

(a) Prove, using induction, that if  $n$  is a positive integer then

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta), \text{ where } i^2 = -1.$$

(b) Hence, or otherwise, find  $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$  in its simplest form.

Q4	Model Solution – 25 Marks	Marking Notes
(a)	<p><b>P(1)</b>  <math>(\cos \theta + i \sin \theta)^1 = \cos(1\theta) + i \sin(1\theta)</math></p> <p><b>P(k):</b> Assume <math>(\cos \theta + i \sin \theta)^k = \cos(k\theta) + i \sin(k\theta)</math></p> <p>Test <b>P(k + 1):</b>  <math>(\cos \theta + i \sin \theta)^{k+1} =</math>  <math>= \cos(k + 1)\theta + i \sin(k + 1)\theta</math></p> <p><math>(\cos \theta + i \sin \theta)^{k+1}</math>  <math>= (\cos \theta + i \sin \theta)^k \cdot (\cos \theta + i \sin \theta)^1</math></p> <p><math>= (\cos(k\theta) + i \sin(k\theta)) \cdot (\cos \theta + i \sin \theta)</math></p> <p><math>= [\cos(k\theta) \cos \theta - \sin(k\theta) \sin \theta]</math>  <math>+ i[\cos(k\theta) \sin \theta + \cos \theta \sin(k\theta)]</math></p> <p><math>= \cos(k + 1)\theta + i \sin(k + 1)\theta</math></p> <p>Thus the proposition is true for <math>n = k + 1</math> provided it is true for <math>n = k</math> but it is true for <math>n = 1</math> and therefore true for all positive integers.</p>	<p><b>Scale 15D (0, 5, 7, 11, 15)</b></p> <p><i>Low Partial Credit:</i>  Step P(1)</p> <p><i>Mid Partial Credit:</i>  Step P(k) or Step P(k + 1)</p> <p><i>High Partial Credit:</i>  Uses Step P(k) to prove Step P(k + 1)</p> <p>Note: Accept Step P(1), Step P(k), Step P(k + 1) in any order</p> <p><i>Full credit -1:</i>  Omits conclusion but otherwise correct</p> <p><i>Full credit:</i>  <math>[r(\cos \theta + i \sin \theta)]^n = r^n (\cos(n\theta) + i \sin(n\theta))</math>  proved correctly</p>
(b)	$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3 = 1 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)^3$ $= \left(\cos(3) \frac{2\pi}{3} + i \sin(3) \frac{2\pi}{3}\right)$ $= (\cos 2\pi + i \sin 2\pi) =$ $1 + 0i$ $= 1$	<p><b>Scale 10C (0, 4, 8, 10)</b></p> <p><i>Low Partial Credit:</i>  Modulus or argument correct  Some correct multiplication  Apply De Moivre correctly with incorrect modulus and argument</p> <p><i>High Partial Credit:</i>  <math>\left(\cos(3) \frac{2\pi}{3} + i \sin(3) \frac{2\pi}{3}\right)</math>  Multiplication correct but un-simplified</p> <p><i>Full credit -1:</i>  <math>\cos 2\pi + i \sin 2\pi</math> or <math>\cos 360^\circ + i \sin 360^\circ</math></p> <p><b>Accept:</b> Answer with reference to cube root of unity</p>

- (a) Prove by induction that  $8^n - 1$  is divisible by 7 for all  $n \in \mathbb{N}$ .



Q4	Model Solution – 25 Marks	Marking Notes
(a)	<p> <math>P_1: 8^1 - 1 = 7</math> (divisible by 7)  <math>P_k</math>: Assume <math>8^k - 1</math> is divisible by 7  <math>8^k - 1 = 7M</math>  <math>8^k = 7M + 1</math>  <math>P_{k+1}: 8^{k+1} - 1 = 8(8^k) - 1</math>  <math>= 8(7M + 1) - 1</math>  <math>= 56M + 7</math>  <math>= 7(8M + 1)</math>  <math>P_{k+1}</math> is divisible by 7   <math>P_1</math> is true  <math>P_k</math> true <math>\Rightarrow P_{k+1}</math> is true            So, <math>P_{k+1}</math> true whenever <math>P_k</math> true.            Since <math>P_1</math> true, then, by induction, <math>P_n</math> is true for all natural numbers <math>\geq 1</math>   <p style="text-align: center;"><b>Or</b></p> <math display="block">P_{k+1} = 8^{k+1} - 1</math> <math display="block">= 8 \cdot 8^k - 1</math> <math display="block">= (7 + 1) \cdot 8^k - 1</math> <math display="block">= 7(8^k) + (8^k - 1)</math> <div style="display: flex; justify-content: space-around; margin-top: 10px;"> <div style="text-align: center;"> <math>\swarrow</math>            Obviously divisible by 7         </div> <div style="text-align: center;"> <math>\searrow</math>            From <math>P_k</math> </div> </div>           So, <math>P_{k+1}</math> true whenever <math>P_k</math> true.            Since <math>P_1</math> true, then, by induction, <math>P_n</math> is true for all natural numbers <math>\geq 1</math> </p>	<p>Scale 15D (0, 4, 7, 11, 15)</p> <p><i>Low Partial Credit</i></p> <ul style="list-style-type: none"> <li><math>P_1</math> step</li> </ul> <p><i>Mid Partial Credit</i></p> <ul style="list-style-type: none"> <li><math>P_k</math> step</li> <li><math>P_{k+1}</math> step</li> </ul> <p><i>High Partial Credit</i></p> <ul style="list-style-type: none"> <li>use of <math>P_k</math> step to prove <math>P_{k+1}</math> step</li> </ul> <p><b>Note:</b> accept <math>P_1</math> step, <math>P_k</math> step and <math>P_{k+1}</math> step in any order</p>

- (a) Prove, by induction, that the sum of the first  $n$  natural numbers,  $1+2+3+\cdots+n$ , is  $\frac{n(n+1)}{2}$ .

- (b) Hence, or otherwise, prove that the sum of the first  $n$  even natural numbers,  $2+4+6+\cdots+2n$ , is  $n^2+n$ .

- (c) Using the results from (a) and (b) above, find an expression for the sum of the first  $n$  odd natural numbers in its simplest form.

- (a) Prove, by induction, that the sum of the first  $n$  natural numbers,  $1+2+3+\dots+n$ , is  $\frac{n(n+1)}{2}$ .

To Prove:  $P(n) = 1+2+3+\dots+n = \frac{n(n+1)}{2}$

$P(1): 1 = \frac{1(1+1)}{2} = 1$ , True

Assume  $P(n)$  is true for  $n = k$ , and prove  $P(n)$  is true for  $n = k + 1$ .

$n = k: 1+2+3+\dots+k = \frac{k(k+1)}{2}$

To prove  $P(k+1) = \frac{(k+1)(k+2)}{2}$

$$\begin{aligned} \text{L.H.S.} &= 1+2+3+\dots+k+(k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} = \text{R.H.S} \end{aligned}$$

But  $P(1)$  is true, so  $P(2)$  is true etc.  
Hence,  $P(n)$  is true for all  $n$ .

- (b) Hence, or otherwise, prove that the sum of the first  $n$  even natural numbers,  $2+4+6+\dots+2n$ , is  $n^2 + n$ .

$a = 2$  and  $d = 2$ .

$$S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(4 + (n-1)2) = \frac{n}{2}(2n+2) = n^2 + n$$

OR

$$\begin{aligned} S_n &= 2 + 4 + 6 + \dots + 2n \\ &= 2(1 + 2 + 3 + \dots + n) \\ &= 2 \left[ \frac{n(n+1)}{2} \right] \\ &= n(n+1) \\ &= n^2 + n \end{aligned}$$

- (c) Using the results from (a) and (b) above, find an expression for the sum of the first  $n$  odd natural numbers in its simplest form.

$$1 + 2 + 3 + \dots + 2n = \frac{2n(2n+1)}{2} = 2n^2 + n$$

$$\Rightarrow (1 + 3 + 5 + \dots n \text{ terms}) + (2 + 4 + 6 + \dots n \text{ terms}) = 2n^2 + n$$

$$\Rightarrow (1 + 3 + 5 + \dots n \text{ terms}) + (n^2 + n) = 2n^2 + n$$

$$\Rightarrow 1 + 3 + 5 + \dots n \text{ terms} = 2n^2 + n - (n^2 + n) = n^2$$

**OR**

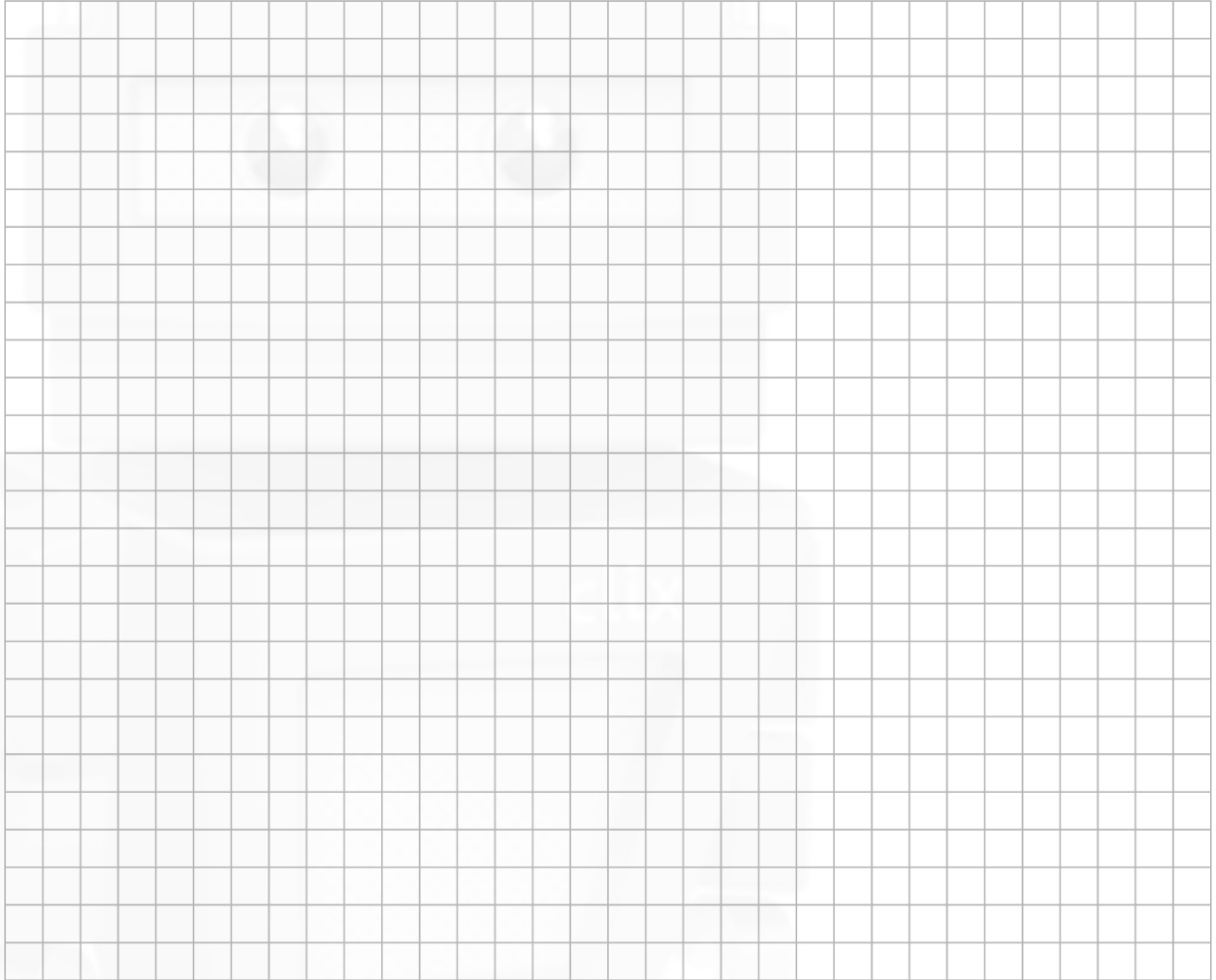
$$S_A = 1 + 2 + 3 + \dots + (2n-1) + (2n) = 2n^2 + n$$

$$S_B = 2 + 4 + 6 + 8 + \dots + 2n = n^2 + n$$

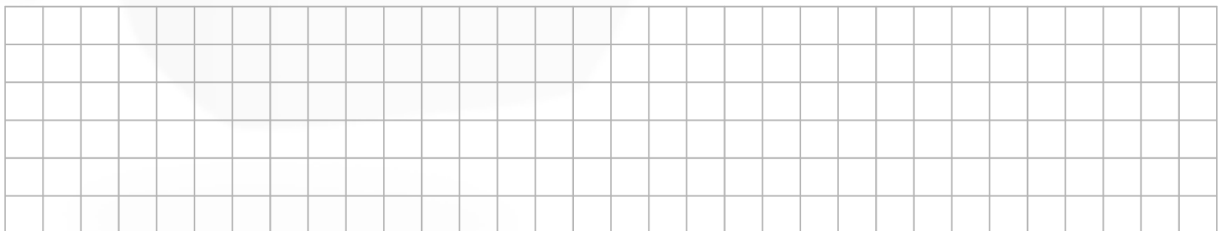
$$S_A - S_B = 1 + 3 + 5 + \dots + (2n-1) = n^2$$

**Question 2****(25 marks)**

- (a) (i) Prove by induction that, for any  $n$ , the sum of the first  $n$  natural numbers is  $\frac{n(n+1)}{2}$ .



- (ii) Find the sum of all the natural numbers from 51 to 100, inclusive.



- (b) Given that  $p = \log_c x$ , express  $\log_c \sqrt{x} + \log_c (cx)$  in terms of  $p$ .





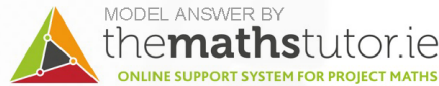
First we check that the statement is true for  $n = 1$ . The sum of the first 1 natural numbers is 1, and when  $n = 1$  we have  $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$ . So the statement is true for  $n = 1$ .  
 Now suppose that the statement is true for some  $n \geq 1$ . So

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}.$$

Now, add  $n + 1$  to both sides and we get

$$\begin{aligned} 1 + 2 + \dots + n + (n + 1) &= \frac{n(n+1)}{2} + (n + 1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

So the sum of the first  $n + 1$  natural numbers is  $\frac{(n+1)((n+1)+1)}{2}$ , which completes the induction step. Therefore, by induction, the statement is true for all natural numbers  $n$ .



(ii) Find the sum of all the natural numbers from 51 to 100, inclusive.

By part (i), we know that

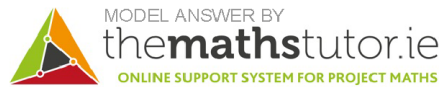
$$1 + 2 + \dots + 100 = \frac{100(101)}{2} = 5050.$$

We also know that

$$1 + 2 + \dots + 50 = \frac{50(51)}{2} = 1275.$$

Subtracting the second equation from the first yields

$$51 + 52 + \dots + 100 = 5050 - 1275 = 3775.$$



(b) Given that  $p = \log_c x$ , express  $\log_c \sqrt{x} + \log_c(cx)$  in terms of  $p$ .

We know that

$$\log_c \sqrt{x} = \log_c x^{\frac{1}{2}} = \frac{1}{2} \log_c x = \frac{1}{2}p$$

using the power law for logarithms.

Also,

$$\log_c(cx) = \log_c c + \log_c x = \log_c c + p$$

using the product rule for logarithms.

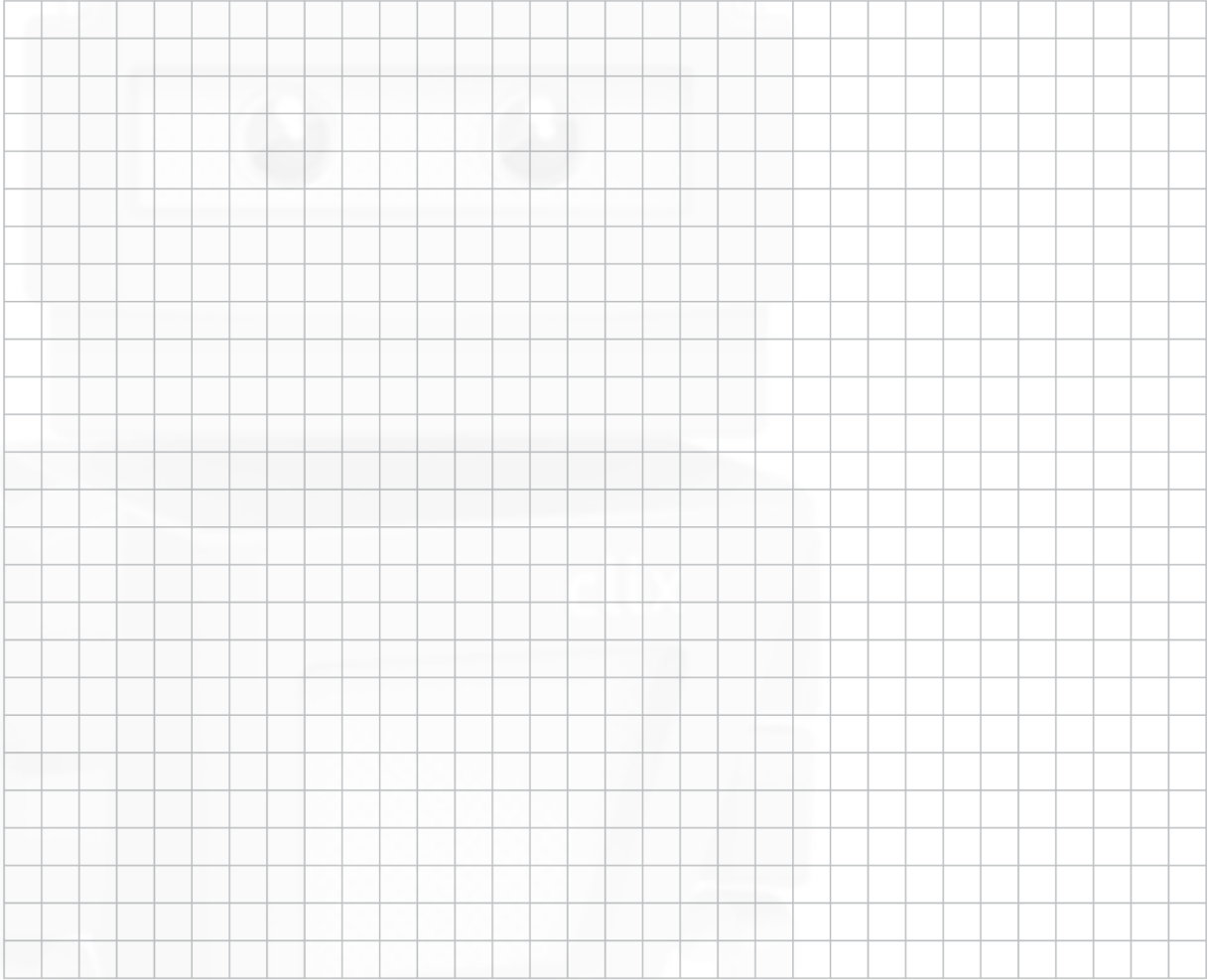
But  $\log_c c = 1$  since  $c^1 = c$ . Therefore

$$\log_c \sqrt{x} + \log_c(cx) = \frac{1}{2}p + 1 + p = \frac{3p}{2} + 1.$$

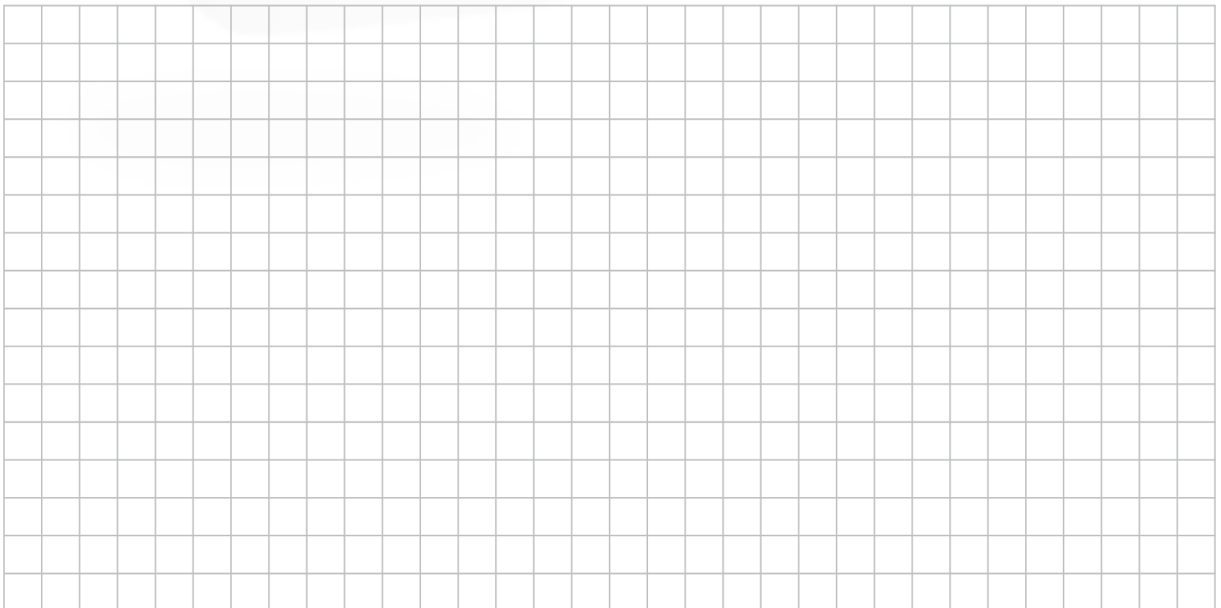
## Question 2

(25 marks)

- (a) Prove by induction that  $\sum_{r=1}^n r = \frac{n(n+1)}{2}$ , for any  $n \in \mathbb{N}$ .



- (b) State the range of values of  $x$  for which the series  $\sum_{r=2}^{\infty} (4x-1)^r$  is convergent, and write the infinite sum in terms of  $x$ .



## Question 2

(25 marks)

- (a) Prove by induction that  $\sum_{r=1}^n r = \frac{n(n+1)}{2}$  for any  $n \in \mathbb{N}$ .

First we check that the statement is true for  $n = 1$ . The sum of the first 1 natural numbers is 1, and when  $n = 1$  we have  $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$ . So the statement is true for  $n = 1$ .

Now suppose that the statement is true for some  $n \geq 1$ . Remember that  $\sum_{r=1}^n r = 1 + 2 + \dots + n$ .

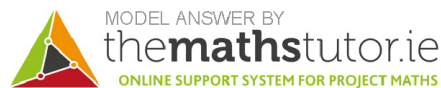
So

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}.$$

Now, add  $n + 1$  to both sides and we get

$$\begin{aligned} 1 + 2 + \dots + n + (n + 1) &= \frac{n(n+1)}{2} + (n + 1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

So the sum of the first  $n + 1$  natural numbers is  $\frac{(n+1)((n+1)+1)}{2}$ , which completes the induction step. Therefore, by induction, the statement is true for all natural numbers  $n$ .



- (b) State the range of values for which the series  $\sum_{r=2}^{\infty} (4x - 1)^r$  is convergent, and write the infinite sum in terms of  $x$ .

This is a geometric series i.e. a series of the form  $T_n = ar^{n-1} = a + ar + ar^2 + ar^3 + \dots$  where  $a = 1$  and  $r = 4x - 1$ .

However, given that it starts from  $r=2$ , this series is missing the first two terms  $a$  and  $ar$  (1 and  $4x - 1$ ). If this series is convergent we must have  $|4x - 1| < 1$  which means

$$\begin{aligned} -1 < 4x - 1 < 1 \\ 0 < 4x < 2 \\ 0 < x < \frac{1}{2} \end{aligned}$$

This is the required range.

The sum to infinity of a geometric series is given by  $S_\infty = \frac{a}{1-r}$  where  $|r| < 1$ . Since this series is missing the first two terms we get

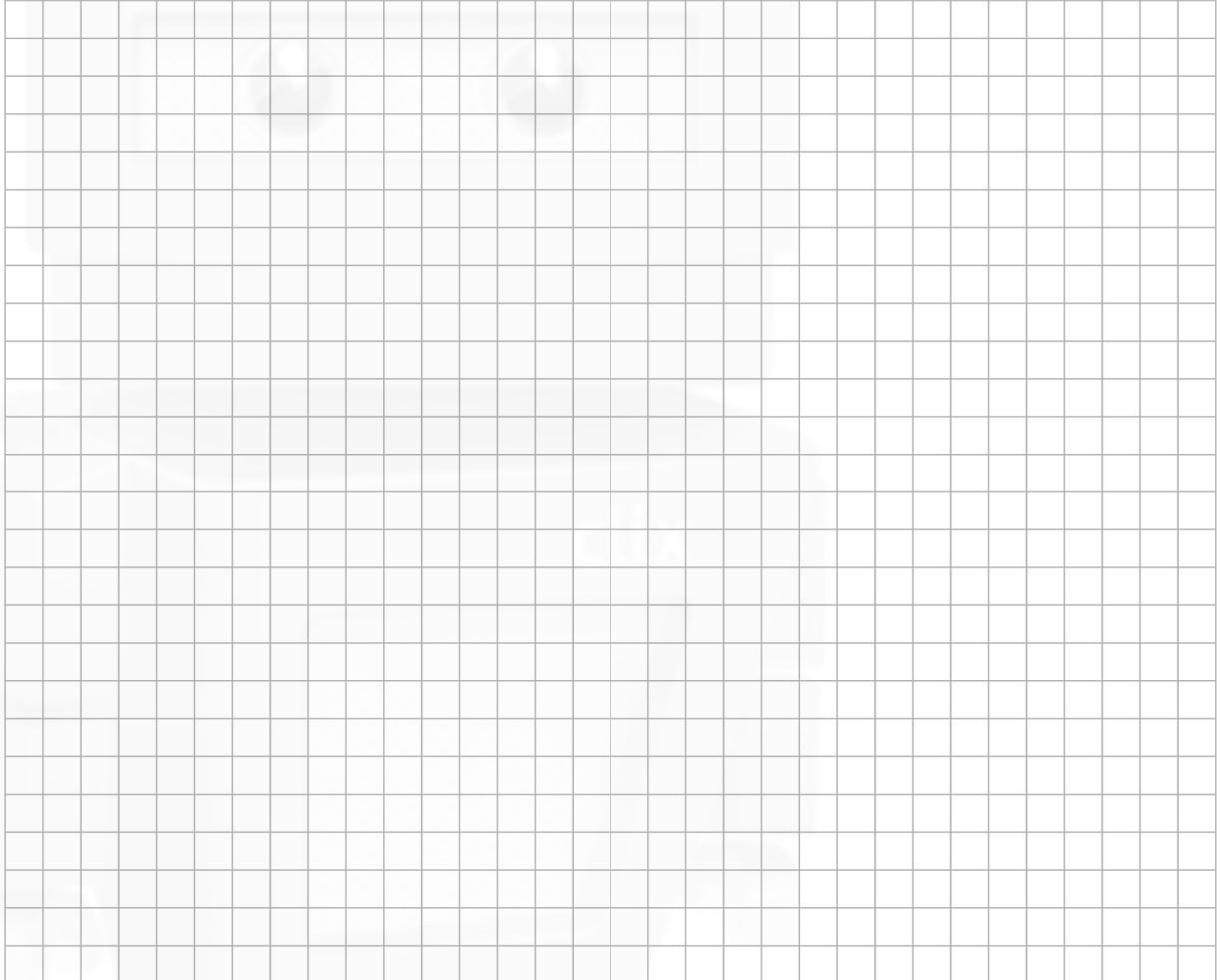
$$\begin{aligned} S_\infty &= \frac{1}{1-(4x-1)} - 1 - (4x-1) \\ &= \frac{1}{2-4x} - 4x \\ &= \frac{1}{2-4x} - \frac{(2-4x)4x}{2-4x} \\ &= \frac{1-8x+16x^2}{2-4x} \end{aligned}$$

## Question 4

(25 marks)

- (a) Prove, by induction, the formula for the sum of the first  $n$  terms of a geometric series. That is, prove that, for  $r \neq 1$ :

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$



- (b) By writing the recurring part as an infinite geometric series, express the following number as a fraction of integers:

$$5.\dot{2}\dot{1} = 5.2121212121\dots$$



$$P(n): a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

Check  $P(1)$ :  $a = \frac{a(1-r)}{1-r}$ , which is true.

Assume  $P(k)$ :  $a + ar + ar^2 + \dots + ar^{k-1} = \frac{a(1-r^k)}{1-r}$

Then:

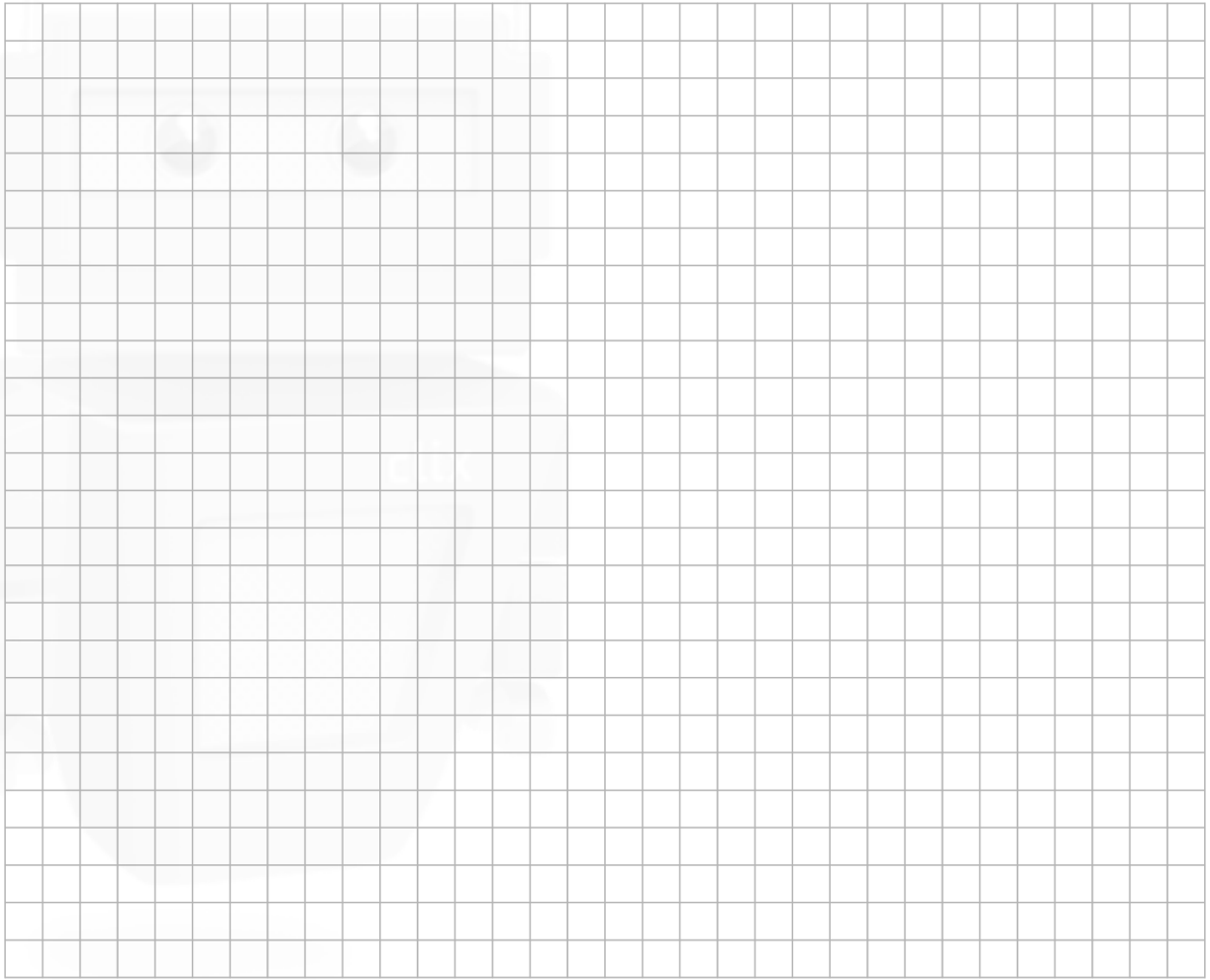
$$\begin{aligned} & \underbrace{a + ar + ar^2 + \dots + ar^{k-1}} + ar^k \\ &= \frac{a(1-r^k)}{1-r} + ar^k \\ &= \frac{a(1-r^k) + ar^k(1-r)}{1-r} \\ &= \frac{a(1 - \cancel{r^k} + \cancel{r^k} - r^{k+1})}{1-r} \\ &= \frac{a(1-r^{k+1})}{1-r} \end{aligned}$$

which establishes  $P(k+1)$ .

Since we have  $P(1) \wedge \{\forall k \in \mathbb{N}, (P(k) \Rightarrow P(k+1))\}$ , it follows that  $P(n)$  holds  $\forall n \in \mathbb{N}$ .

$$\begin{aligned} 5.\dot{2}\dot{1} &= 5 + \frac{21}{100} + \frac{21}{10000} + \frac{21}{1000000} + \dots \\ &= 5 + [\text{geometric series with } a = \frac{21}{100}, r = \frac{1}{100}]. \\ &= 5 + \frac{\frac{21}{100}}{1 - \frac{1}{100}} = 5 + \frac{21}{100-1} = 5 \frac{21}{99} = 5 \frac{7}{33}. \end{aligned}$$

- (a) (i) Prove by induction that, for any  $n$ , the sum of the first  $n$  natural numbers is  $\frac{n(n+1)}{2}$ .



- (ii) Find the sum of all the natural numbers from 51 to 100, inclusive.

