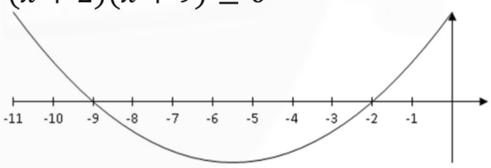
Question 1 **(25 marks)**

(a) Solve the simultaneous equations.

$$\begin{aligned} 2x + 3y - z &= -4 \\ 3x + 2y + 2z &= 14 \\ x \quad\quad - 3z &= -13 \end{aligned}$$

(b) Solve the inequality $\frac{2x-3}{x+2} \geq 3$, where $x \in \mathbb{R}$ and $x \neq -2$.

Q1	Model Solution – 25 Marks	Marking Notes
(a)	$(i)\quad 2x+3y-z=-4 \qquad \times(2)$ $(ii)\quad 3x+2y+2z=14 \qquad \times(-3)$ $4x+6y-2z=-8$ $-9x-6y-6z=-42$ $-5x-8z=-50$ $(iii)\quad x-3z=-13 \qquad \times(5)$ $-5x-8z=-50$ $5x-15z=-65$ $-23z=-115$ $z=5$ $\Rightarrow x=2$ $\Rightarrow y=-1 \qquad \{2,-1,5\}$	**Scale 15D (0, 5, 7, 11, 15)** *Low Partial Credit:* Matches coefficient of 1 variable in 2 equations Writes x in terms of z in eq (*iii*) *Mid Partial Credit:* 1 unknown found with errors Eliminates one unknown 1 unknown found and stops *High Partial Credit:* 2 unknowns found
(b)	$\dfrac{2x-3}{x+2} \geq 3 \qquad \times\ (x+2)^2$ $(2x-3)(x+2) \geq 3(x+2)^2$ $2x^2+x-6 \geq 3x^2+12x+12$ $x^2+11x+18 \leq 0$ $(x+2)(x+9) \leq 0$  $-9 \leq x < -2$	**Scale 10D (0, 3, 5, 8, 10)** *Low Partial Credit* Use of $(x+2)^2$ Relevant work but with linear inequality Squares both sides with some subsequent work (low partial credit at most) *Mid Partial Credit:* Quadratic inequality involving 0 *High Partial Credit* Roots of quadratic found **Note**: Accept $-9 \leq x \leq -2$

Question 2 **(25 marks)**

(a) The first three terms of a geometric series are x^2, $5x - 8$, and $x + 8$, where $x \in \mathbb{R}$.
Use the common ratio to show that $x^3 - 17x^2 + 80x - 64 = 0$.

(b) If $f(x) = x^3 - 17x^2 + 80x - 64$, $x \in \mathbb{R}$, show that $f(1) = 0$, **and** find another value of x for which $f(x) = 0$.

Show:

Other value:

(c) In the case of one of the values of x from part **(b)**, the terms in part **(a)** will generate a geometric series with a finite sum to infinity.
Find this value of x and **hence** find the sum to infinity.

$x =$

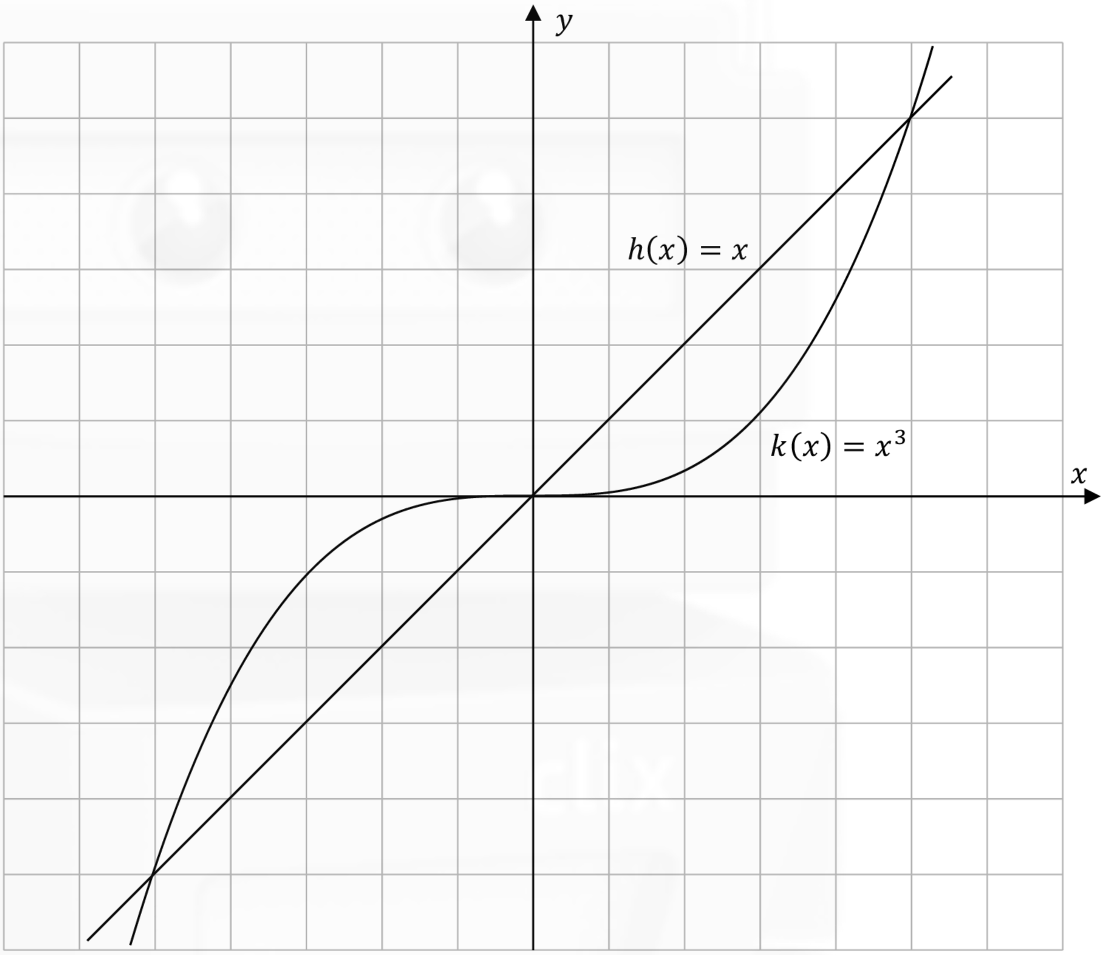
$S_\infty =$

Q2	Model Solution – 25 Marks	Marking Notes
(a)	$\frac{5x-8}{x^2}=\frac{x+8}{5x-8}$ $(5x-8)^2=x^2(x+8)$ $25x^2-80x+64=x^3+8x^2$ $x^3-17x^2+80x-64=0$	**Scale 10C (0, 4, 8, 10)** *Low Partial Credit:* $\frac{5x-8}{x^2}$ or $\frac{x+8}{5x-8}$ Some effort at finding r in a geometric sequence (must use at least one of the terms) $r=\frac{T_n}{T_{n-1}}$ or similar *High Partial Credit:* $\frac{5x-8}{x^2}=\frac{x+8}{5x-8}$ $(5x-8)^2$ and $x^2(x+8)$ *0 credit*: Treats as an arithmetic sequence
(b)	$f(x)=x^3-17x^2+80x-64$ $f(1)=(1)^3-17(1)^2+80(1)-64=0$ $\Rightarrow (x-1)$ is a factor $x^3-17x^2+80x-64=0$ $x^2(x-1)-16x(x-1)+64(x-1)$ $x^2-16x+64=0$ $(x-8)(x-8)=0$ $x=8$	**Scale 10C (0, 4, 8, 10)** *Low Partial Credit:* Shows $f(1)=0$ Any correct substitution *High Partial Credit:* Quotient in quadratic form found Accept $x=8$ without work if $f(1)=0$ has been shown

(c)		
	$\underline{x = 1}$ $1^2, \quad 5(1) - 8, \ \ 1 + 8$ $1, -3, \ 9$ which doesn't have a sum to infinity $(\lvert r \rvert > 1)$ $\underline{x = 8}$ $8^2, \quad 5(8) - 8, \quad 8 + 8$ $64, 32, 16 \ldots .$ $a = 64$ and $r = \frac{1}{2}$ $S_\infty = \frac{a}{1-r} = \frac{64}{1-\frac{1}{2}} = \frac{64}{\frac{1}{2}} = 128$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* Substitution used to identify $x = 8$ as the required value Substitution used to exclude $x = 1$ as the required value Finds $\frac{a}{1-r}$ for $x = 1$ $S_\infty = \frac{x^2}{1 - \frac{5x-8}{x^2}}$ Relevant substitution into correct formula *High Partial Credit:* GP identified (*a* and *r*) If the candidate works with both $x = 1$ **and** $x = 8$ but fails to eliminate $x = 1$ or chooses the incorrect answer **Note**: if $\lvert r \rvert > 1$ then Low Partial Credit at most

Question 6 **(25 marks)**

Parts of the graphs of the functions $h(x) = x$ and $k(x) = x^3$, $x \in \mathbb{R}$, are shown in the diagram below.



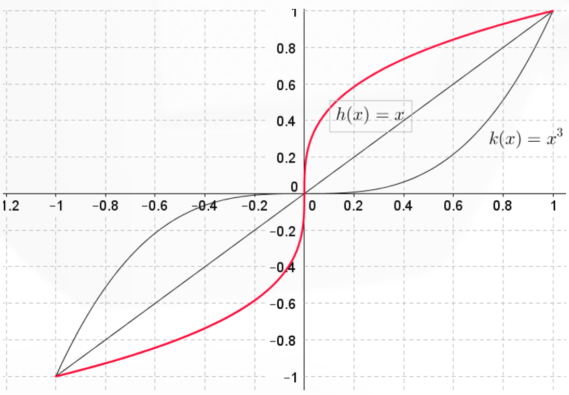
(a) Find the co-ordinates of the points of intersection of the graphs of the two functions.

(b) **(i)** Find the total area enclosed between the graphs of the two functions.

(ii) On the diagram on the previous page, using symmetry or otherwise, draw the graph of k^{-1}, the inverse function of k.

<table>
<tr><th>Q6</th><th>Model Solution – 25 Marks</th><th>Marking Notes</th></tr>
<tr><td>(a)</td><td>$$x^3 = x$$ $$\Rightarrow x^3 - x = 0$$ $$\Rightarrow x(x^2 - 1) = 0$$ $$x(x-1)(x+1) = 0$$ $$x = 0 \text{ or } x = \pm 1$$ (−1, −1), (0, 0), (1, 1)</td><td>Scale 10C (0, 4, 8, 10)
Low Partial Credit:
Equation written
One correct solution from the graph
Solution of the form (a, a) where $a \neq 0, 1$

High Partial Credit:
Equation factorised (3 factors)
2 correct points
x values only</td></tr>
<tr><td>(b)
(i)</td><td>$$2\int_0^1 x - x^3 dx$$ $$= 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right] = 2\left[\frac{1}{2} - \frac{1}{4} - 0\right] =$$ $$\frac{1}{2} \text{ unit}^2$$</td><td>Scale 10C (0, 4, 8, 10)
Low Partial Credit:
Integral indicated
One relevant area found

High Partial Credit:
Integral evaluated at $x = 1$ (upper limit)
$$\int_{-1}^{1} x - x^3 dx = 0$$</td></tr>
<tr><td>(b)
(ii)</td><td></td><td>Scale 5B (0, 2, 5)
Partial Credit:
Incomplete image
2 correct image points
$$k^{-1}(x) = x^{\frac{1}{3}}$$</td></tr>
</table>

(b) Find a, b, c, and d, if $\frac{(n+3)!\,(n+2)!}{(n+1)!(n+1)!} = an^3 + bn^2 + cn + d$, where a, b, c, and $d \in \mathbb{N}$.

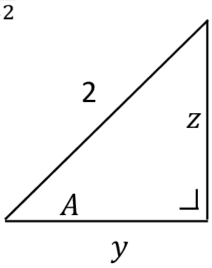
$a =$	$b =$	$c =$	$d =$

Marking Scheme

(b)		
	$\frac{(n+3)!\ (n+2)!}{(n+1)!\,(n+1)!} =$ $(n+3)(n+2)(n+2) =$ $n^3 + 7n^2 + 16n + 12$ **Or** $\frac{(n+3)!\,(n+2)!}{(n+1)!\,(n+1)!} = an^3 + bn^2 + cn + d$ $n = 0 \rightarrow \frac{3!\,.2!}{1!\,1!} = 12 = d$ $n = 1 \rightarrow a + b + c + d = 36$ $n = 2 \rightarrow 8a + 4b + 2c + d = 80$ $n = 3 \rightarrow 27a + 9b + 3c + d = 150$ Solving the simultaneous equations $a = 1, b = 7, c = 16, d = 12$	**Scale 5C (0, 2, 4, 5)** *Low Partial Credit:* Factorial expansion (e.g. $(n+3)! = (n+3)(n+2)(n+1) \ldots \ldots \ldots .1)$ Effort at a numerical value for *n* on both LHS and RHS (method 2) *High Partial Credit:* $(n+3)(n+2)(n+2)$ Four simultaneous equations

(b) Let $\cos A = \frac{y}{2}$, where $0° < A < 90°$. Write $\sin(2A)$ in terms of y.

Marking Scheme

(b)	$2^2 = y^2 + z^2$ $z = \sqrt{4 - y^2}$ $\sin 2A = 2\sin A\cos A$ $2\left(\frac{\sqrt{4-y^2}}{2}\right)\left(\frac{y}{2}\right)$ $= \frac{y\sqrt{4-y^2}}{2}$  **Or** $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$ $\frac{2\frac{\sqrt{4-y^2}}{y}}{1+\frac{4-y^2}{y^2}} = \frac{2y\sqrt{4-y^2}}{y^2+4-y^2} = \frac{y\sqrt{4-y^2}}{2}$	**Scale 5C (0, 2, 4, 5)** *Low Partial Credit:* $\sqrt{4-y^2}$ $2\sin A\cos A$ without substitution $\sin 2A$ expressed in $\tan A$ format Relevant labelled diagram $(2, y, A)$ *High Partial Credit:* Substitution for $\sin A$ or $\cos A$ in formula $\sin A = \left(\frac{\sqrt{4-y^2}}{2}\right)$ $\tan A = \frac{\sqrt{4-y^2}}{y}$

(a) Write the function $f(x) = 2x^2 - 7x - 10$, where $x \in \mathbb{R}$, in the form $a(x+h)^2 + k$, where a, h, and $k \in \mathbb{Q}$.

(b) Hence, write the minimum point of f.

(c) **(i)** Explain why f must have two real roots.

(ii) Write the roots of $f(x) = 0$ in the form $p \pm \sqrt{q}$, where p and $q \in \mathbb{Q}$.

page

Q1	Model Solution – 25 Marks	Marking Notes
(a)	$2\left(x^2-\frac{7}{2}x-5\right)$ $=2\left(\left(x-\frac{7}{4}\right)^2-\frac{129}{16}\right)$ $=2\left(\left(x-\frac{7}{4}\right)^2\right)-\frac{129}{8}$	**Scale 5D (0, 2, 3, 4, 5)** *Low Partial Credit:* • $a=2$ identified explicitly or as factor *Mid partial Credit:* • Completed square *High partial Credit:* • h or k identified from work
(b)	$\left(\frac{7}{4},\quad \frac{-129}{8}\right)$	**Scale 10B (0, 4, 10)** *Partial Credit:* • One relevant co-ordinate identified

(c) (i)	$f(x)$ has min point as $a > 0$ y co-ordinate of min < 0 ⇒graph must cut x-axis twice hence two real roots. **or** $$b^2 - 4ac = 49 + 80 > 0$$ Therefore real roots	**Scale 5B (0, 3, 5)** *Partial Credit:* • Mention of $a > 0$ • $b^2 - 4ac$ • Identifies location of one or two roots, e.g. between 4 and 5.
c (ii)	$$2x^2 - 7x - 10 = 0$$ $$2\left(\left(x - \frac{7}{4}\right)^2\right) - \frac{129}{8} = 0$$ $$\left(x - \frac{7}{4}\right)^2 = \frac{129}{16}$$ $$x - \frac{7}{4} = \pm\frac{\sqrt{129}}{4}$$ $$x = \frac{7}{4} \pm \sqrt{\frac{129}{16}}$$ **OR** $$2x^2 - 7x - 10 = 0$$ $$x = \frac{7 \pm \sqrt{49 + 80}}{4}$$ $$= \frac{7 \pm \sqrt{129}}{4}$$ $$x = \frac{7}{4} \pm \sqrt{\frac{129}{16}}$$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* • Formula with some substitution • Equation rewritten with some transpose *High Partial Credit:* • $x - \frac{7}{4} = \pm\frac{\sqrt{129}}{4}$ or equivalent

Question 5 **(25 marks)**

The function f is such that $f(x) = 2x^3 + 5x^2 - 4x - 3$, where $x \in \mathbb{R}$.

(a) Show that $x = -3$ is a root of $f(x)$ **and** find the other two roots.

(b) Find the co-ordinates of the local maximum point **and** the local minimum point of the function f.

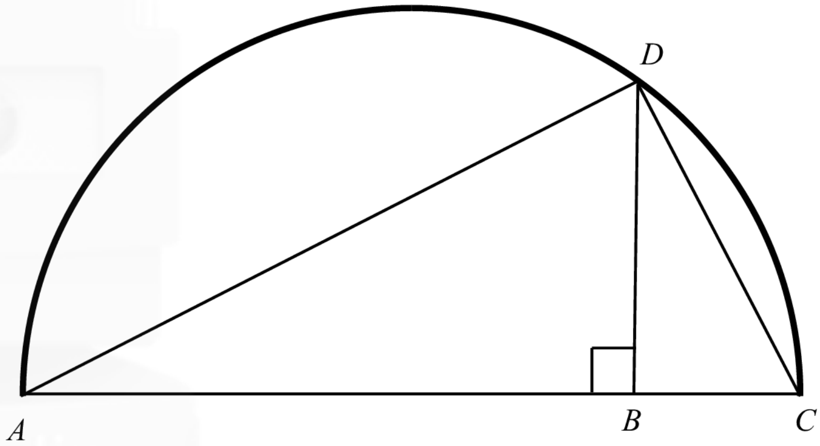
Local maximum point: Local minimum point:

(c) $f(x) + a$, where a is a constant, has only one real root.
Find the range of possible values of a.

(a)	$f(x) = 2x^3 + 5x^2 - 4x - 3$ $f(-3) = 2(-3)^3 + 5(-3)^2 - 4(-3) - 3$ $= -54 + 45 + 12 - 3$ $f(-3) = 0$ $\Rightarrow (x+3)$ is a factor $x+3 \overline{)2x^3 + 5x^2 - 4x - 3}$ quotient $2x^2 - x - 1$ $\underline{2x^3 + 6x^2}$ $-x^2 - 4x$ $\underline{-x^2 - 3x}$ $-x - 3$ $\underline{-x - 3}$ $f(x) = (x+3)(2x^2 - x - 1)$ $f(x) = (x+3)(2x+1)(x-1)$ $x = -3 \quad x = -\frac{1}{2} \quad x = 1$	**Scale 15C (0, 5, 10, 15)** *Low Partial Credit:* • Shows $f(-3) = 0$ *High Partial Credit:* • quadratic factor of $f(x)$ found **Note:** No remainder in division may be stated as reason for $x = -3$ as root
(b)	$y = 2x^3 + 5x^2 - 4x - 3$ $\frac{dy}{dx} = 6x^2 + 10x - 4 = 0$ $3x^2 + 5x - 2 = 0$ $(x+2)(3x-1) = 0$ $3x - 1 = 0 \quad x + 2 = 0$ $x = \frac{1}{3} \quad x = -2$ $f\left(\frac{1}{3}\right) = \frac{-100}{27} \quad f(-2) = 9$ $Max = (-2, 9) \quad Min = \left(\frac{1}{3}, \frac{-100}{27}\right)$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* • $\frac{dy}{dx}$ found (Some correct differentiation) *High Partial Credit* • roots and one y value found **Note:** One of Max/Min must be identified for full credit
(c)	$a > \frac{100}{27}$ or $a < -9$	**Scale 5B (0, 3, 5)** *Partial Credit:* • one value identified • no range identified (from 2 values)

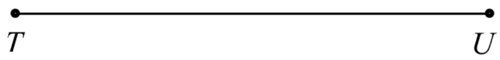
The diagram shows a semi-circle standing on a diameter $[AC]$, and $[BD] \perp [AC]$.

(a) **(i)** Prove that the triangles ABD and DBC are similar.

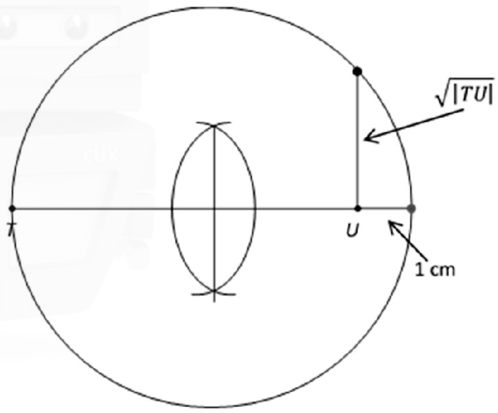


(ii) If $|AB| = x$, $|BC| = 1$, and $|BD| = y$, write y in terms of x.

(b) Use your result from part **(a)(ii)** to **construct** a line segment equal in length (in centimetres) to the square root of the length of the line segment $[TU]$ which is drawn below.



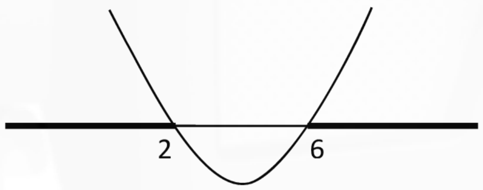
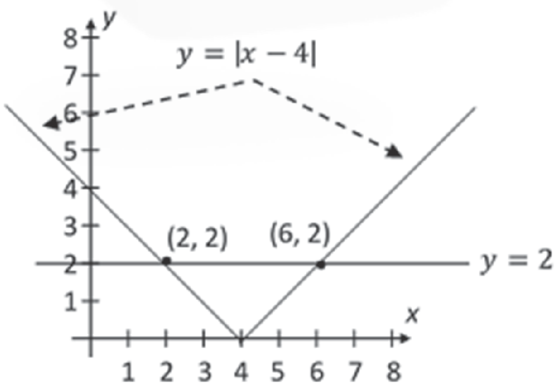
Q4	Model Solution – 25 Marks	Marking Notes
(a) **(i)**	$\lvert\angle ABD\rvert = \lvert\angle CBD\rvert = 90°$.......(i) $\lvert\angle BDC\rvert + \lvert\angle BCD\rvert = 90°$ *...angles in triangle sum to* $180°$ $\lvert\angle ADB\rvert + \lvert\angle BDC\rvert = 90°$ .... *angle in semicircle* $\lvert\angle ADB\rvert + \lvert\angle BDC\rvert = \lvert\angle BDC\rvert + \lvert\angle BCD\rvert$ $\lvert\angle ADB\rvert = \lvert\angle BCD\rvert$........(ii) $\therefore$ *Triangles are equiangular (or similar)* **or** $\lvert\angle ABD\rvert = \lvert\angle CBD\rvert = 90°$.......(i) $\lvert\angle DAB\rvert = \lvert\angle DAC\rvert$ *same angle* $\Rightarrow \lvert\angle ADB\rvert = \lvert\angle DCA\rvert$ *(reasons as above) which is also* $\angle DCB$.........(ii)	Scale 15C (0, 5, 10, 15) *Low Partial Credit* • identifies one angle of same size in each triangle *High Partial Credit* • identifies second angle of same size in each triangle • implies triangles are similar without justifying (ii) in model solution or equivalent
(a) **(ii)**	$\frac{y}{1} = \frac{x}{y}$ $\Rightarrow y^2 = x$ $y = \sqrt{x}$ **or** $\lvert AD\rvert^2 + \lvert DC\rvert^2 = \lvert AC\rvert^2$ $\lvert AD\rvert = \sqrt{x^2 + y^2}$ $\lvert DC\rvert = \sqrt{y^2 + 1}$ $x^2 + y^2 + y^2 + 1 = (x + 1)^2$ $2y^2 = 2x$ $y = \sqrt{x}$ **Or** $\frac{\sqrt{x^2 + y^2}}{\sqrt{y^2 + 1}} = \frac{y}{1} \Rightarrow x^2 + y^2 = y^2(y^2 + 1)$ $y^4 = x^2 \Rightarrow y^2 = x \Rightarrow y = \sqrt{x}$	Scale 5C (0, 2, 4, 5) *Low Partial Credit* • one set of corresponding sides identified • indicates relevant use of Pythagoras *High Partial Credit* • corresponding sides fully substituted • expression in y^2 or y^4, i.e. fails to finish

(b)	Construction 	Scale 5C (0, 2, 4, 5) *Low Partial Credit* • perpendicular line drawn at *U* or *T* • relevant use of 1 cm length • mid point of incorrect extended segment constructed *High Partial Credit* • correct mid-point constructed

(a) Find the range of values of x for which $|x - 4| \geq 2$, where $x \in \mathbb{R}$.

(b) Solve the simultaneous equations:

$$x^2 + xy + 2y^2 = 4$$
$$2x + 3y = -1.$$

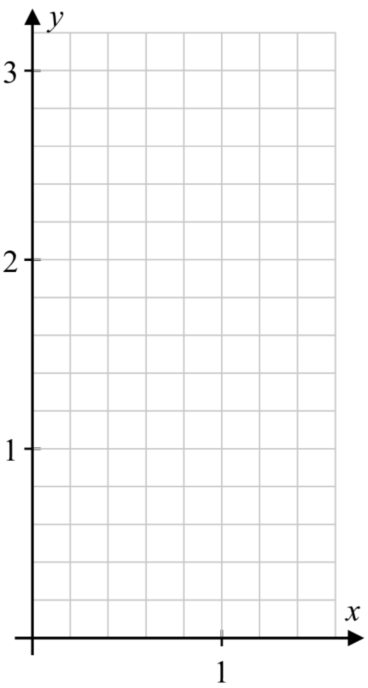
Q2	Model Solution – 25 Marks	Marking Notes
(a)	$x^2 - 8x + 16 \geq 4$ $x^2 - 8x + 12 \geq 0$ $(x-2)(x-6) \geq 0$ $x = 2 \quad x = 6$ $\{x \mid x \leq 2\} \cup \{x \mid x \geq 6\}$ **Or** $x - 4 \geq 2 \cup x - 4 \leq -2$ $x \geq 6 \cup x \leq 2$ **Or** Graphical method (must indicate range on X-axis somehow)  **Or**  $x \leq 2 \cup x \geq 6$	Scale 10C (0, 3, 7, 10) *Low Partial Credit:* • either side squared • one correct linear inequality written • stating range of natural numbers only *High Partial Credit:* • correct solutions to quadratic *Full Credit:* • correct answer without work **Note:** *use of natural numbers in range merits High Partial Credit at most* **Or** Scale 10C (0, 3, 7, 10) *Low Partial Credit:* • any one straight line *High Partial Credit:* • three straight lines

(b)		
	$$x = \frac{-3y-1}{2}$$ $$\left(\frac{-3y-1}{2}\right)^2 + \left(\frac{-3y-1}{2}\right)(y) + 2y^2 = 4$$ $$11y^2 + 4y - 15 = 0$$ $$(11y+15)(y-1) = 0$$ $$y = \frac{-15}{11} \text{ or } y = 1$$ $$x = \frac{-3\left(\frac{-15}{11}\right) - 1}{2} \text{ or } x = \frac{-3(1)-1}{2}$$ $$x = \frac{17}{11} \text{ or } x = -2$$ **or** $$y = \frac{-2x-1}{3}$$ $$x^2 + x\left(\frac{-2x-1}{3}\right) + 2\left(\frac{-2x-1}{3}\right)^2 = 4$$ $$11x^2 + 5x - 34 = 0$$ $$(11x-17)(x+2) = 0$$ $$x = \frac{17}{11} \text{ or } x = -2$$ $$y = \frac{-15}{11} \text{ or } y = 1$$	Scale 15C (0, 5, 10,15) *Low Partial Credit:* • effort to isolate x (or y) *High Partial Credit:* • fully correct substitution into quadratic

(a) **(i)** $f(x) = \frac{2}{e^x}$ and $g(x) = e^x - 1$, where $x \in \mathbb{R}$.
Complete the table below. Write your values correct to two decimal places where necessary.

x	**0**	**0·5**	**1**	**ln(4)**
$f(x) = \frac{2}{e^x}$				
$g(x) = e^x - 1$				

(ii) In the grid on the right, use the table to draw the graphs of $f(x)$ and $g(x)$ in the domain $0 \leq x \leq \ln(4)$. Label each graph clearly.



(iii) Use your graphs to estimate the value of x for which $f(x) = g(x)$.

(b) Solve $f(x) = g(x)$ using algebra.

Previous	Page	Running

Q3	Model Solution – 25 Marks	Marking Notes

(a) (i)

x	0	0·5	1	$\ln(4)$
$f(x)=\dfrac{2}{e^x}$	2	1·21	0·74	0·5
$g(x)=e^x-1$	0	0·65	1·72	3

Scale 5C (0, 2, 4, 5)

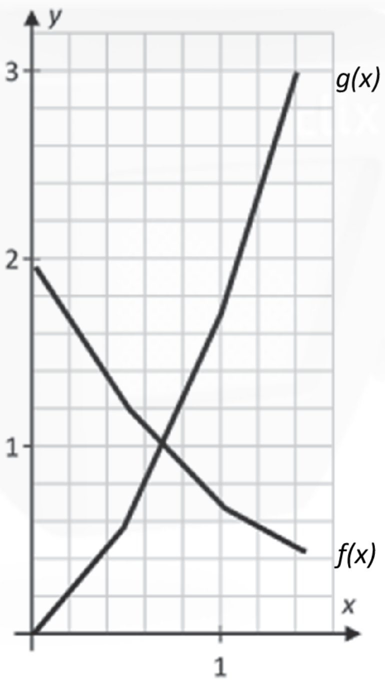
Low Partial Credit

- one entry correct

High Partial Credit

- 5 entries correct

(ii)



Scale 5C (0, 2, 4, 5)

Low Partial Credit

- one plot correct

High Partial Credit

- 5 plots correct
- one correct graph
- no labelling

Notes:

- *straight lines NOT acceptable*
- *one clear label merits full credit*
- *one ambiguous label merits High Partial Credit at most*

(iii)

$f(x)=g(x)$ when $x\approx 0\cdot 7$

Scale 5B (0, 2, 5)

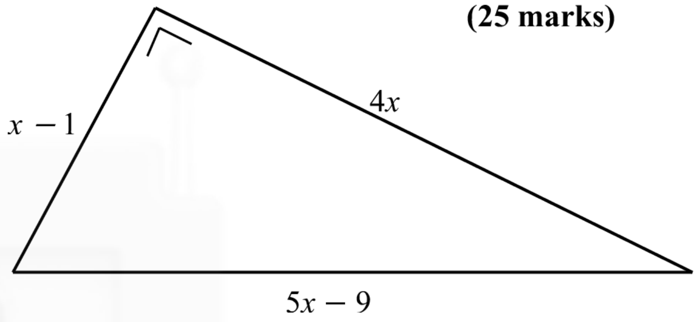
Partial Credit

- point of intersection clearly indicated on graph, but value of x not stated

Q3	Model Solution – Continued	Marking Notes
(b)	$\frac{e^x - 1}{1} = \frac{2}{e^x}$ $e^{2x} - e^x = 2$ $(e^x)^2 - e^x - 2 = 0$ $(e^x - 2)(e^x + 1) = 0$ $e^x = 2$ **or** $e^x = -1$ $x = \ln 2$ or $x = 0{\cdot}693$ **Or** $(e^x)^2 - e^x - 2 = 0$ Let $y = e^x \Rightarrow y^2 - y - 2 = 0$ $y = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}$ $= \frac{1 \pm \sqrt{1+8}}{2}$ $= \frac{1 \pm 3}{2}$ $\Rightarrow y = 2$ or $y = -1$ (not possible) $y = e^x \Rightarrow e^x = 2$ $x = \ln 2$ or $x = 0{\cdot}693$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • substitution correct *High Partial Credit* • correct factors of quadratic • root formula correctly substituted $e^x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}$ **Note:** *oversimplification of equation (i.e. not treating as quadratic) merits Low Partial Credit at most* **Or** Scale 10C (0, 3, 7, 10) *Low Partial Credit* • substitution correct *High Partial Credit* • root formula correctly substituted $y = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}$ **Note:** *oversimplification of equation (i.e. not treating as quadratic) merits Low Partial Credit at most*

Question 5 **(25 marks)**

(a) **(i)** The lengths of the sides of a right-angled triangle are given by the expressions $x - 1$, $4x$, and $5x - 9$, as shown in the diagram. Find the value of x.



(ii) Verify, with this value of x, that the lengths of the sides of the triangle above form a pythagorean triple.

Q5	Model Solution – 25 Marks	Marking Notes
(a) **(i)**	$(5x-9)^2=(x-1)^2+(4x)^2$ $8x^2-88x+80=0$ $x^2-11x+10=0$ $(x-1)(x-10)=0$ $x=1$ or $x=10$ $x=10$	Scale 10D (0, 2, 5, 8, 10) *Low Partial Credit* • any use of Pythagoras *Mid Partial Credit* • fully correct substitution *High Partial Credit* • both roots correct
(a) **(ii)**	Sides=9, 40, 41 $9^2+40^2=41^2$ $81+1600=1681$ $1681=1681$	Scale 5B (0, 2, 5) *Partial Credit* • 9 or 40 or 41 • using 1 or −10 from candidates work

(b) The heptathlon is an Olympic competition. It consists of seven events including the 200 m race and the javelin. The scoring system uses formulas to calculate a score for each event. The table below shows the formulas for two of the events and the values of constants used in these formulas, where x is the time taken (in seconds) or distance achieved (in metres) by the competitor and y is the number of points scored in the event.

Event	x	**Formula**	a	b	c
200 m race	Time (s)	$y = a(b - x)^c$	4·99087	42·5	1·81
Javelin	Distance (m)	$y = a(x - b)^c$	15·9803	3·8	1·04

(i) In the heptathlon, Jessica ran 200 m in 23·8 s and threw the javelin 58·2 m. Use the formulas in the table to find the number of points she scored in each of these events, correct to the nearest point.

200 m	Javelin

(ii) The world record distance for the javelin, in the heptathlon, would merit a score of 1295 points. Find the world record distance for the javelin, in the heptathlon, correct to two decimal places.

(iii) The formula used to calculate the points for the 800 m race, in the heptathlon, is the same formula used for the 200 m race but with different constants.
Jessica ran the 800 m race in 2 minutes and 1·84 seconds which merited 1087 points.
If $a = 0{\cdot}11193$ and $b = 254$ for the 800 m race, find the value of c for this event, correct to two decimal places.

(b) **(i)**	200 m Race: $y = a(b-x)^c$ $y = 4{\cdot}99087(42{\cdot}5 - 23{\cdot}8)^{1{\cdot}81}$ $y = 1000$ Javelin: $y = a(x-b)^c$ $y = 15{\cdot}9803(58{\cdot}2 - 3{\cdot}8)^{1{\cdot}04}$ $y = 1020$	Scale 10D (0, 2, 5, 8, 10) *Low Partial Credit* • some relevant substitution into one formula *Mid Partial Credit* • one value of y found • some relevant substitution into both formulas *High Partial Credit* • one value correct and some relevant substitution into second formula • uses incorrect formula (once only)
(ii)	$y = a(x-b)^c$ $1295 = 15{\cdot}9803(x - 3{\cdot}8)^{1{\cdot}04}$ $81{\cdot}0373 = (x-3{\cdot}8)^{1{\cdot}04} = z^{1{\cdot}04}$ $\log z = \frac{\log 81{\cdot}0373}{1{\cdot}04}$ $z = 68{\cdot}4343 = (x - 3{\cdot}8)$ $x = 72{\cdot}2343 = 72{\cdot}23 \text{ m}$	Scale 5B (0, 2, 5) *Partial Credit* • some relevant substitution into formula
(iii)	$y = a(b-x)^c$ $1087 = 0{\cdot}11193(254 - 121{\cdot}84)^c$ $\frac{1087}{0{\cdot}11193} = (132{\cdot}16)^c$ $\log 9711{\cdot}426 = c \log 132{\cdot}16$ $c = \frac{\log 9711{\cdot}426}{\log 132{\cdot}16} = 1{\cdot}88$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • some relevant substitution into formula *High Partial Credit* • fully correct substitution into formula

(b) A male bee comes from an unfertilised egg, i.e. he has a female parent but he does not have a male parent. A female bee comes from a fertilised egg, i.e. she has a female parent and a male parent.

(i) The following diagram shows the ancestors of a certain male bee. We identify his generation as G_1 and our diagram goes back to G_4. Continue the diagram to G_5.

G_1	G_2	G_3	G_4	G_5
			↗ Female	
		↗ Female		
Male →	Female		↘ Male	
		↘ Male →	Female	

(ii) The number of ancestors of this bee in each generation can be calculated by the formula

$$G_{n+2} = G_{n+1} + G_n,$$

where $G_1 = 1$ and $G_2 = 1$, as in the diagram.
Use this formula to calculate the number of ancestors in G_6 and in G_7.

G_6 G_7

(iii) The number of ancestors in each generation can also be calculated by using the formula

$$G_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

Use this formula to verify the number of ancestors in G_3.

(b) **(i)**	G_5 =Female,Male,Female,Female,Male	Scale 5B (0, 2, 5) *Partial Credit* • one correct entry
(b) **(ii)**	$G_6 = G_5 + G_4 = 5 + 3 = 8$ $G_7 = G_6 + G_5 = 8 + 5 = 13$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • $G_6 = G_5 + G_4$ • $G_7 = G_6 + G_5$ • G_7 or G_6 correct • 8 and/or 13 without work *High Partial Credit* • correct substitution in both
(b) **(iii)**	$G_3 = \frac{(1+\sqrt{5})^3 - (1-\sqrt{5})^3}{2^3\sqrt{5}} = 2$ $(1+\sqrt{5})^3 = \left(1 + 3\sqrt{5} + 3\sqrt{5}^2 + \sqrt{5}^3\right)$ $= 16 + 8\sqrt{5}$ $(1-\sqrt{5})^3 = \left(1 - 3\sqrt{5} + 3\sqrt{5}^2 - \sqrt{5}^3\right)$ $= 16 - 8\sqrt{5}$ $G_3 = \frac{6\sqrt{5} + 2\sqrt{5}^3}{8\sqrt{5}}$ $= \frac{6 + 2\sqrt{5}^2}{8} = \frac{16}{8} = 2 \quad \text{Q.E.D.}$	Scale 5B (0, 2, 5) *Partial Credit* • some correct substitution • using approximate value for $\sqrt{5}$ • $G_3 = 2$ • some effort at cubing **Note:** *use of* $\sqrt{5}$ *as approximation, even if rounded off to 2 at end of work merits at most Partial Credit*

Question 2 **(25 marks)**

Solve the equation $x^3 - 3x^2 - 9x + 11 = 0$.
Write any irrational solution in the form $a + b\sqrt{c}$, where $a, b, c \in \mathbb{Z}$.

Marking Scheme

Question 2 **(25 marks)**

$f(x) = x^3 - 3x^2 - 9x + 11$

$f(1) = 1^3 - 3(1)^2 - 9 + 11 = 0$

$\Rightarrow x = 1$ is a solution.

$(x - 1)$ is a factor

$$
\begin{array}{r|l}
 & x^2 - 2x \quad - 11 \\
\hline
x - 1 & x^3 - 3x^2 - 9x + 11 \\
 & x^3 - x^2 \\
\cline{2-2}
 & \quad -2x^2 - 9x + 11 \\
 & \quad -2x^2 + 2x \\
\cline{2-2}
 & \qquad\quad -11x + 11 \\
 & \qquad\quad -11x + 11 \\
\cline{2-2}
\end{array}
$$

or

$(x-1)(x^2 + Ax - 11) = x^3 - 3x^2 - 9x + 11$

$\Rightarrow x^3 + Ax^2 - x - x^2 - Ax + 1 = x^3 - 3x^2 - 9x + 11$

$\Rightarrow A - 1 = -3$

$\Rightarrow A = -2$

or

	x^2	$-2x$	-11
x	x^3	$-2x^2$	$-11x$
-1	$-x^2$	$2x$	11

Hence, other factor is $x^2 - 2x - 11$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-11)}}{2(1)} = \frac{2 \pm \sqrt{48}}{2} = \frac{2 \pm 4\sqrt{3}}{2} = 1 \pm 2\sqrt{3}$$

Solutions: $\{1,\ 1 + 2\sqrt{3},\ 1 - 2\sqrt{3}\}$

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(a) Solve the equation $x = \sqrt{x + 6},\ x \in \mathbb{R}$.

$$x = \sqrt{x+6}$$
$$\Rightarrow x^2 = x + 6$$
$$\Rightarrow x^2 - x - 6 = 0$$
$$\Rightarrow (x+2)(x-3) = 0$$
$$\Rightarrow x = -2, \quad x = 3$$
$$x = -2: \quad -2 \neq \sqrt{-2+6} = \sqrt{4} = 2 \ \text{✗}$$
$$x = 3: \quad 3 = \sqrt{3+6} = \sqrt{9} = 3 \ \checkmark$$

Question 3 **(25 marks)**

A cubic function f is defined for $x \in \mathbb{R}$ as

$$f : x \mapsto x^3 + (1-k^2)x + k, \quad \text{where } k \text{ is a constant.}$$

(a) Show that $-k$ is a root of f.

(b) Find, in terms of k, the other two roots of f.

(c) Find the set of values of k for which f has exactly one real root.

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(a) Show that $-k$ is a root of f.

Substituting $-k$ for x we obtain

$$\begin{aligned} f(-k) &= (-k)^3 + (1-k^2)(-k) + k \\ &= -k^3 - k + k^3 + k \\ &= 0 \end{aligned}$$

Therefore $-k$ is a root of f.

(b) Find, in terms of k, the other two roots of f.

Since $-k$ is a root of f we know, by the Factor Theorem, that $(x+k)$ is a factor of $f(x)$. Now we carry out long division to find the other factor.

$$
\begin{array}{r|l}
 & x^2 - kx + 1 \\
\hline
x+k & x^3 + 0x^2 + (1-k^2)x + k \\
 & x^3 + kx^2 \\
\hline
 & -kx^2 + (1-k^2)x \\
 & -kx^2 - k^2x \\
\hline
 & x + k \\
 & x + k \\
\hline
 & 0
\end{array}
$$

So

$$x^3+(1-k^2)x+k=(x+k)(x^2-kx+1).$$

Therefore the other two roots of f are solutions of the equation

$$x^2-kx+1=0.$$

Using the quadratic formula we get

$$x=\frac{k\pm\sqrt{k^2-4}}{2}.$$

So the other two roots of f are

$$\frac{k+\sqrt{k^2-4}}{2} \text{ and } \frac{k-\sqrt{k^2-4}}{2}.$$

(c) Find the set of values of k for which f has exactly one real root.

From the solution to part (b), we see that f has exactly one real root if and only if $k^2-4<0$. This is equivalent to $k^2<4$ or

$$-2<k<2$$

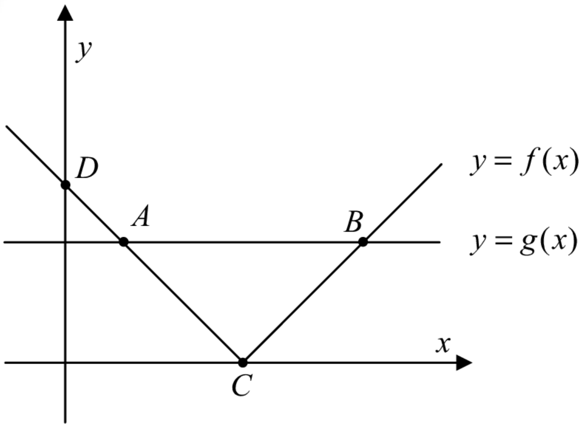
Question 4 **(25 marks)**

(a) Solve the simultaneous equations:

$$2x+8y-3z=-1$$
$$2x-3y+2z=2$$
$$2x+y+z=5.$$

(b) The graphs of the functions $f:x\mapsto|x-3|$ and $g:x\mapsto 2$ are shown in the diagram.

(i) Find the co-ordinates of the points A, B, C and D.



$A=(\quad,\quad)$ $B=(\quad,\quad)$

$C=(\quad,\quad)$ $D=(\quad,\quad)$

(ii) Hence, or otherwise, solve the inequality $|x-3|<2$.

We can subtract the second equation from the first:

$$\begin{array}{rcl} 2x+8y-3z &=& -1 \\ 2x-3y+2z &=& 2 \\ \hline 11y-5z &=& -3 \end{array}$$

Similarly, we subtract the third equation from the first:

$$\begin{array}{rcl} 2x+8y-3z &=& -1 \\ 2x+y+z &=& 5 \\ \hline 7y-4z &=& -6 \end{array}$$

Now we solve the simultaneous equations

$$\begin{array}{rcl} 11y-5z &=& -3 \\ 7y-4z &=& -6 \end{array}$$

Multiply the first by 7, the second by 11 and subtract:

$$\begin{array}{rcl} 77y-35z &=& -21 \\ 77y-44z &=& -66 \\ \hline 9z &=& 45 \end{array}$$

Therefore $z = \frac{45}{9} = 5$. Now substitute $z = 5$ into $7y - 4z = -6$ to get $7y - 4(5) = -6$ or $7y = -6 + 20 = 14$ Therefore $y = 2$.
Finally substitute $y = 2$ and $z = 5$ into $2x + 8y - 3z = -1$ to get $2x + 8(2) - 3(5) = -1$ or $2x = -1 - 8(2) + 3(5) = -2$ So $x = -1$.
So the solution is

$$x = -1, y = 2, z = 5.$$

Now we can check this by substituting into the original equations and verifying that they are all true:

$$\begin{array}{rcl} 2(-1)+8(2)-3(5) &=& -1 \\ 2(-1)-3(2)+2(5) &=& 2 \\ 2(-1)+(2)+(5) &=& 5. \end{array}$$

(b) The graphs of the functions $f : x \mapsto |x-3|$ and $g : x \mapsto 2$ are shown in the diagram.

(i) Find the co-ordinates of the points A, B, C and D.

D is on the y-axis, so its x-co-ordinate is 0. Now $f(0) = |0-3| = |-3| = 3$. So $D = (0,3)$.
$C = (,0)$ (on the x-axis), so we solve $|x-3| = 0$ to find the x-co-ordinate. Now $|x-3| = 0 \Leftrightarrow x-3 = 0 \Leftrightarrow x = 3$. So $C = (3,0)$.
A and B both have y-co-ordinate 2, so we solve $|x-3| = 2$. Now $|x-3| = 2 \Leftrightarrow \pm(x-3) = 2$. So either

$$(x-3) = 2 \text{ or } -(x-3) = 2.$$

In the first case $x = 5$ and in the second case $-x+3 = 2$ or $x = 1$. So $A = (1,2)$ and $B = (5,2)$.

$$A = (1,2) \quad B = (5,2)$$
$$C = (3,0) \quad D = (0,3)$$

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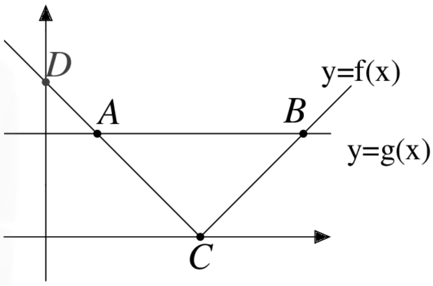


(ii) Hence, or otherwise, solve the inequality $|x-3| < 2$.

The solution set of the inequality corresponds to the values of x for which the graph of f is below the graph of g. From the diagram and calculations above, we see that the solution set is

$$1 < x < 5.$$

MODEL ANSWER BY thematsthutor.ie ONLINE SUPPORT SYSTEM FOR PROJECT MATHS

(a) Find the set of all real values of x for which $2x^2 + x - 15 \geq 0$.

(b) Solve the simultaneous equations;

$$x + y + z = 16$$

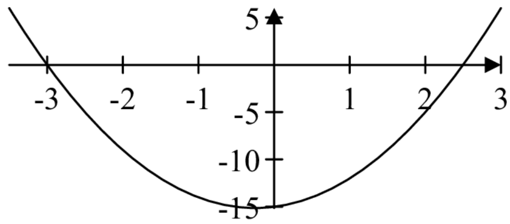
$$\frac{5}{2}x + y + 10z = 40$$

$$2x + \frac{1}{2}y + 4z = 21.$$

$2x^2 + x - 15 = 0$

$\Rightarrow (2x-5)(x+3) = 0 \Rightarrow x = 2\frac{1}{2}$ or $x = -3$

$2x^2 + x - 15 \geq 0$ for $\{x \mid x \leq -3\} \cup \{x \mid x \geq 2\frac{1}{2}\}$



OR

$f(x) = 2x^2 + x - 15 = (2x-5)(x+3)$

$(2x-5)(x+3) = 0$

$\Rightarrow x = \dfrac{5}{2}$ or $x = -3$

(i): $x \geq -3$ and $x \geq \dfrac{5}{2} \Rightarrow x \geq \dfrac{5}{2}$

(ii): $x \leq -3$ and $x \leq \dfrac{5}{2} \Rightarrow x \leq -3$

Solution Set: $\{x \mid x \leq -3\} \cup \{x \mid x \geq \frac{5}{2}\}$

$$\begin{aligned} x + y + z &= 16 \\ \tfrac{5}{2}x + y + 10z &= 40 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2x + 2y + 2z &= 32 \\ 5x + 2y + 20z &= 80 \\ \hline 3x \qquad + 18z &= 48 \end{aligned}$$

$$\begin{aligned} x + y + z &= 16 \\ 4x + y + 8z &= 42 \\ \hline 3x \qquad + 7z &= 26 \end{aligned}$$

$$\begin{aligned} 3x + 18z &= 48 \\ 3x + 7z &= 26 \\ \hline 11z &= 22 \quad \Rightarrow \quad z = 2 \end{aligned}$$

$3x + 7z = 26 \Rightarrow 3x + 7(2) = 26 \Rightarrow 3x = 12 \Rightarrow x = 4$

$x + y + z = 16 \Rightarrow 4 + y + 2 = 16 \Rightarrow y = 10$

A stadium can hold 25 000 people. People attending a regular event at the stadium must purchase a ticket in advance. When the ticket price is €20, the expected attendance at an event is 12 000 people. The results of a survey carried out by the owners suggest that for every €1 reduction, from €20, in the ticket price, the expected attendance would increase by 1000 people.

(a) If the ticket price was €18, how many people would be expected to attend?

(b) Let x be the ticket price, where $x \leq 20$. Write down, in terms of x, the expected attendance at such an event.

(c) Write down a function f that gives the expected income from the sale of tickets for such an event.

(d) Find the price at which tickets should be sold to give the maximum expected income.

(e) Find this maximum expected income.

(f) Suppose that tickets are instead priced at a value that is expected to give a full attendance at the stadium. Find the difference between the income from the sale of tickets at this price and the maximum income calculated at **(e)** above.

(g) The stadium was full for a recent special event. Two types of tickets were sold, a single ticket for €16 and a family ticket (2 adults and 2 children) for a certain amount. The income from this event was €365 000. If 1000 more family tickets had been sold, the income from the event would have been reduced by €14 000. How many family tickets were sold?

(a) If the ticket price was €18, how many people would be expected to attend?

$$12000 + (20 - 18)1000 = 14000$$

(b) Let x be the ticket price, where $x \le 20$. Write down, in terms of x, the expected attendance at such an event.

$$12000 + (20 - x)1000 = 32000 - 1000x$$

(c) Write down a function f that gives the expected income from the sale of tickets for such an event.

$$f(x) = (32000 - 1000x)x$$

(d) Find the price at which tickets should be sold to give the maximum expected income.

$$f(x) = (32000 - 1000x)x$$

$$f'(x) = 32000 - 2000x = 0 \Rightarrow x = €16$$

(e) Find this maximum expected income.

$$f(x) = (32000 - 1000x)x$$

$$f(16) = (32000 - 16000)16 = €256\,000$$

(f) Suppose that tickets are instead priced at a value that is expected to give a full attendance at the stadium. Find the difference between the income from the sale of tickets at this price and the maximum income calculated at **(e)** above.

$$32000 - 1000x = 25000 \Rightarrow 1000x = 7000 \Rightarrow x = 7$$

$$f(x) = (32000 - 1000x)x \Rightarrow f(7) = (32000 - 7000)7 = 175\,000$$

Difference: €256 000 – €175 000 = €81 000

(g) The stadium was full for a recent special event. Two types of tickets were sold; a single ticket for €16 and a family ticket (2 adults and 2 children) for a certain amount. The income from this event was €365 000. If 1000 more family tickets had been sold the income from the event would have been reduces by €14 000. How many family tickets were sold?

Single ticket: €16; Family ticket €y
Number of single tickets: p; Number of family tickets: $\frac{25000-p}{4}$

$$16p + \frac{25000-p}{4}y = 365000$$
$$16(p-4000) + \left(\frac{25000-p}{4} + 1000\right)y = 351000 \Rightarrow 16p + \frac{29000-p}{4}y = 415000$$

$$\frac{29000-p}{4}y - \frac{25000-p}{4}y = 50000 \Rightarrow 4000y = 200000 \Rightarrow y = 50$$

$$16p + \frac{25000-p}{4}50 = 365000 \Rightarrow 7p = 105000 \Rightarrow p = 15000$$

Number of family tickets: $\frac{25000-p}{4} = \frac{25000-15000}{4} = 2500$

OR

x = number of single tickets
f = number of family tickets
y = cost of family ticket

$$x + 4f = 25000$$
$$16x + fy = 365000$$
$$16(x-4000) + (f+1000)y = 351000$$

$$16x - 64000 + fy + 1000y = 351000$$
$$16x + fy = 365000$$
$$1000y = 50000$$
$$y = 50$$

Question 1 **(25 marks)**

(a) Solve the simultaneous equations:

$$a^2 - ab + b^2 = 3$$
$$a + 2b + 1 = 0$$

(b) Find the set of all real values of x for which $\dfrac{2x-5}{x-3} \leq \dfrac{5}{2}$.

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$$a = -2b - 1$$

$$(-2b-1)^2 + (2b+1)b + b^2 = 3$$
$$7b^2 + 5b - 2 = 0$$
$$(7b-2)(b+1) = 0$$
$$b = \tfrac{2}{7} \quad \text{or} \quad b = -1$$
$$a = \tfrac{-11}{7} \quad \text{or} \quad a = 1$$

Solution: $\{b = \tfrac{2}{7}$ and $a = \tfrac{-11}{7}\}$ or $\{b = -1$ and $a = 1\}$.

(b) Find the set of all real values of x for which $\dfrac{2x-5}{x-3} \leq \dfrac{5}{2}$.

Multiply across by $2(x-3)^2$, which is non-negative:

$$2(x-3)(2x-5) \leq 5(x-3)^2$$
$$4x^2 - 22x + 30 \leq 5x^2 - 30x + 45$$
$$0 \leq x^2 - 8x + 15$$
$$0 \leq (x-5)(x-3)$$

$x \geq 5$ or $x < 3$.

OR

$$\frac{2x-5}{x-3} - \frac{5}{2} \leq 0$$
$$\frac{2(2x-5) - 5(x-3)}{2(x-3)} \leq 0$$
$$\frac{-x+5}{2(x-3)} \leq 0$$

	$x < 3$	$3 < x < 5$	$x > 5$
$-x+5$	$+$	$+$	$-$
$x-3$	$-$	$+$	$+$
$\dfrac{-x+5}{2(x-3)}$	$-$	$+$	$-$

$x \geq 5$ or $x < 3$.

Question 4 **(25 marks)**

(a) Solve the simultaneous equations,

$$2x + 8y - 3z = -1$$
$$2x - 3y + 2z = 2$$
$$2x + y + z = 5.$$

Marking Scheme