)g'^3g^{(3)})+x^2z^2g^7g''((2 \\
& +87x^2z^2g'^2)g''^2-g'(1+11x^2z^2g'^2)g^{(3)})+g^4(-4x^2z^2g'^6 \\
& -54x^4z^4g'^8+(5x^2+13z^2)g'^2g''^2-x^2z^2g''^4-2g'^4(7+x^2z^2(25x^2 \\
& -3z^2)g''^2)-2(2x^2+3z^2)g'^3g^{(3)}+2x^2z^2(5x^2+2z^2)g'^5g^{(3)} \\
& +x^2z^2g'g''^2g^{(3)})+g^6(g''^2(2-x^2z^2(-5g'^2+179x^2z^2g'^4+(6x^2 \\
& +z^2)g''^2))+g'(-3+x^2z^2(-4g'^2+11x^2z^2g'^4+(2x^2 \\
& +z^2)g''^2))g^{(3)})\big)
\end{aligned}
$$

$$
\begin{aligned}
(\Delta^{IV}\Lambda^2)_4 = {} & \frac{Q^3}{2x^2z^2g^2(-2g'^2+gg'')^4}\big(-2(7x^4+2x^2z^2+7z^4)g'^8+2(13x^4+12x^2z^2 \\
& +13z^4)gg'^6g''+4x^4z^4g^8g''^2(-9g''^2+g'g^{(3)})+g^2g'^4(-4(x^2 \\
& +z^2)g'^2-16x^2z^2(x^2+z^2)g'^4-5(3x^2+z^2)(x^2+3z^2)g''^2-(x^4 \\
& -10x^2z^2+z^4)g'g^{(3)})+g^3g'^2g''(2(x^2+z^2)g'^2+66x^2z^2(x^2 \\
& +z^2)g'^4+(5x^4+36x^2z^2+5z^4)g''^2-(x^4+12x^2z^2+z^4)g'g^{(3)}) \\
& +g^5g''(16x^2z^2g'^4+576x^4z^4g'^6+2(x^2+z^2)g''^2+g'^2(12 \\
& +65x^2z^2(x^2+z^2)g''^2)-(x^2+z^2)g'g^{(3)}-17x^2z^2(x^2+z^2)g'^3g^{(3)}) \\
& +2x^2z^2g^7g''(2(1+59x^2z^2g'^2)g''^2-g'(1+10x^2z^2g'^2)g^{(3)}) \\
& +g^4(-16x^2z^2g'^6-234x^4z^4g'^8+5(x^2+z^2)g'^2g''^2-6x^2z^2g''^4 \\
& -2g'^4(7+53x^2z^2(x^2+z^2)g''^2)-4(x^2+z^2)g'^3g^{(3)}+16x^2z^2(x^2 \\
& +z^2)g'^5g^{(3)}+2x^2z^2g'g''^2g^{(3)})+g^6(g''^2(2+x^2z^2(-3g'^2 \\
& -553x^2z^2g'^4-11(x^2+z^2)g''^2))+g'(-3+x^2z^2(-2g'^2 \\
& +21x^2z^2g'^4+3(x^2+z^2)g''^2))g^{(3)})\big)
\end{aligned}
$$

olmak üzere,

$$\Delta^{IV}\Lambda^2 = ((\Delta^{IV}\Lambda^2)_1, (\Delta^{IV}\Lambda^2)_2, (\Delta^{IV}\Lambda^2)_3, (\Delta^{IV}\Lambda^2)_4) \qquad (35)$$

dir.

Sonuç. E^4'te 2. tip homothetical hiperyüzey (34)'ün dördüncü Laplace-Beltrami (LB^{IV}) operatörü, $g(y) = \frac{c_3}{y+c_4}$, $(c_3, c_4 \in \mathbb{R}, c_3 \neq 0)$, için hesaplanamaz.

Son olarak da, $h(z)$ diferensiyellenebilir bir fonksiyon olmak üzere,

$$\Lambda^3 : \mathrm{E}^3 \longrightarrow \mathrm{E}^4$$

$$(x, y, z) \longrightarrow (x, y, z, x.y.h(z)) \tag{36}$$

ile verilen 3. tip homothetical hiperyüzey için yine uzun işlemlerden sonra şu teoremi elde ederiz:

Teorem. E^4'te 3. tip homothetical hiperyüzey (36)'nın dördüncü Laplace-Beltrami (LB^{IV}) operatörü,

$$R = \sqrt{1 + (x^2 + y^2)h^2(z) + x^2y^2h'^2(z)} \text{ için}$$

$$\begin{aligned}(\Delta^{IV}\Lambda^3)_1 = \frac{R^3}{2x^2y^3h^3(-2h'^2 + hh'')^4}&\big(2(11x^2 - 7y^2)(x^2 + y^2)h'^8 - 2(12x^4 + 9x^2y^2 \\ &- 13y^4)hh'^6h'' + 2x^4y^4h^8h''^2(-7h''^2 + h'h^{(3)}) + h^3h'^2h''(2(-6x^2 \\ &+ y^2)h'^2 + 6x^2y^2(-3x^2 + 4y^2)h'^4 + y^2(7x^2 + 5y^2)h''^2 - y^2(7x^2 \\ &+ y^2)h'h^{(3)}) + h^2h'^4((8x^2 - 4y^2)h'^2 + (8x^4y^2 - 4x^2y^4)h'^4 + (11x^4 \\ &+ 6x^2y^2 - 15y^4)h''^2 - (3x^4 - 4x^2y^2 + y^4)h'h^{(3)}) + h^5h''(2x^2y^2h'^4 \\ &+ 158x^4y^4h'^6 + 2y^2h''^2 + h'^2(12 + x^2y^2(7x^2 + 36y^2)h''^2) \\ &- y^2h'h^{(3)} - x^2y^2(7x^2 + 12y^2)h'^3h^{(3)}) + x^2y^2h^7h''((2 \\ &+ 87x^2y^2h'^2)h''^2 - h'(1 + 11x^2y^2h'^2)h^{(3)}) + h^4(-4x^2y^2h'^6 \\ &- 54x^4y^4h'^8 + (13x^2 + 5y^2)h'^2h''^2 - x^2y^2h''^4 + 2h'^4(-7 \\ &+ x^2y^2(3x^2 - 25y^2)h''^2) - 2(3x^2 + 2y^2)h'^3h^{(3)} + 2x^2y^2(2x^2 \\ &+ 5y^2)h'^5h^{(3)} + x^2y^2h'h''^2h^{(3)}) + h^6(h''^2(2 - x^2y^2(-5h'^2 \\ &+ 179x^2y^2h'^4 + (x^2 + 6y^2)h''^2)) + h'(-3 + x^2y^2(-4h'^2 + 11x^2y^2h'^4 \\ &+ (x^2 + 2y^2)h''^2))h^{(3)})\big),\end{aligned}$$

$$(\Delta^{IV}\Lambda^3)_2=\frac{R^3}{2x^3y^2h^3(-2h'^2+hh'')^4}\big(2(-7x^4+4x^2y^2+11y^4)h'^8+2(13x^4-9x^2y^2-12y^4)hh'^6h''+2x^4y^4h^8h''^2(-7h''^2+h'h^{(3)})+h^3h'^2h''(2(x^2-6y^2)h'^2+6(4x^4y^2-3x^2y^4)h'^4+x^2(5x^2+7y^2)h''^2-x^2(x^2+7y^2)h'h^{(3)})+h^2h'^4(-4(x^2-2y^2)h'^2+(-4x^4y^2+8x^2y^4)h'^4+(-15x^4+6x^2y^2+11y^4)h''^2-(x^4-4x^2y^2+3y^4)h'h^{(3)})+h^5h''(2x^2y^2h'^4+158x^4y^4h'^6+2x^2h''^2+h'^2(12+x^2y^2(36x^2+7y^2)h''^2)-x^2h'h^{(3)}-x^2y^2(12x^2+7y^2)h'^3h^{(3)})+x^2y^2h^7h''((2+87x^2y^2h'^2)h''^2-h'(1+11x^2y^2h'^2)h^{(3)})+h^4(-4x^2y^2h'^6-54x^4y^4h'^8+(5x^2+13y^2)h'^2h''^2-x^2y^2h''^4-2h'^4(7+x^2y^2(25x^2-3y^2)h''^2)-2(2x^2+3y^2)h'^3h^{(3)}+2x^2y^2(5x^2+2y^2)h'^5h^{(3)}+x^2y^2h'h''^2h^{(3)})+h^6(h''^2(2-x^2y^2(-5h'^2+179x^2y^2h'^4+(6x^2+y^2)h''^2))+h'(-3+x^2y^2(-4h'^2+11x^2y^2h'^4+(2x^2+y^2)h''^2))h^{(3)})\big),$$

$$(\Delta^{IV}\Lambda^3)_3=\frac{R^3}{2x^3y^3h^2\left(-2h'^2+hh''\right)^4}\big(-2(7x^4+2x^2y^2+7y^4)h'^7+4(4x^4+3x^2y^2+4y^4)hh'^5h''+2x^4y^4h^7h'h''(3h''^2+h'h^{(3)})+2h^3h'h''(-2(x^2+y^2)h'^2+6x^2y^2(x^2+y^2)h'^4-x^2y^2h''^2+x^2y^2h'h^{(3)})+h^2h'^3(8(x^2+y^2)h'^2-4x^2y^2(x^2+y^2)h'^4-(9x^4+10x^2y^2+9y^4)h''^2+(3x^4+2x^2y^2+3y^4)h'h^{(3)})+h^4h'(8x^2y^2h'^4-54x^4y^4h'^6-9(x^2+y^2)h''^2+2h'^2(11-5x^2y^2(x^2+y^2)h''^2)+6(x^2+y^2)h'h^{(3)}+2x^2y^2(x^2+y^2)h'^3h^{(3)})+2h^5h'h''(-10+x^2y^2(-2h'^2+46x^2y^2h'^4-(x^2+y^2)h''^2+(x^2+y^2)h'h^{(3)}))+h^6(3h^{(3)}-x^2y^2h'((9+49x^2y^2h'^2)h''^2+h'(-6+x^2y^2h'^2)h^{(3)}))\big),$$

$$(\Delta^{IV}\Lambda^3)_4=\frac{R^3}{2x^2y^2h^2(-2h'^2+hh'')^4}\big(-2(7x^4+2x^2y^2+7y^4)h'^8+2(13x^4+12x^2y^2+13y^4)hh'^6h''+4x^4y^4h^8h''^2(-9h''^2+h'h^{(3)})+h^2h'^4(-4(x^2+y^2)h'^2-16x^2y^2(x^2+y^2)h'^4-5(3x^2+y^2)(x^2+3y^2)h''^2-(x^4-10x^2y^2+y^4)h'h^{(3)})+h^3h'^2h''(2(x^2+y^2)h'^2+66x^2y^2(x^2+y^2)h'^4+(5x^4+36x^2y^2+5y^4)h''^2-(x^4+12x^2y^2+y^4)h'h^{(3)})+h^5h''(16x^2y^2h'^4+576x^4y^4h'^6+2(x^2+y^2)h''^2+h'^2(12+65x^2y^2(x^2+y^2)h''^2)-(x^2+y^2)h'h^{(3)}-17x^2y^2(x^2+y^2)h'^3h^{(3)})+2x^2y^2h^7h''(2(1+59x^2y^2h'^2)h''^2-h'(1+10x^2y^2h'^2)h^{(3)})+h^4(-16x^2y^2h'^6-234x^4y^4h'^8+5(x^2+y^2)h'^2h''^2-6x^2y^2h''^4-2h'^4(7+53x^2y^2(x^2+y^2)h''^2)-4(x^2+y^2)h'^3h^{(3)}+16x^2y^2(x^2+y^2)h'^5h^{(3)}+2x^2y^2h'h''^2h^{(3)})+h^6(h''^2(2+x^2y^2(-3h'^2-553x^2y^2h'^4-11(x^2+y^2)h''^2))+h'(-3+x^2y^2(-2h'^2+21x^2y^2h'^4+3(x^2+y^2)h''^2))h^{(3)})\big)$$

olmak üzere,

$$\Delta^{IV}\Lambda^3=((\Delta^{IV}\Lambda^3)_1,(\Delta^{IV}\Lambda^3)_2,(\Delta^{IV}\Lambda^3)_3,(\Delta^{IV}\Lambda^3)_4) \qquad (37)$$

dir.

Sonuç. E^4'te 3. tip homothetical hiperyüzey (36)'nın dördüncü Laplace-Beltrami (LB^{IV}) operatörü, $h(z) = \frac{c_5}{z+c_6}$, $(c_5, c_6 \in \mathbb{R}, c_5 \neq 0)$, için hesaplanamaz.

3. ÖRNEK

Bu bölümde ve 1. tip homothetical hiperyüzey için bir örnek vererek, bu hiperyüzeyin dördüncü Laplace-Beltrami operatörünü hesaplayacağız.

E^4'te 1. tip homothetical hiperyüzey (16)'da $f(x) = x^2$ olarak alınırsa,

$$\Lambda^1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$$

$$(x, y, z) \longrightarrow (x, y, z, x^2yz) \tag{38}$$

hiperyüzeyinin LB^{IV} operatörü,

$$(\Delta^{IV}\Lambda^1)_1 = \frac{1}{108x^5y^3z^3}(4x^4 + 5x^2y^2 - 44y^4 + (5x^2 - 4y^2)(1 + x^4y^2)z^2 - 4(11 + x^4y^2 + 26x^8y^4)z^4)\big(1 + 4x^2y^2z^2 + x^4(y^2 + z^2)\big)^{3/2},$$

$$(\Delta^{IV}\Lambda^1)_2 = \frac{1}{108x^6y^2z^3}(-5x^4 + 14x^2y^2 + 136y^4 - 2(2x^2 - 7y^2)(1 + x^4y^2)z^2 - 4(11 + x^4y^2 + 26x^8y^4)z^4)\big(1 + 4x^2y^2z^2 + x^4(y^2 + z^2)\big)^{3/2},$$

$$(\Delta^{IV}\Lambda^1)_3 = \frac{-1}{108x^6y^3z^2}(5x^4 + 4x^2y^2 + 44y^4 + 2(x^2 + y^2)(-7 + 2x^4y^2)z^2 + 2(-68 - 7x^4y^2 + 52x^8y^4)z^4)\big(1 + 4x^2y^2z^2 + x^4(y^2 + z^2)\big)^{3/2},$$

$$(\Delta^{IV}\Lambda^{1})_4 = \frac{-1}{108x^4y^2z^2}(5x^4 + 4x^2y^2 + 44y^4 + 2(x^2 + y^2)(2 + 11x^4y^2)z^2 + 2(22 + 11x^4y^2 + 304x^8y^4)z^4)(1 + 4x^2y^2z^2 + x^4(y^2 + z^2))^{3/2}$$

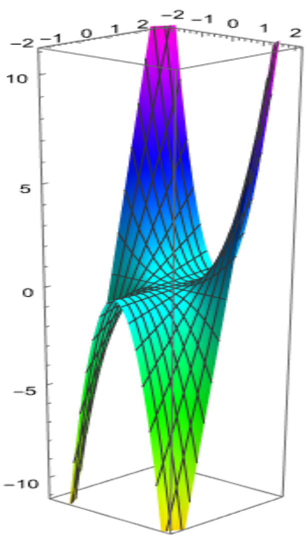
olmak üzere

$$\Delta^{IV}\Lambda^{1} = ((\Delta^{IV}\Lambda^{1})_1, (\Delta^{IV}\Lambda^{1})_2, (\Delta^{IV}\Lambda^{1})_3, (\Delta^{IV}\Lambda^{1})_4)$$

olarak hesaplanır.

Aşağıdaki şekilde, $z = 2$ için 1. tip homothetical hiperyüzey (38)'in, $x_1x_2x_4$-uzayına izdüşümü görülebilir.

Şekil



4. SONUÇ

Bu çalışmada, 4-boyutlu Öklid uzayında üç farklı tip homothetical hiperyüzeyin dördüncü Laplace-Beltrami operatörleri, dördüncü temel formlar yardımıyla hesaplanarak, bu hiperyüzeylerin dördüncü Laplace-Beltrami operatörleri için bazı karakterizasyonlar verilmiştir. Ayrıca örnek olarak özel bir homothetical hiperyüzey tanımlanmış ve bu hiperyüzeyin dördüncü Laplace-Beltrami operatörü elde edilmiştir.

Bu çalışma yardımıyla, 4-boyutlu farklı uzaylarda homothetical hiperyüzeylerin dördüncü Laplace-Beltrami operatörlerinin elde edilebileceği ve bu hiperyüzeyler için de farklı karakterizasyonlar verilebileceği kanaatindeyiz.

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ON THE SHAPE OF TRIANGLES IN DUAL SPACE AND DUAL LORENTZ SPACE

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1. INTRODUCTION

Specific points of triangles in Euclidean geometry has attracted many mathematician. The similarities between the two triangles, can be easily expressed using the concept of shape. The geometry of triangles have focused on the application of equivalence classes (Encheva, 1999,2002; Lester,1996,1996,1997). The notion of shape of triangle can be extended to shapes of a triangle in Minkowski 3-space. Using split quaternions, we may consider equivalence classes of triangles with respect to screw motion (rotation and translation). Thus, the shape of a triangle is a split quaternion. Similarities between two triangles can be easily expressed in terms of shapes (Artzy, 1994; Kimberling, 1994; Sato, 1998).

A dual number is described as $\mathbb{A} = a + \varepsilon a^*$, where $a, a^* \epsilon\ \mathbb{R}$ with $\varepsilon^2 = 0$, $\varepsilon \neq 0$ (Guggenheimer, 2012). The dual quaternion γ is defined by

$$\gamma = \gamma_0 + \gamma_1 i_1 + \gamma_2 i_2 + \gamma_3 i_3 \quad (1)$$

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where $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ are dual numbers, and i_1, i_2, i_3 are quaternionic units which satisfy the non-commutative multiplication rules

$$i_1^2 = i_2^2 = i_3^2 = -1, i_1 i_2 = -i_2 i_1 = i_3$$

and γ may be given as $\gamma = S_\gamma + V_\gamma$, where

$$S_\gamma = \gamma_0,\ V_\gamma = \gamma_1 i_1 + \gamma_2 i_2 + \gamma_3 i_3$$

are the scalar and vector parts, respectively. The product of dual quaternions γ and δ is defined as

$$\gamma\ \delta = S_\gamma S_\delta + \langle V_\gamma, V_\delta \rangle + S_\gamma V_\delta + S_\delta V_\gamma + V_\gamma \times V_\delta$$

where $\langle\ ,\ \rangle$ and $\times$ are inner and vector products, respectively, for quaternion γ. Then norm of γ is defined to be

$$\|\gamma\| = \sqrt{\langle \gamma, \gamma \rangle} = \sqrt{\gamma_0^2 + \gamma_1^2 + \gamma_2^2 + \gamma_3^2}$$

and norm of vector part is as

$$\|V_\gamma\| = \sqrt{\langle V_\gamma, V_\gamma \rangle} = \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}$$

From Equality (1), we can write

$$\gamma = \gamma_0 + \lambda \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}$$

where $\lambda = \frac{\gamma_1 i + \gamma_2 j + \gamma_3 k}{\sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}}$ is unit pure dual quaternion and $\langle \lambda, \lambda \rangle = 1$. Then it can be written that

$$\gamma = \|\gamma\| \left(\cos\tilde{\phi} + \lambda \sin\tilde{\phi} \right)$$

Here the dual angle $\tilde{\phi} = \phi + \varepsilon\phi^*$ makes rotation about the dual axis λ as ϕ and makes translation as ϕ^*, where $\cos\tilde{\phi} = \cos\phi - \varepsilon\phi^* \sin\phi$,

$\sin\tilde{\phi} = \sin\phi + \varepsilon\phi^*\cos\phi$, (Veldkamp, 1976; Wittenburg, 1984; Akyar, 2008).

The dual split quaternion (Ozdemir, 2006; Atasoy, 2017; Kula, 2006, 2007) as an expression of the form

$$\tilde{\gamma} = \tilde{\gamma}_0 + \tilde{\gamma}_1 i + \tilde{\gamma}_2 j + \tilde{\gamma}_3 k = \mathbb{A} + \varepsilon\mathbb{A}^*$$

where $\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3$ are dual numbers, and i_1, i_2, i_3 are quaternionic units which satisfy the non-commutative multiplication rules

$$i_1^2 = -i_2^2 = -i_3^2 = -1, i_1 i_2 = -i_2 i_1 = i_3$$

with

$$\mathbb{A} = a_0 + a_1 i + a_2 j + a_3 k \text{ and } \mathbb{A}^* = a_0^* + a_1^* i + a_2^* j + a_3^* k$$

split quaternion components are spacelike if $(l < 0)$, timelike if $(l > 0)$ or lightlike if $(l = 0)$ where $l = \tilde{\gamma}_0^2 + \tilde{\gamma}_1^2 - \tilde{\gamma}_2^2 - \tilde{\gamma}_3^2$.

Classify the dual split quaternion $\tilde{\gamma}$ according to the given polar forms,respectively.

- If $\tilde{\gamma}$ is spacelike dual split quaternion then

  $\tilde{\gamma} = \|\tilde{\gamma}\|\left(\sinh\tilde{\phi} + \hat{\mu}\cosh\tilde{\phi}\right)$, where

$$\sinh\tilde{\phi} = \frac{\tilde{\gamma}_0}{\|\tilde{\gamma}\|}, \cosh\tilde{\phi} = \frac{\sqrt{-\tilde{\gamma}_1^2 + \tilde{\gamma}_2^2 + \tilde{\gamma}_3^2}}{\|\tilde{\gamma}\|}, \tilde{\phi} = \phi + \varepsilon\phi^*$$

  dual angle and

$$\hat{\mu} = \frac{\tilde{\gamma}_1 i + \tilde{\gamma}_2 j + \tilde{\gamma}_3 k}{\sqrt{-\tilde{\gamma}_1^2 + \tilde{\gamma}_2^2 + \tilde{\gamma}_3^2}}$$

  is spacelike unit vector.

- If $\tilde{\gamma}$ is timelike dual split quaternion with spacelike vector part then

$$\tilde{\gamma} = \|\tilde{\gamma}\|\left(\cosh\tilde{\phi} + \hat{\mu}\sinh\tilde{\phi}\right)$$

where

$$\cosh\tilde{\phi} = \frac{\tilde{\gamma}_0}{\|\tilde{\gamma}\|},\ \sinh\tilde{\phi} = \frac{\sqrt{-\tilde{\gamma}_1^2 + \tilde{\gamma}_2^2 + \tilde{\gamma}_3^2}}{\|\tilde{\gamma}\|},\ \tilde{\phi} = \phi + \varepsilon\phi^*$$

dual angle and $\hat{\mu} = \frac{\tilde{\gamma}_1 i + \tilde{\gamma}_2 j + \tilde{\gamma}_3 k}{\sqrt{-\tilde{\gamma}_1^2 + \tilde{\gamma}_2^2 + \tilde{\gamma}_3^2}}$ is spacelike unit vector.

- If $\tilde{\gamma}$ is timelike dual split quaternion with timelike vector part then

$$\tilde{\gamma} = \|\tilde{\gamma}\|\left(\cos\tilde{\phi} + \hat{\mu}\sin\tilde{\phi}\right)$$

where $\cos\tilde{\phi} = \frac{\tilde{\gamma}_0}{\|\tilde{\gamma}\|}$, $\sin\tilde{\phi} = \frac{\sqrt{\tilde{\gamma}_1^2 - \tilde{\gamma}_2^2 - \tilde{\gamma}_3^2}}{\|\tilde{\gamma}\|}$ and $\hat{\mu} = \frac{\tilde{\gamma}_1 i + \tilde{\gamma}_2 j + \tilde{\gamma}_3 k}{\sqrt{\tilde{\gamma}_1^2 - \tilde{\gamma}_2^2 - \tilde{\gamma}_3^2}}$ is spacelike unit vector. $\hat{\mu}^2 = \hat{\mu}\hat{\mu} = -1$.

- If $\tilde{\gamma}$ is unit dual split quaternion with lightlike dual split vector part then $\tilde{\gamma} = 1 + \hat{\mu}$ where $\hat{\mu}$ is lightlike (null) dual split vector.

2. A SPHERICAL TRIANGLE SHAPE ON THE UNIT SPHERE

Let X, Y be any two unit dual vectors and γ be dual quaternion, respectively, for

$$\gamma X = Y \tag{2}$$

If we multiplied both sides of Equality (2) with X^{-1} we obtain

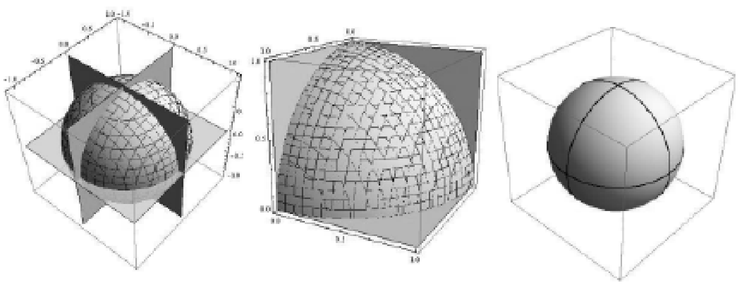
$$\begin{aligned} \gamma &= YX^{-1} = Y\frac{\bar{X}}{\|X\|} = Y(-X) = -YX \\ &= -(-< Y, X > + Y \wedge X) \\ &= < Y, X > - Y \wedge X \\ &= < X, Y > + X \wedge Y \end{aligned}$$

A screw motion is the combination of rotation and translation movements and can be represented by dual quaternions. A spherical triangle ΔXYZ is obtained by non-linear points X, Y and Z on the unit sphere and arc between X and Y is indicated with $arc(XY)$, then

$$\gamma = < X, Y > + X \wedge Y \tag{3}$$

Similarly, $\delta = < X, Z > + X \wedge Z$ is calculated and $\lambda = \delta\gamma^{-1}$ is shape is ΔXYZ.

Figure 1. A Spherical Triangle Model on a Unit Sphere



Example 1. *(Application of a spherical triangle in dual space) Let $X(1,0,0)$, $Y(0,1,0)$ and $Z(0,0,1)$ be points on the unit sphere that generated a spherical triangle ΔXYZ, we will find the shape of ΔXYZ.*

From Equality (3), dual quaternion corresponds to $arc(XY)$ will be obtained as

$$\gamma = < X, Y > + X \wedge Y = 0 + \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = k$$

In this case, if we write γ in the form of

$$\begin{aligned} \gamma &= 0 + k.1 = \cos\tilde{\phi} + \mu\sin\tilde{\phi} \\ &= \cos(\phi + \varepsilon\phi^*) + k\sin(\phi + \varepsilon\phi^*) \end{aligned}$$

then $\phi^* = 0$ and $\phi = \frac{\pi}{2}$, that is, if dual quaternion γ is multiplied from the left with a unit vector X then X turns around k axis about $\frac{\pi}{2}$, don't translate and dual vector Y is obtained

$$\gamma X = ki = j = Y$$

Similarly, the dual quaternion corresponding to the $arc(YZ)$

$$\delta = < X, Z > + X \wedge Z = -j = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$$

That is, if dual quaternion δ is multiplied from the left with a unit vector X then X turns around j axis about $-\frac{\pi}{2}$, don't translate and dual vector Z is obtained. $\gamma\delta = \lambda$ is calculated and a dual vector γ is multiplied from the left with λ and a dual vector δ is obtained. Therefore, $\lambda = i$ is a shape of spherical triangle ΔXYZ (see Figure 1).

Example 2. *(Application of a shape of triangle ΔXYZ in dual space) Consider unit dual vectors $X = \frac{i}{\sqrt{2}} + \frac{j}{\sqrt{2}} + \varepsilon j, Y = j + \varepsilon k$ and $Z = k$ that generated a triangle ΔXYZ, we will find the shape of ΔXYZ.*

For dual quaternion γ we obtain

$$\begin{aligned}
\gamma \quad &= < X, Y > + X \times Y \\
&= < \frac{i}{\sqrt{2}} + \frac{j}{\sqrt{2}}, j > + \varepsilon\left(< j, j > + < \frac{i}{\sqrt{2}} + \frac{j}{\sqrt{2}}, k >\right) \\
&+ \left(\frac{i}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right) \times j + \varepsilon\left(\left(\frac{i}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right) \times k + j \times j\right) \\
&= \frac{-1}{\sqrt{2}} - \varepsilon + \frac{k}{\sqrt{2}} + \varepsilon\left(\frac{-j + i}{\sqrt{2}}\right) \\
&= \left(-\frac{1}{\sqrt{2}} + \frac{k}{\sqrt{2}}\right) + \varepsilon\left(-1 + \frac{-j + i}{\sqrt{2}}\right) \\
&= \left(\frac{-1}{\sqrt{2}} + \frac{k}{\sqrt{2}}\right) + \varepsilon\left(\frac{-1 + \frac{-j + i}{\sqrt{2}}}{\sqrt{2}}\right)\sqrt{2}
\end{aligned}$$

So, if γ rotate X vector $\frac{\pi}{4}$ around the axis of k and translate along the axis $\frac{-1+\frac{-j+i}{\sqrt{2}}}{\sqrt{2}}$ about $\sqrt{2}$ units then constant to Y. Then $\delta = arc(XY)$ is obtained.

$$\begin{aligned} \delta &= < X, Z > + X \times Z \\ &= \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j + \varepsilon i \end{aligned}$$

So, if δ rotate X vector $\frac{\pi}{2}$ around the axis of $\frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$ and translate along the axis i about 1 units then con|s to Z. Then, $\delta = arc(XZ)$ is expressed in the form. Then, $\lambda\gamma = \delta$ is satisfied and $\lambda = \delta\gamma^{-1}$ is calculated.

$$\begin{aligned} \lambda &= \left(\frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} i + \varepsilon i\right)\left(\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} k - \frac{1}{\sqrt{2}} \varepsilon i - \frac{1}{\sqrt{2}} \varepsilon j + \varepsilon k\right) \\ &= j + \varepsilon - \frac{2}{\sqrt{2}} \varepsilon i \\ &= (0 + j.1) + \frac{1 - \sqrt{2} i}{\sqrt{3}} \varepsilon \sqrt{3} \end{aligned}$$

equality is provided. λ is multiplied from the left with λ, for to obtain δ. Thus, $\lambda = j + \varepsilon - \frac{2}{\sqrt{2}} \varepsilon i$ is the shape of spherical triangle ΔXYZ. Let P, Q be any two dual split vectors and γ be dual split quaternion, respectively and if we multiplied both sides of $\gamma P = Q$ with P^{-1} we will have

$$\begin{aligned} \gamma &= QP^{-1} = Q \frac{\bar{P}}{\parallel P \parallel^2} = Q \frac{(-P)}{\parallel P \parallel^2} = \frac{-QP}{\parallel P \parallel^2} \\ &= -\frac{1}{\parallel P \parallel^2} (- < Q, P > + Q \wedge P) \\ &= \frac{1}{\parallel P \parallel^2} (< Q, P > - Q \wedge P) \\ &= \frac{1}{\parallel P \parallel^2} (< P, Q > + P \wedge Q) \end{aligned}$$

For every dual split vectors P and Q, there are corresponding lines in Lorentzian space. Lines between P, Q and P, R are shown as $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ respectively, then

$$\tilde{\gamma} = \frac{1}{\| \overrightarrow{PQ} \|^2} \left(< \overrightarrow{PQ}, \overrightarrow{PR} > + \overrightarrow{PQ} \wedge \overrightarrow{PR} \right)$$

is shape for dual triangle ΔPQR. Similarly, shape for ΔQPR dual triangle is given with dual quaternion

$$\overline{p} = \frac{1}{\|\overrightarrow{QP}\|^2} \left(< \overrightarrow{QP}, \overrightarrow{QR} > + \overrightarrow{QP} \wedge \overrightarrow{QR} \right).$$

Example 3. *(Application of a shape of triangle ΔPQR in dual space) Let us consider $P = i + \varepsilon j, Q = \varepsilon k$ and $R = i + \varepsilon k$ dual vectors that formed a triangle ΔPQR, we find the shape of ΔPQR.*

We can write that $\gamma \overrightarrow{PQ} = \overrightarrow{PR}$ and

$$\gamma = \overrightarrow{PR}\, \overrightarrow{PQ}^{-1} = \varepsilon(k - j)\big(i + \varepsilon(j - k)\big) = (j + k)\varepsilon$$

We obtain the $\gamma = \varepsilon(j + k)$ equation where $\overrightarrow{PQ} = -i + \varepsilon(k - j)$ and $\overrightarrow{PR} = \varepsilon(k - j)$. This is shape of ΔPQR dual triangle. Then, we can see that

$$\gamma \overrightarrow{PQ} = (j + k)\varepsilon\big(-i + \varepsilon(k - j)\big) = (k - j)\varepsilon = \overrightarrow{PR}$$

3. A DUAL LORENTZIAN TRIANGLE SHAPE IN DUAL LORENTZIAN SPACE

Let A, B be any two dual lorentz vectors and γ be dual split quaternion, respectively, for

$$\gamma A = B \tag{4}$$

If we multiplied both sides of Equality (4) with A^{-1} we obtain

$$\begin{aligned}\gamma &= BA^{-1} = B\frac{\bar{A}}{\|A\|^2} = B\frac{(-A)}{\|A\|^2} = \frac{-BA}{\|A\|^2} \\ &= -\frac{1}{\|A\|^2}(-< B, A > + B \wedge A) \\ &= \frac{1}{\|A\|^2}(< B, A > - B \wedge A) \\ &= \frac{1}{\|A\|^2}(< A, B > + A \wedge B)\end{aligned}$$

The combination of rotation and translation movements creates a screw motion which can be represented by dual split quaternions in Lorentzian space. A non-linear on the Lorentz sphere A, B and C create a spherical triangle. Let dual split vectors A and B, line correspond to in Lorentzian space, then

$$\overline{q} = \frac{1}{\left\|\overrightarrow{AB}\right\|^2}\left(< \overrightarrow{AB}, \overrightarrow{AC} > - \overrightarrow{AB} \wedge \overrightarrow{AC}\right)$$

is the shape of triangle ΔABC in dual Lorentz. Similarly, the shape for dual Lorentz triangle ΔBAC is given with dual split quaternion

$$\overline{p} = \frac{1}{\left\|\overrightarrow{BA}\right\|^2}\left(< \overrightarrow{BA}, \overrightarrow{BC} > - \overrightarrow{BA} \wedge \overrightarrow{BC}\right).$$

Let $\overline{q} = q + \varepsilon r$ and $\overline{p} = p + \varepsilon t$ be dual split quaternions, then

$$\langle \overline{p}, \overline{q} \rangle = \langle p, q \rangle + \varepsilon(\langle r, p \rangle + \langle q, t \rangle)$$

$$\overline{p} \times \overline{q} = p \times q + \varepsilon(p \times r + t \times q)$$

$\overline{p}$ is timelike, spacelike and lightlike if p is timelike, spacelike and lightlike, respectively.

- If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are dual timelike vectors, then $\overrightarrow{AB} \wedge_L \overrightarrow{AC}$ is a dual spacelike vector and

$$\left\langle \overrightarrow{AB}, \overrightarrow{AC} \right\rangle_L = -\left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AC}\right\| \cosh \hat{A}$$

$$\left\|\overrightarrow{AB} \wedge_L \overrightarrow{AC}\right\| = \left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AC}\right\|\text{sinh}\hat{A}$$

where $\hat{A} = a + \varepsilon a^*$ is the hyperbolic dual angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then

$$\overline{p} = -\frac{\|A - C\|}{\|A - B\|}\left(\text{cosh}\hat{A} + \hat{l}\text{sinh}\hat{A}\right)$$

where $\hat{l} = \frac{(A-B)\times(A-C)}{\|(A-B)\times(A-C)\|}$ is a dual split axis.

- If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are dual spacelike vectors satisfying

$$\left\|\left\langle\overrightarrow{AB}, \overrightarrow{AC}\right\rangle_L\right\| < \left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AC}\right\|$$

then $\overrightarrow{AB} \wedge_L \overrightarrow{AC}$ is a timelike dual split vector and

$$\left\langle\overrightarrow{AB}, \overrightarrow{AC}\right\rangle_L = \left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AC}\right\|\text{cos}\hat{A}$$

$$\left\|\overrightarrow{AB} \wedge_L \overrightarrow{AC}\right\| = \left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AC}\right\|\text{sin}\hat{A}$$

Then

$$\overline{p} = -\frac{\|A - C\|}{\|A - B\|}\left(\text{cos}\hat{A} - \hat{l}\text{sin}\hat{A}\right)$$

where $\hat{l} = \frac{(A-B)\times(A-C)}{\|(A-B)\times(A-C)\|}$ is a dual split axis.

- If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are dual spacelike vectors satisfying

$$\left\|\left\langle\overrightarrow{AB}, \overrightarrow{AC}\right\rangle_L\right\| > \left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AC}\right\|$$

then $\overrightarrow{AB} \wedge_L \overrightarrow{AC}$ is a spacelike dual split vector and

$$\left\langle\overrightarrow{AB}, \overrightarrow{AC}\right\rangle_L = \left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AC}\right\|\text{cosh}\hat{A}$$

$$\left\|\overrightarrow{AB} \wedge_L \overrightarrow{AC}\right\| = \left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AC}\right\|\text{sinh}\hat{A}$$

Then

$$\overline{p} = \frac{\|A - C\|}{\|A - B\|}\left(\text{cosh}\hat{A} - \hat{l}\text{sinh}\hat{A}\right)$$

where $\hat{l} = \frac{(A-B)\times(A-C)}{\|(A-B)\times(A-C)\|}$ is a dual split axis. The shape of a dual split triangle in dual Lorentzian space contains the information about its dual angles and ratios of side lengths, for instance, if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are dual split timelike vectors, then

$$\cosh\hat{A} = -\frac{Re\overline{p}}{\|\overline{p}\|},\ \sinh\hat{A} = \frac{\|Im\overline{p}\|}{\|\overline{p}\|},\ \|\overline{p}\| = \frac{\|A-C\|}{\|A-B\|}.$$

If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are spacelike dual split vectors satisfying

$$\left\|\left\langle\overrightarrow{AB},\overrightarrow{AC}\right\rangle_L\right\| < \left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AC}\right\|$$

then

$$\cos\hat{A} = -\frac{Re\overline{p}}{\|\overline{p}\|},\ \sin\hat{A} = -\frac{\|Im\overline{p}\|}{\|\overline{p}\|},\ \|\overline{p}\| = \frac{\|A-C\|}{\|A-B\|}.$$

If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are spacelike dual split vectors satisfying the

$$\left\|\left\langle\overrightarrow{AB},\overrightarrow{AC}\right\rangle_L\right\| > \left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AC}\right\|$$

then

$$\cosh\hat{A} = \frac{Re\overline{p}}{\|\overline{p}\|},\ \sinh\hat{A} = -\frac{\|Im\overline{p}\|}{\|\overline{p}\|},\ \|p\| = \|\overline{p}\| = \frac{\|A-C\|}{\|A-B\|}.$$

Therefore, the normal dual split vector to the dual split triangle plane is $Im\overline{p}$.

Example 4. *(Application of a shape of triangle ΔABC in dual Lorentz space) Let $A = i + \varepsilon j, B = \varepsilon k$ and $C = 2i + \varepsilon k$ be dual split vectors that formed a triangle ΔABC, we find the shape of* ΔABC. We can find

$$\overline{p} = -1 + 2\varepsilon(j+k)$$
$$\overrightarrow{AB} = -i + \varepsilon(k-j)$$
$$\overrightarrow{AC} = i + \varepsilon(k-j)$$

where $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are timelike dual split vectors, then $\overrightarrow{AB} \wedge_L \overrightarrow{AC}$ is a dual spacelike vector and

$$\langle \overrightarrow{AB}, \overrightarrow{AC} \rangle_L = -\|\overrightarrow{AB}\|\|\overrightarrow{AC}\|\cosh\hat{A} = -\cosh\hat{A}$$

$$\|\overrightarrow{AB} \wedge_L \overrightarrow{AC}\| = \|\overrightarrow{AB}\|\|\overrightarrow{AC}\|\sinh\hat{A} = \sinh\hat{A}$$

then

$$\overline{p} = -\frac{\|A - C\|}{\|A - B\|}\left(\cosh\hat{A} + \hat{l}\sinh\hat{A}\right)$$

where $\hat{l} = \frac{(A-B)\times(A-C)}{\|(A-B)\times(A-C)\|}$ dual split axis. This is shape of ΔABC dual Lorentz triangle.

4. CONCLUSION

In this study, we obtained representations for shape of triangles in dual space and dual Lorentz space. The use of quaternions in the study of triangles in dual space and dual Lorentz space allows for a deeper understanding of the geometry of these spaces. By representing the shape of triangles in these spaces using these mathematical objects, it is possible to perform calculations and make predictions about the behavior of objects in these spaces (Ozdemir,, 2006; Atasoy, 2017; Kula, 2006, 2007).

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SEVERAL OF THE DIFFERENTIAL OPERATORS ON THE SEMI-RIEMANNIAN MANIFOLDS

Derya SAĞLAM[1]

We present the definitions of some differential operators on the Euclidean space $\mathbb{R}^3$ by expanding them on a semi-Riemannian manifold M $(dimM = n)$. The operators are gradient, divergence, Hessian and Laplacian.

Let $F(M)$ be the set of all smooth real-valued functions on M. $F(M)$ is a commutative ring. The set $\mathrm{X}(M)$ of all smooth vector fields on M is a module over the ring $F(M)$. Let $\mathrm{X}^*(M)$ be the set of all (smooth) one-forms on M.

Definition 1. The vector field that is metrically equivalent to df is called the gradient of a function $f \in F(M)$ and is denoted by $gradf$ or ∇f.

Accordingly, gradf is the vector field that satisfies the equation

$$< \nabla f, V > = df(V) = Vf$$

where $V \in \mathrm{X}(M)$ and $f \in \mathrm{F}(M)$.

Let $(x_1, x_2, \ldots, x_n)$ be a local coordinate system of M. And then $\partial_i = \frac{\partial}{\partial x_i}$ is a coordinate vector field. We know that the differential of a function $f \in F(M)$ is

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$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

Now let's try to find the gradf vector field. If the vector field that is metrically equivalent to an one-form θ is V, then the vector field that is metrically equivalent to the one-form $f\theta$ is fV, for $f \in F(M)$. If the vector fields that are metrically equivalent to one-forms θ_1, θ_2 are V_1, V_2, respectively, then the vector field that is metrically equivalent to the one-form, $\theta_1 + \theta_2$ is $V_1 + V_2$. Moreover, the vector field that is metrically equivalent to the one-form dx_j is in the following equation

$$\sum_{j=1}^{n} g^{ij}\, \partial_j,$$

where g^{ij} is the (i,j) entry of the inverse matrix of the matrix

$$[g_{ij}]_{n\times n} = [< \partial_i, \partial_j >]_{n\times n}.$$

Thus, the vector field that is metrically equivalent to df is

$$gradf = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \left(\sum_{j=1}^{n} g^{ij}\, \partial_j \right) = \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x_i} \partial_j. \qquad (1)$$

Let $(u_1, u_2, \ldots, u_n)$ be a local coordinate system of the semi-Euclidean space $\mathbb{R}_q^n$. Since

$$g_{ij} = g^{ij} = \delta_{ij}\varepsilon_j,$$

from the equation (1), we get

$$gradf = \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial u_i} \partial_j = \sum_{i,j=1}^{n} \delta_{ij}\varepsilon_j \frac{\partial f}{\partial u_i} \partial_j = \sum_{i=1}^{n} \varepsilon_i \frac{\partial f}{\partial u_i} \partial_i,$$

where δ_{ij} Kronecker delta is given by

$$\delta_{ij} = \begin{cases} 1, & if\ i = j \\ 0, & if\ i \neq j \end{cases}$$

and $\varepsilon_i =< \partial_i, \partial_i >= \pm 1$. Since $\varepsilon_i = 1$ in the Euclidean space $\mathbb{R}^n$, then the vector field $gradf$ is in the following equation

$$gradf = \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} \partial_i.$$

Since df is a $(0,1)$ tensor on $\mathrm{X}(M)$, then $\uparrow_1^1 df$ is a $(1,0)$ tensor on $\mathrm{X}(M)$. Since every vector field is a $(1,0)$ tensor on $\mathrm{X}(M)$, then $gradf$ is a $(1,0)$ tensor on $\mathrm{X}(M)$. And then from the following equations

$$(\uparrow_1^1 df\,)(dx_i) = df\left(\sum_{k=1}^{n} g^{ik}\,\partial_k\right) = \sum_{k=1}^{n} g^{ik}\, df(\partial_k)$$

$$= \sum_{k=1}^{n} g^{ik}\,\partial_k(f) = \sum_{k=1}^{n} g^{ik}\frac{\partial f}{\partial x_k}$$

and

$$(gradf\,)(dx_i) = \left(\sum_{k,j=1}^{n} g^{kj}\frac{\partial f}{\partial x_k}\partial_j\right)(dx_i)$$

$$= \sum_{k,j=1}^{n} g^{kj}\frac{\partial f}{\partial x_k}\partial_j(dx_i) = \sum_{k,j=1}^{n} g^{kj}\frac{\partial f}{\partial x_k}\delta_{ij}$$

$$= \sum_{k=1}^{n} g^{ki}\frac{\partial f}{\partial x_k} = \sum_{k=1}^{n} g^{ik}\frac{\partial f}{\partial x_k}$$

we obtain $\uparrow_1^1 df = gradf$.

If A is a (r,s) tensor on $\mathrm{X}(M)$, then its covariant differential DA is a $(r,s+1)$ tensor on $\mathrm{X}(M)$. Accordingly, we can define the divergence of A.

Definition 2. A divergence of a (r,s) tensor A is a contraction of the new covariant slot of DA with one of its original slot and denoted by $divA$.

These contractions are C^{i}_{s+1} or $C_{j(s+1)}$, not otherwise. The divergence of a tensor A may be more than one, depending on the contractions. But in these two special cases it is unique.

Case 1. Let's calculate the divergence of a vector field V. V is a (1,0) tensor and its covariant differential DA is a (1,1) tensor on X(M). Then we can just get the contraction C^1_1. and $divV = C^1_1(DV) \in F(M)$. If A is a (1,1) tensor on X(M), then

$$C^1_1 A = \sum_{i=1}^{n} A^i_i = \sum_{i=1}^{n} A(dx_i, \partial_i).$$

Hence, for a frame field $\{E_1, E_2, \dots E_n\}$, from the equations

$$\langle D_{E_i}V, E_i\rangle = \langle \sum_{j=1}^{n} a_j\, E_j, E_i\rangle = \sum_{j=1}^{n} a_j\, \langle E_j, E_i\rangle = \sum_{j=1}^{n} a_j\, \delta_{ij}\varepsilon_j$$
$$= a_i\varepsilon_i$$

and

$$(DV)^i_i = (DV)((E_i)^*, E_i) = (D_{E_i}V)(E_i)^* = (E_i)^*(D_{E_i}V)$$
$$= (E_i)^*\left(\sum_{j=1}^{n} a_j\, E_j\right) = \sum_{j=1}^{n} a_j\, (E_i)^*(E_j)$$
$$= \sum_{j=1}^{n} a_j\, \delta_{ij} = a_i,$$

then the components of DV is

$$(DV)^i_i = a_i = \varepsilon_i\langle D_{E_i}V, E_i\rangle.$$

Thus we obtain

$$divV = C_1^1(DV) = \sum_{i=1}^{n}(DV)_i^i = \sum_{i=1}^{n}\varepsilon_i\langle D_{E_i}V, E_i\rangle. \qquad (2)$$

$\left\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \dots, \frac{\partial}{\partial u_n}\right\}$ is the natural coordinate frame of the semi-Euclidean space $\mathbb{R}_q^n$. In $\mathbb{R}_q^n$, a vector field V is following by

$$V = \sum_{i=1}^{n} V_i\,\partial_i,$$

where $\partial_i = \frac{\partial}{\partial u_i}$. We will denote the components of DV by $V_{;j}^i = (DV)_j^i$. Since

$$\begin{aligned}
D_{\partial_i}V = D_{\partial_i}\left(\sum_{j=1}^{n} V_j\,\partial_j\right) &= \sum_{j=1}^{n} D_{\partial_i}\left(V_j\partial_j\right) \\
&= \sum_{j=1}^{n}\left(\partial_i\left(V_j\right)\partial_j + V_j D_{\partial_i}\partial_j\right) \\
&= \sum_{j=1}^{n}\left(\partial_i\left(V_j\right)\partial_j + V_j\sum_{k=1}^{n}\Gamma_{ij}^k\,\partial_k\right) \\
&= \sum_{j=1}^{n}\partial_i\left(V_j\right)\partial_j + \sum_{j,k=1}^{n} V_j\,\Gamma_{ij}^k\partial_k \\
&= \sum_{k=1}^{n}\partial_i(V_k)\partial_k + \sum_{j,k=1}^{n} V_j\,\Gamma_{ij}^k\partial_k \\
&= \sum_{k=1}^{n}\left(\partial_i(V_k) + \sum_{j=1}^{n} V_j\,\Gamma_{ij}^k\right)\partial_k
\end{aligned}$$

and

$$(du_i)(D_{\partial_i}V) = (du_i)\left(\sum_{k=1}^{n}\left(\partial_i(V_k) + \sum_{j=1}^{n} V_j\,\Gamma_{ij}^{k}\right)\partial_k\right)$$
$$= \sum_{k=1}^{n}\left(\partial_i(V_k) + \sum_{j=1}^{n} V_j\,\Gamma_{ij}^{k}\right)(du_i)(\partial_k)$$
$$= \sum_{k=1}^{n}\left(\partial_i(V_k) + \sum_{j=1}^{n} V_j\,\Gamma_{ij}^{k}\right)\delta_{ik}$$
$$= \partial_i(V_i) + \sum_{j=1}^{n} V_j\,\Gamma_{ij}^{i} = \frac{\partial V_i}{\partial u_i} + \sum_{j=1}^{n} V_j\,\Gamma_{ij}^{i},$$

we obtain

$$divV = \sum_{i=1}^{n}(DV)_i^i = \sum_{i=1}^{n} V_{;i}^{i} = \sum_{i=1}^{n}(DV)(du_i, \partial_i)$$
$$= \sum_{i=1}^{n}(D_{\partial_i}V)(du_i) = \sum_{i=1}^{n}(du_i)(D_{\partial_i}V)$$
$$= \sum_{i=1}^{n}\left(\frac{\partial V_i}{\partial u_i} + \sum_{j=1}^{n} V_j\,\Gamma_{ij}^{i}\right).$$

Finally, since $\Gamma_{ij}^{k} = 0$ for all $i, j, k = 1, \dots, n$ in $\mathbb{R}_q^n$, we have

$$divV = \sum_{i=1}^{n}\frac{\partial V_i}{\partial u_i}.$$

Case 2. We assume that A is a symmetric $(0,2)$ tensor on $\mathrm{X}(M)$. Then the tensor $divA$ is also obtained with contractions C_{13} or C_{23} of its covariant differential DA. Also DA is a $(0,3)$ tensor and $divA$ is a $(0,1)$ tensor on $\mathrm{X}(M)$. If A is $(0, s)$ tensor on $\mathrm{X}(M)$, then for a frame field $\{E_1, E_2, \dots E_n\}$

$$(C_{ab}A)(V_1, \dots, V_{s-2}) = \sum_{m=1}^{n} \varepsilon_m \, A(V_1, \dots, E_m, \dots, E_m, \dots, V_{s-2}),$$

where the vector fields E_m in the ath and bth slot, respectively. Moreover $C_{ab} = C_{ba}$, $C_{13}(DA)$ and $C_{23}(DA)$ are $(0,1)$ tensors on $\mathrm{X}(M)$. Since a tensor A is symmetric, then the tensor $D_V A$ is symmetric for any vector field V. Actually, for all $X, Y \in \mathrm{X}(M)$

$$\begin{aligned}(D_V A)(X,Y) &= D_V\big(A(X,Y)\big) - A(D_V X, Y) - A(X, D_V Y) \\ &= D_V\big(A(Y,X)\big) - A(Y, D_V X) - A(D_V Y, X) \\ &= D_V\big(A(Y,X)\big) - A(D_V Y, X) - A(Y, D_V X) \\ &= (D_V A)(Y,X)\end{aligned}$$

(see page 45 in [1]). Thus, from the equations

$$\begin{aligned}\big(C_{13}(DA)\big)(V) &= \sum_{m=1}^{n} \varepsilon_m \, (DA)(E_m, V, E_m) \\ &= \sum_{m=1}^{n} \varepsilon_m \, (D_{E_m} A)(E_m, V)\end{aligned}$$

and

$$\begin{aligned}\big(C_{23}(DA)\big)(V) &= \sum_{m=1}^{n} \varepsilon_m \, (DA)(V, E_m, E_m) \\ &= \sum_{m=1}^{n} \varepsilon_m \, (D_{E_m} A)(V, E_m)\end{aligned}$$

for all $V \in \mathrm{X}(M)$, we obtain $C_{13}(DA) = C_{23}(DA) = divA$. Hence we obtain

$$(divA)(V) = \sum_{i=1}^{n} \varepsilon_i \, (D_{E_i} A)(E_i, V). \tag{3}$$

In addition, the components of $divA$ is by following

$$(divA)_i = \left(C_{13}(DA)\right)_i = \sum_{j,k=1}^{n} g^{jk}\,(DA)_{jik} = \sum_{j,k=1}^{n} g^{jk}\,A_{ji;k}$$
$$= \sum_{k=1}^{n}\left(\sum_{j=1}^{n} g^{jk}\,A_{ji;k}\right) = \sum_{k=1}^{n} A^{k}_{i;k}.$$

Definition 3. The Hessian of a smooth function f is its second covariant differential and is denoted by H^f. Then $H^f = D(Df)$.

Lemma 4. The Hessian of a smooth function f is a symmetric $(0,2)$ tensor on X(M)such that

$$H^f(V,W) = VWf - (D_V W)f = \langle (D_V(gradf), W\rangle. \qquad (4)$$

Proof. Since $[V,W] = D_V W - D_W V = VW - WV,$ then $VW - D_V W = WV - D_W V$. Thus

$$VWf - (D_V W)f = WVf - (D_W V)f$$

and

$$H^f(V,W) = \left(D(Df)\right)(V,W) = WVf - (D_W V)f = H^f(W,V).$$

This equation proves that H^f is symmetric. Finally, from the following equation

$$V\langle gradf, W\rangle = \langle D_V(gradf), W\rangle + \langle gradf, D_V W\rangle,$$

we have

$$\langle D_V(gradf), W\rangle = V\langle gradf, W\rangle - \langle gradf, D_V W\rangle$$
$$= VWf - (D_V W)f = H^f(V,W).$$

Specifically, we calculate the Hessian of a differentiable function $f: \mathbb{R}^3 \to \mathbb{R}$. Let $V = \sum_{i=1}^{3} a_i\,\partial_i$ and $W = \sum_{j=1}^{3} b_j\,\partial_j$ Then, from the equation (4), we get

$$\begin{aligned} H^f(V,W) &= W(Vf) - (D_W V)f \\ &= \sum_{j=1}^{3} b_j\,\partial_j\left(\sum_{i=1}^{3} a_i\,\partial_i f\right) - \left(\sum_{i,j=1}^{3} b_j\,D_{\partial_j} a_i \partial_i\right) f \\ &= \sum_{i,j=1}^{3} b_j\,\partial_j(a_i\partial_i f) - \sum_{i,j=1}^{3} b_j\,(D_{\partial_j} a_i \partial_i) f \\ &= \sum_{i,j=1}^{3} b_j\,[a_i\partial_j(\partial_i f) - a_i(D_{\partial_j}\partial_i) f] \\ &= \sum_{i,j=1}^{3} b_j \left[a_i\partial_j(\partial_i f) - a_i\left(\sum_{k=1}^{3}\Gamma_{ji}^k\,\partial_k\right) f\right]. \end{aligned}$$

Since $\Gamma_{ji}^k = 0$ in $\mathbb{R}^3$, then we obtain

$$H^f(V,W) = \sum_{i,j=1}^{3} a_i\partial_j(\partial_i f)\,b_j = \sum_{i,j=1}^{3} a_i\,\frac{\partial^2 f}{\partial u_i \partial u_j}\,b_j.$$

Definition 5. The Laplacian of a differentiable function f is a new function and is defined the divergence of its gradient. We denote it by Δf. Then

$$\Delta f = div(gradf) \in F(M).$$

We assume that A is a $(0,2)$ tensor on $\mathrm{X}(M)$. Since

$$\begin{aligned} (\uparrow_1^1 A)_j^i = (\uparrow_1^1 A)(dx_i, \partial_j) &= A\left(\sum_{k=1}^{n} g^{ik}\,\partial_k, \partial_j\right) \\ &= \sum_{k=1}^{n} g^{ik}\,A(\partial_k, \partial_j) = \sum_{k=1}^{n} g^{ik}\,A_{kj} = \sum_{k=1}^{n} g^{ki}\,A_{kj}, \end{aligned}$$

we have

$$(C\,\uparrow_1^1)A = \sum_{i=1}^{n} (\uparrow_1^1 A)_i^i = \sum_{i=1}^{n}\sum_{k=1}^{n} g^{ki}\,A_{ki} = \sum_{i,k=1}^{n} g^{ki}\,A_{ki} = C_{12}A$$

Since the covariant differential commutes with type-changing, we find

$$\Delta f = div(gradf) = CD(gradf) = CD(\uparrow_1^1 df) = C \uparrow_1^1 Ddf$$
$$= (C \uparrow_1^1)H^f = C_{12}H^f. \qquad (5)$$

According to the coordinate system of $(x_1, x_2, \ldots, x_n)$ on M, we get

$$\begin{aligned}\Delta f = C_{12}H^f &= \sum_{i,j=1}^{n} g^{ij} H_{ij}^f = \sum_{i,j=1}^{n} g^{ij} H^f\left(\partial_i, \partial_j\right) \\ &= \sum_{i,j=1}^{n} g^{ij} \left(\partial_i \partial_j f - (D_{\partial_i}\partial_j)f\right) \\ &= \sum_{i,j=1}^{n} g^{ij} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \left(\sum_{k=1}^{n} \Gamma_{ij}^k \partial_k\right) f\right) \\ &= \sum_{i,j=1}^{n} g^{ij} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial f}{\partial x_k}\right).\end{aligned}$$

Also, in the semi-Euclidean space $\mathbb{R}_q^n$

$$\Delta f = \sum_{i,j=1}^{n} \delta_{ij} \varepsilon_j \frac{\partial^2 f}{\partial u_i \partial u_j} = \sum_{i=1}^{n} \varepsilon_i \frac{\partial^2 f}{\partial {u_i}^2}$$

and in the Euclidean $\mathbb{R}^n$

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial {u_i}^2}.$$

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CHARACTERIZATIONS OF SPECIAL RULED SURFACES IN $\mathbb{R}_1^3$

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1. INTRODUCTION

One of the differential geometry topics that is most frequently studied is the theory of surfaces and transformations.

Ruled surfaces created by moving a straight line in constant motion have long been a strong subject for line geometry in Minkowski space. Therefore it has been studied by many authors [5-17]. Yüksel considered in [18] spacelike ruled surfaces through parallel vector fields, known as alternating or parallel frames of curves, according to the Bishop frame introduced by L. R. Bishop in 1975 in [1] and gave some characterizations for spacelike ruled surfaces generated by Bishop vectors. However, the ruled surfaces according to the Bishop frame have been considered by many authors [3, 4]. In this study, we provide some characterizations of timelike ruled surfaces in Minkowski 3-space generated by Bishop vectors.

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2. BASIC CONCEPTS

Consider that a Minkowski 3-dimensional space $\mathbb{R}_1^3$, with the metric tensor

$$J = \langle .,. \rangle = -dx_1^2 + dx_2^2 + dx_3^2.$$

The norm of $u \in \mathbb{R}_1^3$ is defined by $\|u\| = \sqrt{|\langle u,u \rangle|}$

One of the three Lorentzian properties that a vector $u \in \mathbb{R}_1^3$ can have;

it can be spacelike if $\langle u,u \rangle > 0$ or $u = 0$;

timelike if $\langle u,u \rangle < 0$;

lightlike (or null) if $\langle u,u \rangle = 0$ and $u \neq 0$.

Let $\eta: J \to \mathbb{R}_1^3$, $\eta(s) = \left(\eta_1(s), \eta_2(s), \eta_3(s)\right)$ is a regular curve in $\mathbb{R}_1^3$. If $\eta'(s)$ is a timelike vector for all $s \in J \subset \mathbb{R}$, we call the curve η a timelike curve.

If the Lorentz metric on a surface in Minkowski 3-space is a negative definite, then the surface is referred to as a timelike surface [10].

A surface that is swept out by a straight line A moving along a curve η is known as a ruled surface. The surface's rullings refer to the different locations of the generating line A. The parameterization of such a surface is as follows in the ruled form:

$$\omega(s,u) = \eta(s) + uA(s)$$

where A is the director vector and $\eta(s)$ is the base curve. A surface that follows a fixed ruling and has a constant tangent plane is referred to as a developable surface. Skew surfaces are the remaining ruled surfaces [13]. The parameterization of the timelike-ruled surface S in $\mathbb{R}_1^3$ is given by

$$\omega: I \times \mathbb{R} \to \mathbb{R}_1^3$$
$$(s,u) \to \omega(s,u) = \eta(s) + uA(s) \qquad (1)$$

where A is orthogonal to the tangent vector field $T(s)$ of the base curve η. Here $\omega: I \to \mathbb{R}_1^3$, is a differentiable timelike curve parametrized by its arc length in $\mathbb{R}_1^3$ of the director curve.

The moving Frenet frame along the regular curve with the arc-length parameter s is shown as $\{T, N, B\}$.

The binormal vector B, the principal normal vector N, and the tangent vector T constitute the structure of the Frenet trihedron. When η is a timelike curve with a spacelike binormal, the following characteristics of the Frenet frame are present:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \varkappa & 0 \\ \varkappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where

$$\langle T, T \rangle = -1 \,, \; \langle N, N \rangle = \langle B, B \rangle = 1$$

and first curvature and second curvature $\varkappa(s), \tau(s)$ respectively.

An alternate method for creating a moving frame that is well-defined even in cases when the curve's second derivative vanishes is the Bishop frame, also known as the parallel transport frame. To represent the parallel transport of an orthonormal frame along a curve, just parallel transport each frame component. Any convenient arbitrary basis and the tangent vector are used for the remaining section of the frame.

Assume that the timelike curve $\eta(s)$ has a Bishop frame $\{T, M_{1,} M_2\}$, where $T(s)$ is the timelike unit tangent vector, $M_1(s)$ is the spacelike unit normal vector, and $M_2(s)$ is the spacelike unit binormal vector. Therefore, the vectors $\{T, M_{1,} M_2\}$'s scalar and cross products are given by

$$\langle T,T\rangle = -1\,, \langle M_1,M_1\rangle = \langle M_2,M_2\rangle = 1$$

$$\langle T,M_1\rangle = \langle T,M_2\rangle = \langle M_1,M_2\rangle = 0$$

$$T \times M_1 = M_2$$

$$M_1 \times M_2 = -T$$

$$M_2 \times T = M_1$$

The Bishop frame derivate equations $\{T', M_1', M_2'\}$ are as follows

$$\begin{bmatrix} T'(s) \\ {M_1}'(s) \\ {M_2}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_1(s) & \varepsilon_2(s) \\ \varepsilon_1(s) & 0 & 0 \\ \varepsilon_2(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ M_1(s) \\ M_2(s) \end{bmatrix} \tag{2}$$

Someone can demonstrate that

$$\varkappa = \sqrt{\varepsilon_1^2 + \varepsilon_2^2} \tag{3}$$

$$\tau = \Theta'(\mathrm{s}), \qquad \Theta(\mathrm{s}) = \operatorname{argtanh}\frac{\varepsilon_2}{\varepsilon_1} \tag{4}$$

For the polar coordinates $\varkappa$, Θ with $\int \tau(s)ds$ to correctly relate to a cartesian coordinate system, ε_1 and ε_2 [2].

Remark 2.1.1 Based on the argtanh function's definition, we presume that $|\varepsilon_2/\varepsilon_1| < 1$.

The ruled surface $\omega(s,u)$ has the distribution parameter, mean curvature, and Gaussian curvature provided by

$$P_A = \frac{det(T,A,D_T A)}{\langle D_T A, D_T A\rangle} \tag{5}$$

$$H = \frac{1}{2}\left[\frac{Gl+En-2Fm}{EG-F^2}\right] \tag{6}$$

where D is Levi Civita connection on $\mathbb{R}_1^3$ [11].

Theorem 2.1.2 The development of a timelike surface occurs only when the distribution parameter of a timelike ruled surface is zero [13].

The striction curve's parametrization on the ruled surface is provided by

$$\bar{\eta}\,(s) = \eta(s) - \frac{\langle T, D_T A\rangle}{\langle D_T A, D_T A\rangle} A(s). \qquad (7)$$

3. MAIN RESULTS

Let $\{T, M_{1,} M_2\}$ be the Bishop frame of a timelike curve $\eta: J \to \mathbb{R}_1^3$, where $T, M_{1,} M_2$ are the curve's tangent, principal normal, and binormal vectors of the curve η, respectively. M_1 and M_2 are spacelike vectors whereas T is a timelike vector.

In $\mathbb{R}_1^3$, the moving space H and the fixed space $H'$are represented by the orthogonal coordinate systems $\{O; T, M_{1,} M_2\}$ and $\{O'; e_1, e_{2,} e_3\}$.

Suppose that A is a vector that is unit spacelike or unit timelike.

$$A \in Sp\{T(s), M_1(s), M_2(s)\}$$

$$A = a_1 T(s) + a_2 M_1(s), +a_3 M_2(s) \qquad (8)$$

such that

$$\langle A, A\rangle = \mp 1.$$

It is possible to derive the distribution parameter P_A of the timelike ruled surface generated by the moving space H's straight line A. When we differentiate (8) with respect to s, we obtain

$$D_T A = a_1 T'(s) + a_2 M_1'(s) + a_3 M_2'(s),$$

$$-a_1^2 + a_2^2 + a_3^2 = \mp 1.$$

Substituting the Bishop derivative equation we obtain following equation

$$D_T A = (a_2\varepsilon_1 + a_3\varepsilon_2)T(s) + a_1\varepsilon_1 M_1(s) + a_1\varepsilon_2 M_2(s).$$

$$det(T, A, D_T A) = \begin{vmatrix} 1 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_2\varepsilon_1 + a_3\varepsilon_2 & a_1\varepsilon_1 & a_1\varepsilon_2 \end{vmatrix}$$

$$= a_1 a_2 \varepsilon_2 - a_1 a_3 \varepsilon_1, \tag{9}$$

$$\langle D_T A, D_T A \rangle = -(a_2\varepsilon_1 + a_3\varepsilon_2)^2 + a_1^2\varepsilon_1^2 + a_1^2\varepsilon_2^2. \tag{10}$$

If Equation (9) and Equation (10) are substituted in Equation (5) we obtain

$$P_A = \frac{a_1(a_2\varepsilon_2 - a_3\varepsilon_1)}{-(a_2\varepsilon_1 + a_3\varepsilon_2)^2 + a_1^2(\varepsilon_1^2 + \varepsilon_2^2)},$$

$$-a_1^2 + a_2^2 + a_3^2 = \mp 1 \tag{11}$$

Theorem 3.1.1 Suppose that S is a timelike ruled surface, parametrized by $\omega(s,u) = \eta(s) + uA(s)$. If and only if the base curve η is a planar curve with curvatures such that

$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{a_3}{a_2},$$

or if the director vector A is located in the plane created by $M_1(s)$ and $M_2(s)$, then S is developable.

Proof 3.1.1 Let $\omega(s,u) = \eta(s) + uA(s)$ be a timelike ruled surface denoted by S and from Theorem 2.1.2

$$a_1(a_2\varepsilon_2 - a_3\varepsilon_1) = 0$$

is obtained. So we have

$$a_1 = 0 \text{ or } a_2\varepsilon_2 - a_3\varepsilon_1 = 0.$$

Thus

$$A \in Sp\{M_1(s), M_2(s)\}$$

or

$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{a_3}{a_2} = \text{constant}.$$

From Equation (4) $\tau = 0$ then η is a planar curve.

3.2. Characterizations of Ruled Surfaces Whose Director Vector is Bishop Vectors for Special Cases

Suppose that S is a timelike ruled surface, parametrized by $\omega(s,u) = \eta(s) + uA(s)$ and, A is the director vector of the base curve η.

3.2.1 The Case $\boldsymbol{A} = \boldsymbol{T}$ (Timelike)

In the present situation

$$a_1 = 1, \ \ a_2 = a_3 = 0,$$

from Equation (11)

$$P_T = 0.$$

As a result, the next theorem is evident.

Theorem 3.2.2 During the one-parameter spatial motion H/H', the timelike ruled surface in the fixed space H' that is generated by the tangent line T of the curve η(s) in the moving space H is developable.

3.2.3 The Case $\boldsymbol{A} = \boldsymbol{M_1}$ (Spacelike)

In this case,

$$a_2 = 1, \ \ a_1 = a_3 = 0$$

thus from Equation (11)

$$P_{M_1} = 0.$$

3.2.4 The Case $\boldsymbol{A} = \boldsymbol{M_2}$ (Spacelike)

In this case,

$$a_3 = 1, \ \ a_1 = a_2 = 0$$

thus from Equation (11)

$$P_{M_2} = 0.$$

Corollary 3.2.5 It is possible to construct the developable timelike ruled surface in the fixed space H' during the one-parameter spatial motion H/H', which is generated by the normal and binormal line M_1, M_2 of th curve $\eta(s)$ in the moving space.

3.2.6 The Case $\boldsymbol{A} \in \boldsymbol{Sp}\{\boldsymbol{T}, \boldsymbol{M_1}\}$

In this case,

$$a_3 = 0$$

So, the director vector A is given by

$$A = a_1 T + a_2 M_1, \quad -a_1^2 + a_2^2 = \mp 1.$$

The ruled surface's distribution parameter is provided by

$$P_A = \frac{a_1 a_2 \varepsilon_2}{-a_2^2 \varepsilon_{1+}^2 a_1^2 (\varepsilon_{1+}^2 \varepsilon_2^2)},$$

$$P_A = \mp \frac{a_1 \sqrt{\mp 1 + a_1^2}\varepsilon_2}{-a_2^2 \varepsilon_{1+}^2 a_1^2 (\varepsilon_{1+}^2 \varepsilon_2^2)}.$$

Theorem 2.1.2 shows us that the distribution parameter must be zero in order for a timelike surface to be developable, both required and sufficient. Then,

if $$P_A = 0,$$

$$a_1 = 0 \quad or \quad \varepsilon_2 = 0.$$

if $a_1 = 0$ this is case 3.2.3. If the second Bishop curvature $\varepsilon_2 = 0$, it is possible to say that the η basis curve is planar.

3.2.7 The Case$\boldsymbol{A} \in \boldsymbol{Sp}\{\boldsymbol{T}, \boldsymbol{M_2}\}$

In this case,

$$a_2 = 0$$

So, the director vector A is given by

$$A = a_1 T + a_3 M_2, \quad -a_1^2 + a_3^2 = \mp 1.$$

The ruled surface's distribution parameter is provided by

$$P_A = -\frac{a_1 a_3 \varepsilon_1}{-a_3^2\varepsilon_2^2 + a_1^2(\varepsilon_{1+}^2 \varepsilon_2^2)},$$

$$P_A = \mp\frac{a_1\sqrt{\mp 1 + a_1^2}\varepsilon_1}{-a_3^2\varepsilon_{1+}^2 a_1^2(\varepsilon_{1+}^2\varepsilon_2^2)}$$

It is known from Theorem 2.1.2 that the distribution parameter must be zero in order for a timelike surface to be developable. Then

if $$P_A = 0$$

$$\varepsilon_1 \neq 0 \quad and \quad a_1 = 0.$$

So this is case 3.2.4.

3.2.8 The Case $A \in Sp\{M_1, M_2\}$

In this case,

$$a_1 = 0.$$

So, the director vector A is given by

$$A = a_2 M_1 + a_3 M_2, \quad a_2^2 + a_3^2 = \mp 1.$$

Then

$$P_A = 0.$$

From Theorem 2.1.2 it is obvious that the timelike ruled surface is developable.

Since the timelike ruled surfaces generated by the direction vectors T, M_1, M_2 are developable ruled surfaces, their Gaussian curvatures are zero. Now let us calculate the mean curvatures of these surfaces.

Proposition 3.2.9 Let S_T be a timelike ruled surface generated by the curve η's tangent line T. The mean curvature can be calculated as follows from Equation (6)

$$H_T = \frac{1}{2}\frac{\epsilon(\varepsilon_1'\varepsilon_2 - \varepsilon_1\varepsilon_2')}{u(\varepsilon_1^2+\varepsilon_2^2)^{3/2}}, \qquad \epsilon = \mp 1.$$

Thus from Equation (3) and Equation (4) we obtain

$$H_T = \frac{1}{2}\frac{\epsilon\tau}{\varkappa^3} \tag{12}$$

Corollary 3.2.10 If and only if η is a planar curve, then the surface S_T is minimal.

Proof 3.2.10 Assume that the timelike-ruled surface S_T to be minimal. In this particular situation, we find from Equation (12),

$$\tau = 0.$$

Let η, on the other hand, be a planar curve. Therefore, $H_T = 0$ is implied by $\tau = 0$.

Proposition 3.2.11 Let S_{M_1} be a timelike ruled surface generated by the curve η's normal line M_1. The mean curvature can be calculated as follows from Equation (6)

$$H_{M_1} = -\frac{1}{2}\frac{\epsilon\varepsilon_2}{(1+u\varepsilon_1)} \qquad \epsilon = \mp 1$$

Corollary 3.2.12 The Bishop frame states that the normal line M_1 in $\mathbb{R}_1^3$ does not yield a minimal timelike ruled surface.

Proposition 3.2.13 Let S_{M_2} be a timelike ruled surface generated by the curve η's binormal line M_2. The mean curvature can be calculated as follows from (6)

$$H_{M_2} = \frac{1}{2}\frac{\epsilon\varepsilon_1}{(1+u\varepsilon_2)} \qquad \epsilon = \mp 1$$

Corollary 3.2.14 The Bishop frame states that the normal line M_2 in $\mathbb{R}_1^3$ does not yield a minimal timelike ruled surface.

Proposition 3.2.15 Given the parametrization

$\omega(s,u) = \eta(s) + uA(s)$, have S constitute a timelike ruled surface.

Thus

$$H_A = \frac{1}{2}\left[\frac{\mathrm{Gl+En-2Fm}}{\mathrm{EG-F^2}}\right],$$

where

$$E = \langle \omega_s, \omega_s\rangle = -\big(1 + u(\varepsilon_1 a_2 + \varepsilon_2 a_3)\big)^2 + u^2 a_1^2(\varepsilon_1^2 - \varepsilon_2^2)$$

$$F = \langle \omega_s, \omega_u\rangle = -a_1$$

$$G = \langle \omega_u, \omega_u\rangle = \mp 1,$$

$$l = \frac{1}{\|\omega_s \times \omega_u\|}\langle \omega_{ss}, \omega_s \times \omega_u\rangle$$

$$= \left\{\begin{matrix} (\varepsilon_2 a_2 - \varepsilon_1 a_3) + 2u\big((a_2^2 - a_3^2)\varepsilon_1\varepsilon_2 - a_2 a_3(\varepsilon_1^2 - \varepsilon_2^2)\big) \\ +u a_1\big((a_2\varepsilon_2' - a_3\varepsilon_1') + u\delta(\varepsilon_1'\varepsilon_2 - \varepsilon_1\varepsilon_2')\big) \\ +u^2\varepsilon_1 a_3\big((a_1^2 - a_2^2)\varepsilon_1^2 + (a_1^2 + 2a_2^2 - a_3^2)\varepsilon_2^2\big) \\ +u^2\varepsilon_2 a_2\big((a_3^2 - a_1^2)\varepsilon_2^2 - (a_1^2 + 2a_3^2 - a_2^2)\varepsilon_1^2\big) \end{matrix}\right\}$$

$$m = \frac{1}{\|\omega_s \times \omega_u\|}\langle \omega_{su}, \omega_s \times \omega_u\rangle = \frac{a_1(\varepsilon_2 a_2 - \varepsilon_1 a_3)}{\|\omega_s \times \omega_u\|}$$

$$n = \frac{1}{\|\omega_s \times \omega_u\|}\langle \omega_{uu}, \omega_s \times \omega_u\rangle = 0$$

where

$$N = \frac{\omega_s \times \omega_u}{\|\omega_s \times \omega_u\|}$$

$$= \frac{1}{\|\omega_s \times \omega_u\|}\left\{\begin{matrix} [u a_1(\varepsilon_2 a_2 - \varepsilon_1 a_3)]T \\ +[-a_3(1 + u(\varepsilon_1 a_2 + \varepsilon_2 a_3) + u a_1^2\varepsilon_2)]M_1 \\ +[a_2(1 + u(\varepsilon_1 a_2 + \varepsilon_2 a_3) - u a_1^2\varepsilon_1)]M_2 \end{matrix}\right\}$$

represents the timelike ruled surface S's unit normal vector.

Proposition 3.2.16 Consider that the timelike ruled surface S, which is obtained using parameterization

$\omega(s,u) = \eta(s) + uA(s)$. The curvature functions ε_1 and ε_2 of the base curve η satisfy the following equation if the base curve of S is also a striction curve:

$$a_2\varepsilon_1 + a_3\varepsilon_2 = 0 \tag{13}$$

Proof 3.2.16 Let the base curve η be the striction curve. Thus, from Equation (7),

$$\langle T, D_T A\rangle = 0$$

Then we obtain

$$a_2\varepsilon_1 + a_3\varepsilon_2 = 0.$$

Corollary 3.2.17 Let S be a timelike ruled surface, given by the parametrization $\omega(s,u) = \eta(s) + uA(s)$. η is a planar curve if S's base curve is a striction curve as well.

Proof 3.2.17 Let the base curve η be also striction curve.Thus from Equation (13)

$$a_2\varepsilon_1 + a_3\varepsilon_2 = 0.$$

Hence we get

$$\frac{\varepsilon_1}{\varepsilon_2} = -\frac{a_3}{a_2} = constant.$$

From Equation (4), η is a planar curve.

Proposition 3.2.18 Let S be a timelike ruled surface given by

the parametrization (1). If the base curve of S is also an asymptotic curve then

$$\big(1 + u(\varepsilon_1 a_2 + \varepsilon_2 a_3)\big)(a_2\varepsilon_1 + a_3\varepsilon_2) = 0 \tag{14}$$

Proof 3.2.18 Assumedly, the asymptotic curve is the base curve of the surface S. Then,

$$\langle \eta'', N\rangle = 0. \tag{15}$$

From Equation (15), we obtain

$$\big(1 + u(\varepsilon_1 a_2 + \varepsilon_2 a_3)\big)(a_2\varepsilon_1 + a_3\varepsilon_2) = 0.$$

Theorem 3.2.19 Assume that the surface S has an asymptotic base curve. The timelike ruled surface S is developable if its base curve is also a striction curve.

Proof 3.2.19 Assume that the surface S has a base curve that is both an asymptotic and striction curve. By using Equations (13) and (14)

we get

$$(a_2\varepsilon_1 + a_3\varepsilon_2) = 0$$

From Equation (11), the timelike ruled surface S is developable.

Proposition 3.2.20 Consider that the timelike ruled surface S, which is obtained using parametrization (1) $\omega(s,u) = \eta(s) + uA(s)$. Regarding the timelike ruled surfaces, we derive the following conclusions

a. If and only if $l = 0$, the (S)'s s-parameter curve is also an asymptotic curve.

b. Also, S's u-parameter curve is an asymptotic curve.

Proof 3.2.20 (a) When s-parameter curve of S is asymptotically formed, then

$$l = \frac{1}{\|\omega_s \times \omega_u\|}\langle \omega_{ss}, \omega_s \times \omega_u \rangle = 0,$$

thus proof is clear.

(b) If the u-parameter curve of S is also an asymptotic curve, then

$$n = \frac{1}{\|\omega_s \times \omega_u\|}\langle \omega_{uu}, \omega_s \times \omega_u \rangle = 0,$$

$$m = \frac{1}{\|\omega_s \times \omega_u\|}\langle \omega_{su}, \omega_s \times \omega_u \rangle = \frac{a_1(\varepsilon_2 a_2 - \varepsilon_1 a_3)}{\|\omega_s \times \omega_u\|} = 0$$

u-parameter curve of (S) is an asymptotic curve.

Theorem 3.2.21 Consider S be a timelike ruled surface which is developable, given by the parametrization

$\omega(s,u) = \eta(s) + uA(s)$. An asymptotic curve is also represented by the s-parameter curve of S, provided that S is a minimal surface.

Proof 3.2.21 Suppose that the surface S's *s*-parameter curve is an asymptotic curve. Consequently,

$$l = \frac{1}{\|\omega_s \times \omega_u\|} \langle \omega_{ss}, \omega_s \times \omega_u \rangle = 0$$

where *N* is the surface S's unit normal vector field. Given that S is a developable timelike ruled surface,

$$m = \frac{1}{\|\omega_s \times \omega_u\|} \langle \omega_{su}, \omega_s \times \omega_u \rangle = \frac{a_1(\varepsilon_2 a_2 - \varepsilon_1 a_3)}{\|\omega_s \times \omega_u\|} = 0$$

Thus

$$H_A = 0$$

is acquired.

In contrast, consider S to be a minimal surface. From Equations (6), we obtain

$$\mathrm{l} - 2\mathrm{Fm} = 0$$

Considering that S is a developable timelike ruled surface, we get

$$l = 0$$

The proof has become complete.

Proposition 3.2.22 Assume that the surface S's base curve is geodesic. Next, the ensuing formulas are derived.

a. $ua_1^2(\varepsilon_{1+}^2\varepsilon_2^2) - \left(1 + u(\varepsilon_1 a_2 + \varepsilon_2 a_3)\right)(a_2\varepsilon_1 + a_3\varepsilon_2) = 0$

b. $ua_1(\varepsilon_2 a_2 - \varepsilon_1 a_3)\varepsilon_1 = 0$

c. $ua_1(\varepsilon_2 a_2 - \varepsilon_1 a_3)\varepsilon_2 = 0$ (16)

Proof 3.2.22 Assuming that the surface *S*'s base curve η is a geodesic curve,

$$\eta'' \times N = 0,$$

$$\eta'' = \varepsilon_1 M_1 + \varepsilon_2 M_2$$

and

$$N = \frac{1}{\|\omega_s \times \omega_u\|}\left\{\begin{matrix} [ua_1(\varepsilon_2 a_2 - \varepsilon_1 a_3)]T \\ +[-a_3(1 + u(\varepsilon_1 a_2 + \varepsilon_2 a_3) + ua_1^2\varepsilon_2)]M_1 \\ +[a_2(1 + u(\varepsilon_1 a_2 + \varepsilon_2 a_3) - ua_1^2\varepsilon_1)]M_2 \end{matrix}\right\}$$

We get

$$\eta'' \times N =$$

$$\left\{\begin{matrix} \left(ua_1^2(\varepsilon_{1+}^2\varepsilon_2^2) - \left(1 + u(\varepsilon_1 a_2 + \varepsilon_2 a_3)\right)(a_2\varepsilon_1 + a_3\varepsilon_2)\right)T \\ (ua_1(\varepsilon_2 a_2 - \varepsilon_1 a_3)\varepsilon_1)\varepsilon_2 M_1 \\ (ua_1(\varepsilon_2 a_2 - \varepsilon_1 a_3)\varepsilon_1)\varepsilon_1 M_2 \end{matrix}\right\} = 0.$$

Equations (a), (b), and (c) are therefore satisfied.

Theorem 3.2.23 The timelike ruled surface S is developable if its base curve is geodesic.

Proof 3.2.23 Let the base curve η of the surface *S* be a geodesic. From Equation (16), we get

$$a_1(a_2\varepsilon_2 - a_3\varepsilon_1) = 0.$$

From Equation (11) the timelike-ruled surface *S* is developable.

Corollary 3.2.24 Let A be the director vector of the timelike ruled surface S. If η is geodesic, then the striction line of η is also geodesic.

4. RESULTS

In this paper, we give some characterizations for timelike ruled surfaces generated by a straight line in Bishop frame in Minkowski space. In the case of

$$A = a_1 T(s) + a_2 M_1(s) + a_3 M_2(s)$$

as the direction vector of the surface and for special cases,

$$A = T(s)\ , A = M_1(s), A = M_2(s),$$

$$A\ \in Sp\ \{T, M_1\ \}, A\ \in Sp\ \{T, M_2\ \}, A\ \in Sp\ \{M_1, M_2\ \}$$

We calculate the distribution parameter and the mean curvature of the surfaces and analyze the developability and minimality for each surface obtained. In addition, we obtain some results on the developability of the surface when the base curve of the ruled surface is an asymptotic curve and a geodesic curve.

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A SHORT NOTE ON NON-SELF I NONEXPANSİVE MAP

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1. INTRODUCTION AND PRELIMINARIES

In nonlinear phenomena, one of the most profitable tools is the theory of fixed points, which holds a great variety of technics in many fields such as computer science, engineering, control theory, game theory, social sicences, economics and pure sciences (Ahmada et al., 2023), (Ahmed et al., 2021), (Zou et al., 2023), (Border, 1985), (Debnath et al., 2021), (Kreps, 1989), (Byrne, 2004).

Let $V \neq \emptyset$ be a convex subset of a Banach space Z, $T: V \to V$ is a map and $F_T(\kappa) = \{\kappa \in V: T\kappa = \kappa\}$. A map $T: V \to V$ is called to be nonexpansive if $\|T\varrho - T\tau\| \leq \|\varrho - \tau\|, \ \forall \varrho, \tau \in V$. Two maps $T: V \to V$ is called to be $I-$nonexpansive on V if $\|T\varrho - T\tau\| \leq \|I\varrho - I\tau\|, \ \forall \varrho, \tau \in V$.

The author established a general class of nonexpansive maps said $I-$nonexpansive maps and showed best approximation result for this class in Banach spaces (Shahzad, 2004). The authors proved the weak convergence of Mann iteration for $I-$nonexpansive maps Banach spaces via Opial's condition (Rhoades and Temir, 2006). The authors presented convergence result for non-self $I-$nonexpansive map (Chornphrom and Phonin, 2009). The authors discussed some convergence theorem

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for I −nonexpansive maps in abstract spaces (Gunduz and Akbulut, 2016).

A subset of V of Z is called to be a retract of Z if there is a continuous map $P: Z \to V$ such that $P\varrho = \varrho$ for $\forall \varrho \in V$. A map $P: Z \to Z$ is called to be a retraction if $P^2 = P$. So, if a map P is a retraction, then $P\tau = \tau$ for $\forall \tau$ in the range of P.

Inspired by these facts, a renewed type of iteration process is introduced in this writing. This process can be seen as a generalization for iterative schemes of Jubair et al. The iteration technique is described in the following manner.

Let Z be a normed space, $V \neq \emptyset$ be a convex subset of Z which is also a nonexpansive retract of Z. Let $T: V \to Z$ be given non-self nonexpansive map. Then for $\varrho_1 \in V$ and $n \geq 1$, reckon the sequences $\{\varrho_n\}$ defined by

$$\varrho_{n+1} = PT(g_n), \tag{1.1}$$
$$g_n = P\big((1-c_n)f_n + c_n T f_n\big)$$
$$f_n = P\big((1-a_n)\tau_n + a_n T \tau_n\big)$$
$$\tau_n = P\big((1-b_n)T\varrho_n + b_n T z_n\big)$$
$$z_n = P\big((1-d_n)\varrho_n + d_n I \varrho_n\big),$$

here $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} \subseteq [0,1]$.

If $c_n = 0$ and P be the identity map, then (1.1) reduces to the iterative given by

$$\varrho_{n+1} = T(f_n), \tag{1.2}$$
$$f_n = (1-a_n)\tau_n + a_n T \tau_n$$
$$\tau_n = (1-b_n)T\varrho_n + b_n T z_n$$
$$z_n = (1-d_n)\varrho_n + d_n I \varrho_n,$$

here $\{a_n\}, \{b_n\}, \{d_n\} \subseteq [0,1]$ (Jubair et al., 2022).

If $a_n = c_n = 0$ and P be the identity map, then (1.1) reduces to the S −iteration process defined by

$$\varrho_{n+1} = (1-b_n)T\varrho_n + b_nTz_n, \quad (1.3)$$
$$z_n = (1-d_n)\varrho_n + d_nT\varrho_n,$$

here $\{b_n\}, \{d_n\} \subseteq [0,1]$ (Agarwal et al., 2007).

If $a_n = b_n = c_n = 0$ and P be the identity map, then (1.1) reduces to the normal $S-$iteration process defined by

$$\varrho_{n+1} = T\big((1-d_n)\varrho_n + d_nT\varrho_n,\big), \quad (1.4)$$

here $\{d_n\} \subseteq [0,1]$ (Sahu, 2011).

If $a_n = b_n = c_n = d_n = 0$ and P be the identity map, then (1.1) reduces to the Picard iteration process defined by

$$\varrho_{n+1} = T(\varrho_n), \quad (1.5)$$

(Picard, 1890).

In this paper, convergence result is established for a new iteration for non-self $I-$nonexpansive map in Banach spaces. Based on this iteration process, a numerical example is provided to validate our results via Matlab Program Software.

2. MAIN RESULTS

Theorem 1. Let Z be a real Banach space, let $V \neq \emptyset$ be a closed convex subset of Z which is also a nonexpansive retract of Z. Let T be a $I-$nonexpansive self-map of V and I be nonexpansive map of V and $F_T = F(T) \cap F(I) \neq \emptyset$. Let $\{\varrho_n\}$ be given by (1.1), where $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} \subseteq [0,1]$. Then

(a_1) $lim_{n\to\infty}\|\varrho_n - \xi\|$ exists, $\forall\xi \in F_T$;

(a_2) $\{\varrho_n\} \to \xi \in F_T$ iff $liminf_{n\to\infty} d(\varrho_n, F_T) = 0$.

Proof. (a_1) For any $\xi \in F_T \neq \emptyset$, we obtain

$$\|\varrho_{n+1} - \xi\| = \|PT(g_n) - P\xi\| \quad (2.1)$$
$$= \|g_n - \xi\|$$

$$\begin{aligned}
&= \left\|P\left((1-c_n)f_n + c_n T f_n\right) - P\xi\right\| \\
&\leq \|f_n - \xi\| \\
&= \left\|P\left((1-a_n)\tau_n + a_n T\tau_n\right) - P\xi\right\| \\
&\leq \|\tau_n - \xi\| \\
&= \left\|P\left((1-b_n)T\varrho_n + b_n T z_n\right) - P\xi\right\| \\
&\leq (1-b_n)\|T\varrho_n - \xi\| + b_n\|Tz_n - \xi\| \\
&\leq (1-b_n)\|\varrho_n - \xi\| + b_n\|z_n - \xi\| \\
&\leq (1-b_n)\|\varrho_n - \xi\| \\
&+ b_n\left\|P\left((1-d_n)\varrho_n + d_n I\varrho_n\right) - P\xi\right\| \\
&\leq \|\varrho_n - \xi\|
\end{aligned}$$

which show that $\{\|\varrho_n - \xi\|\}$ is decreasing, and this proves part (a_1).

(a_2) The necessity is apparent. In fact, if $\{\varrho_n\} \to \xi \in F_T$ $(n \to \infty)$, then $d(\varrho_n, F_T) = inf_{\xi \in F_T} d(\varrho_n, \xi) \leq \|\varrho_n - \xi\| \to 0 (n \to \infty)$.

Next, we prove sufficiency. As $\{\|\varrho_n - \xi\|\}$ is bounded, (2.1) implies $inf_{\xi \in F_T}\|\varrho_{n+1} - \xi\| \leq inf_{\xi \in F_T}\|\varrho_n - \xi\|$, that is, $d(\varrho_{n+1}, F_T) \leq d(\varrho_n, F_T)$. Due to (a_1), we have $lim_{n\to\infty} d(\varrho_n, F_T)$ exist. But by hypothesis $liminf_{n\to\infty} d(\varrho_n, F_T) = 0$, hence we get $lim_{n\to\infty} d(\varrho_n, F_T) = 0$. We shall exhibit that $\{\varrho_n\}$ is a Cauchy sequences in V. Because $lim_{n\to\infty} d(\varrho_n, F_T) = 0$, for $\varepsilon > 0$, there exists n_0 in $\mathbb{N}$ such that for all $n \geq n_0$, $d(\varrho_n, F_T) < \frac{\varepsilon}{2}$. Particularly, $inf\left\{\left\|\varrho_{n_0} - \xi\right\| : \xi \in F_T\right\} < \frac{\varepsilon}{2}$. Therefore, there exists $\xi_* \in F_T$ such that $\left\|\varrho_{n_0} - \xi_*\right\| < \frac{\varepsilon}{2}$. Next, for $m, n \geq n_0$,

$$\|\varrho_{n+m} - \varrho_n\| \leq \|\varrho_{n+m} - \xi_*\| + \|\varrho_n - \xi_*\| < \varepsilon$$

which show that $\{\varrho_n\}$ is a Cauchy sequence in V. As V is closed in Z, there exists a point ξ in V such that $lim_{n\to\infty}\varrho_n = \xi$. Now $lim_{n\to\infty} d(\varrho_n, F_T) = 0$ gives that $d(\varrho_n, F_T) = 0$. Hence $\xi \in F_T$.

Remark 1. Since I-nonexpansive map reduces to nonexpansive map, Theorem 1 in this work extend the result of Theorem 21 of Jubair et al.

Inspired by Gunduz&Akbulut, we apply to testify Theorem 1 numerical example below (Gunduz&Akbulut, 2016).

Example 1. Let Z is the real line with the usual norm $|.|$, $V=[0,\infty)$ and P be the identity map. Suppose that $Tx=((1+2x)/4)$ and $Ix=-x+1$. Then it can be easily seen that T is a I −nonexpansive map and I is a nonexpansive map on V with common fixed point set $\{0.5000\}$. Set $a_n=\frac{1}{5+5n}$, $b_n=\frac{1}{6+6n}$, $c_n=\frac{1}{7+7n}$ and $d_n=\frac{1}{8+8n}$. Hence, all hypotheses of Theorem 1 are satisfied. Next, we indicate that $\varrho_n\to 0.5000$. For $n=0$ and $\varrho_0\in V$, we obtain $a_0=\frac{1}{5}$, $b_0=\frac{1}{6}$, $c_0=\frac{1}{7}$ and $d_0=\frac{1}{8}$ and calculate ϱ_1 from

$$\varrho_1=PT(g_0),$$
$$g_0=P\big((1-c_0)f_0+c_0Tf_0\big)$$
$$f_0=P\big((1-a_0)\tau_0+a_0T\tau_0\big)$$
$$\tau_0=P\big((1-b_0)T\varrho_0+b_0Tz_0\big)$$
$$z_0=P\big((1-d_0)\varrho_0+d_0I\varrho_0\big).$$

Similarly, $\varrho_2,\varrho_3,\cdots,\varrho_n,\cdots$. We get the first ten values of $\{\varrho_n\}$ as in following table for initial term $\varrho_0=0.1769$, $\varrho_0=0.6143$, $\varrho_0=0.9772$ and $\varrho_0=1.3588$, resp. One can see that $\varrho_n\to 0.5000$ from the table. This means that Theorem 1 is appliable.

n	$\varrho_0 = 0.1769$	$\varrho_0 = 0.6143$	$\varrho_0 = 0.9772$	$\varrho_0 = 1.3588$
1	0.4353	0.5229	0.5955	0.5851
2	0.4853	0.5052	0.5217	0.5194
3	0.4966	0.5012	0.5051	0.5046
4	0.4992	0.5003	0.5012	0.5011
5	0.4998	0.5001	0.5003	0.5003
6	0.5000	0.5000	0.5001	0.5001
7	0.5000	0.5000	0.5000	0.5000
8	0.5000	0.5000	0.5000	0.5000
9	0.5000	0.5000	0.5000	0.5000
10	0.5000	0.5000	0.5000	0.5000

3. CONCLUSION

Our results extend, generalize and unify various known results in the existing literature (Mann, 195), (Ishikawa, 1974), (Agarwal et al., 2007), (Sahu,2011), (Jubair et al., 2022). Within the future scope of the idea, reader may indicate that the new iteration defined by (1.1) present strong convergence theorems for G −nonexpansive map on Banach spaces with directed graph.

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