2018

(a) Prove, using induction, that if n is a positive integer then

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta), \text{ where } i^2 = -1.$$

(b) Hence, or otherwise, find $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$ in its simplest form.

<table>
<tr><th>Q4</th><th>Model Solution – 25 Marks</th><th>Marking Notes</th></tr>
<tr><td>(a)</td><td>

P(1)
$(\cos\theta + i\sin\theta)^1 = \cos(1\theta) + i\sin(1\theta)$

$\boldsymbol{P(k)}$: Assume $(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$

Test $\boldsymbol{P(k+1)}$:
$(\cos\theta + i\sin\theta)^{k+1} =$
$= \cos(k+1)\theta + i\sin(k+1)\theta$

$(\cos\theta + i\sin\theta)^{k+1}$
$= (\cos\theta + i\sin\theta)^k.(\cos\theta + i\sin\theta)^1$

$= (\cos(k\theta) + i\sin(k\theta)).(\cos\theta + i\sin\theta)$

$= [\cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta]$
$+i[\cos(k\theta)\sin\theta + \cos\theta\sin(k\theta)]$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

Thus the proposition is true for $n = k+1$ provided it is true for $n = k$ but it is true for $n = 1$ and therefore true for all positive integers.
</td><td>

Scale 15D (0, 5, 7, 11, 15)
Low Partial Credit:
Step $P(1)$

Mid Partial Credit:
Step $P(k)$ or Step $P(k+1)$

High Partial Credit:
Uses Step $P(k)$ to prove Step $P(k+1)$

Note: Accept Step $P(1)$, Step $P(k)$, Step $P(k+1)$ in any order

Full credit –1:
Omits conclusion but otherwise correct

Full credit:
$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos(n\theta) + i\sin(n\theta))$
proved correctly
</td></tr>
<tr><td>(b)</td><td>

$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3 = 1\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3$
$= \left(\cos(3)\frac{2\pi}{3} + i\sin(3)\frac{2\pi}{3}\right)$
$= (\cos 2\pi + i\sin 2\pi) =$
$1 + 0i$
$= 1$
</td><td>

Scale 10C (0, 4, 8, 10)
Low Partial Credit:
Modulus or argument correct
Some correct multiplication
Apply De Moivre correctly with incorrect modulus and argument

High Partial Credit:
$\left(\cos(3)\frac{2\pi}{3} + i\sin(3)\frac{2\pi}{3}\right)$
Multiplication correct but un-simplified

Full credit –1:
$\cos 2\pi + i\sin 2\pi$ or $\cos 360° + i\sin 360°$

Accept: Answer with reference to cube root of unity
</td></tr>
</table>

(a) Prove by induction that $8^n - 1$ is divisible by 7 for all $n \in \mathbb{N}$.

Q4	Model Solution – 25 Marks	Marking Notes
(a)	P_1: $8^1 - 1 = 7$ (divisible by 7) P_k: Assume $8^k - 1$ is divisible by 7 $8^k - 1 = 7M$ $8^k = 7M + 1$ P_{k+1}: $8^{k+1} - 1 = 8(8^k) - 1$ $= 8(7M + 1) - 1$ $= 56M + 7$ $= 7(8M + 1)$ P_{k+1} is divisible by 7 P_1 is true P_k true $\implies$ P_{k+1} is true So, P_{k+1} true whenever P_k true. Since P_1 true, then, by induction, P_n is true for all natural numbers ≥ 1 **Or** $P_{k+1} = 8^{k+1} - 1$ $= 8.8^k - 1$ $= (7+1).8^k - 1$ $= 7(8^k) + (8^k - 1)$ Obviously divisible by 7 — From P_k So, P_{k+1} true whenever P_k true. Since P_1 true, then, by induction, P_n is true for all natural numbers ≥ 1	Scale 15D (0, 4, 7, 11, 15) *Low Partial Credit* • P_1 step *Mid Partial Credit* • P_k step • P_{k+1} step *High Partial Credit* • use of P_k step to prove P_{k+1} step **Note:** *accept* P_1 step, P_k step *and* P_{k+1} step *in any order*

(a) Prove, by induction, that the sum of the first n natural numbers, $1+2+3+\cdots+n$, is $\dfrac{n(n+1)}{2}$.

(b) Hence, or otherwise, prove that the sum of the first n even natural numbers, $2+4+6+\cdots+2n$, is n^2+n.

(c) Using the results from **(a)** and **(b)** above, find an expression for the sum of the first n odd natural numbers in its simplest form.

page	running

(a) Prove, by induction, that the sum of the first n natural numbers, $1+2+3+\cdots+n$, is $\frac{n(n+1)}{2}$.

To Prove: $P(n)=1+2+3+\cdots+n=\frac{n(n+1)}{2}$

$P(1):\ 1\ =\ \frac{1(1+1)}{2}=1$, True

Assume P(n) is true for n = k, and prove P(n) is true for n = k + 1.

n = k: $1+2+3+\cdots+k=\frac{k(k+1)}{2}$

To prove P(k+1) $=\frac{(k+1)}{2}(k+2)$

L.H.S. $=1+2+3+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}$

$=\frac{(k+1)}{2}(k+2)=$ R.H.S

But P(1) is true, so P(2) is true etc.
Hence, P(n) is true for all n.

(b) Hence, or otherwise, prove that the sum of the first n even natural numbers, $2+4+6+\cdots+2n$, is n^2+n.

a = 2 and d = 2.

$S_n=\frac{n}{2}\left(2a+(n-1)d\right)=\frac{n}{2}\left(4+(n-1)2\right)=\frac{n}{2}\left(2n+2\right)=n^2+n$

OR

$$\begin{aligned}S_n &= 2+4+6+\ldots\ldots\ldots\ldots+2n\\ &= 2(1+2+3+\ldots\ldots\ldots\ldots+n)\\ &= 2\left[\frac{n(n+1)}{2}\right]\\ &= n(n+1)\\ &= n^2+n\end{aligned}$$

(c) Using the results from **(a)** and **(b)** above, find an expression for the sum of the first n odd natural numbers in its simplest form.

$$1+2+3+\cdots+2n=\frac{2n(2n+1)}{2}=2n^2+n$$

$$\Rightarrow (1+3+5+\cdots n \text{ terms})+(2+4+6+\cdots n \text{ terms})=2n^2+n$$

$$\Rightarrow (1+3+5+\cdots n \text{ terms})+(n^2+n)=2n^2+n$$

$$\Rightarrow 1+3+5+\cdots n \text{ terms} = 2n^2+n-(n^2+n)=n^2$$

OR

$$S_A = 1+2+3+\ldots\ldots\ldots\ldots+(2n-1)+(2n) \qquad = 2n^2+n$$

$$S_B = 2+4+6+8+\ldots\ldots\ldots+2n \qquad = n^2+n$$

$$S_A - S_B = 1+3+5+\ldots\ldots\ldots+(2n-1) \qquad = n^2$$

Question 2 **(25 marks)**

(a) **(i)** Prove by induction that, for any n, the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

(ii) Find the sum of all the natural numbers from 51 to 100, inclusive.

(b) Given that $p = \log_c x$, express $\log_c \sqrt{x} + \log_c (cx)$ in terms of p.

First we check that the statement is true for $n = 1$. The sum of the first 1 natural numbers is 1, and when $n = 1$ we have $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$. So the statement is true for $n = 1$. Now suppose that the statement is true for some $n \geq 1$. So

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Now, add $n + 1$ to both sides and we get

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

So the sum of the first $n+1$ natural numbers is $\frac{(n+1)((n+1)+1)}{2}$, which completes the induction step. Therefore, by induction, the statement is true for all natural numbers n.

(ii) Find the sum of all the natural numbers from 51 to 100, inclusive.

By part (i), we know that

$$1 + 2 + \cdots + 100 = \frac{100(101)}{2} = 5050.$$

We also know that

$$1 + 2 + \cdots + 50 = \frac{50(51)}{2} = 1275.$$

Subtracting the second equation from the first yields

$$51 + 52 + \cdots + 100 = 5050 - 1275 = 3775.$$

(b) Given that $p = \log_c x$, express $\log_c \sqrt{x} + \log_c(cx)$ in terms of p.

We know that

$$\log_c \sqrt{x} = \log_c x^{\frac{1}{2}} = \frac{1}{2}\log_c x = \frac{1}{2}p$$

using the power law for logarithms.
Also,

$$\log_c(cx) = \log_c c + \log_c x = \log_c c + p$$

using the product rule for logarithms.
But $\log_c c = 1$ since $c^1 = c$. Therefore

$$\log_c \sqrt{x} + \log_c(cx) = \frac{1}{2}p + 1 + p = \frac{3p}{2} + 1.$$

Question 2 **(25 marks)**

(a) Prove by induction that $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$, for any $n \in \mathbb{N}$.

(b) State the range of values of x for which the series $\sum_{r=2}^{\infty} (4x-1)^r$ is convergent, and write the infinite sum in terms of x.

Question 2 **(25 marks)**

(a) Prove by induction that $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$.

First we check that the statement is true for $n = 1$. The sum of the first 1 natural numbers is 1, and when $n = 1$ we have $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$. So the statement is true for $n = 1$.

Now suppose that the statement is true for some $n \geq 1$. Remember that $\sum_{r=1}^{n} r = 1+2+\cdots+n$. So

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

Now, add $n+1$ to both sides and we get

$$\begin{aligned} 1+2+\cdots+n+(n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1)+2(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

So the sum of the first $n+1$ natural numbers is $\frac{(n+1)((n+1)+1)}{2}$, which completes the induction step. Therefore, by induction, the statement is true for all natural numbers n.

(b) State the range of values for which the series $\sum_{r=2} (4x-1)^r$ is convergent, and write the infinite sum in terms of x.

This is a geometric series i.e. a series of the form $T_n = ar^{n-1} = a + ar + ar^2 + ar^3 + \dots$ where $a = 1$ and $r = 4x - 1$.
However, given that it starts from r=2, this series is missing the first two terms a and ar (1 and $4x - 1$). If this series is convergent we must have $|4x - 1| < 1$ which means

$$\begin{aligned} -1 < 4x - 1 < 1 \\ 0 < 4x < 2 \\ 0 < x < \frac{1}{2} \end{aligned}$$

This is the required range.

The sum to infinity of a geometric series is given by $S_\infty = \dfrac{a}{1-r}$ where $|r| < 1$. Since this series is missing the first two terms we get

$$\begin{aligned} S_\infty &= \frac{1}{1-(4x-1)} - 1 - (4x - 1) \\ &= \frac{1}{2-4x} - 4x \\ &= \frac{1}{2-4x} - \frac{(2-4x)4x}{2-4x} \\ &= \frac{1 - 8x + 16x^2}{2-4x} \end{aligned}$$

Question 4 **(25 marks)**

(a) Prove, by induction, the formula for the sum of the first n terms of a geometric series. That is, prove that, for $r \neq 1$:

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

(b) By writing the recurring part as an infinite geometric series, express the following number as a fraction of integers:

$$5{\cdot}\dot{2}\dot{1} = 5{\cdot}2121212121\ldots$$

$P(n): \quad a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$

Check $P(1)$: $\quad a = \dfrac{a(1-r)}{1-r}$, which is true.

Assume $P(k)$: $\quad a + ar + ar^2 + \cdots + ar^{k-1} = \dfrac{a(1-r^k)}{1-r}$

Then: $\quad \underbrace{a + ar + ar^2 + \cdots + ar^{k-1}} + ar^k$

$$= \frac{a(1-r^k)}{1-r} + ar^k$$

$$= \frac{a(1-r^k) + ar^k(1-r)}{1-r}$$

$$= \frac{a(1-\cancel{r^k}+\cancel{r^k}-r^{k+1})}{1-r}$$

$$= \frac{a(1-r^{k+1})}{1-r}$$

which establishes $P(k+1)$.

Since we have $P(1) \wedge \{\forall k \in \mathbb{N}, (P(k) \Rightarrow P(k+1))\}$, it follows that $P(n)$ holds $\forall n \in \mathbb{N}$.

$$5\cdot\dot{2}\dot{1} = 5 + \frac{21}{100} + \frac{21}{10000} + \frac{21}{1000000} + \cdots$$

$= 5 +$ [geometric series with $a = \frac{21}{100}$, $\quad r = \frac{1}{100}$].

$$= 5 + \frac{\frac{21}{100}}{1-\frac{1}{100}} = 5 + \frac{21}{100-1} = 5\tfrac{21}{99} = 5\tfrac{7}{33}.$$

(a) **(i)** Prove by induction that, for any n, the sum of the first n natural numbers is $\dfrac{n(n+1)}{2}$.

(ii) Find the sum of all the natural numbers from 51 to 100, inclusive.

Marking Scheme