Question 1 **(25 marks)**

In a competition Mary has a probability of $\frac{1}{20}$ of winning, a probability of $\frac{1}{10}$ of finishing in second place, and a probability of $\frac{1}{4}$ of finishing in third place. If she wins the competition she gets €9000. If she comes second she gets €7000 and if she comes third she gets €3000. In all other cases she gets nothing. Each participant in the competition must pay €2000 to enter.

(a) Find the **expected value** of Mary's loss if she enters the competition.

(b) Each of the 3 prizes in the competition above is increased by the same amount (€x) but the entry fee is unchanged.
For example, if Mary wins the competition now, she would get €(9000 + x).
Mary now expects to break even.
Find the value of x.

Q1	Model Solution – 25 Marks	Marking Notes
(a)	$\frac{1}{20}(9000)+\frac{1}{10}(7000)+\frac{1}{4}(3000)$ $= 1900$ $E(x) = 2000 - 1900 = 100$ **Or** $E(x)=\frac{1}{20}(-7000)+\frac{1}{10}(-5000)$ $+\frac{1}{4}(-1000)+\frac{3}{5}(2000)$ $= -350 - 500 - 250 + 1200 = 100$ So expected gain for organisers of competition and therefore a loss for Mary of 100	**Scale 15C (0, 4, 11, 15)** *Low Partial Credit:* $E(x)$ partially formulated (1 or 2 terms) *High Partial Credit:* $E(x)$ fully formulated (sum of all three/all four terms)

(b)	$$\frac{1}{20}(9000+x)+\frac{1}{10}(7000+x)$$ $$+\frac{1}{4}(3000+x)=2000$$ $$\left(1900+\frac{8}{20}x\right)=2000$$ $$\frac{8}{20}x=100$$ $$x=250$$ **Or** From **(a)** to break even it will take €100. $$\frac{x}{20}+\frac{x}{10}+\frac{x}{4}=100$$ $$\frac{x+2x+5x}{20}=100$$ $$\frac{8}{20}x=100$$ $$x=250$$ **Or** $$E(x)=\frac{1}{20}(-7000-x)$$ $$+\frac{1}{10}(-5000-x)$$ $$+\frac{1}{4}(-1000-x)+\frac{3}{5}(2000)=0$$ $$-7000-x-10\,000-2x-5000-5x$$ $$+24\,000=0$$ $$2000=8x\Rightarrow 250=x$$	**Scale 10D (0, 3, 5, 8, 10)** *Low Partial Credit:* Any relevant use of x, excluding $(9000+x)$ *Mid Partial Credit:* $E(x)$ fully formulated (LHS). $\left(1900+\frac{8}{20}x\right)$ or equivalent and stops. $\frac{x}{20}+\frac{x}{10}+\frac{x}{4}$ *High Partial Credit* Relevant equation in x *Low Partial Credit:* Any relevant use of x e.g. $(-7000+x)$ *Mid Partial Credit:* $E(x)$ fully formulated (LHS). $\left(100-\frac{8}{20}x\right)$ or equivalent and stops. *High Partial Credit* Relevant equation in x

Question 3 **(25 marks)**

(a) A security code consists of six digits chosen at random from the digits 0 to 9.
The code may begin with zero and digits may be repeated.

For example

0	7	1	7	3	7

is a valid code.

(i) Find how many of the possible codes will end with a zero.

(ii) Find how many of the possible codes will contain the digits 2 0 1 8 together and in this order.

Marking Scheme

Q3	Model Solution – 25 Marks	Marking Notes
(a) (i)	$10^5 \times 1$ or $100\ 000$	**Scale 15C (0, 4, 11, 15)** *Low Partial Credit:* Some use of 10. Identifies that 5 other digits are required to complete code. *High Partial Credit* 9^5 or equivalent 10^6
(a) (ii)	$1 \times 10 \times 10 + 10 \times 1 \times 10 + 10 \times 10 \times 1$ $3 \times 10 \times 10$ or 3×10^2 or 300	**Scale 5B (0, 2, 5)** *Partial Credit:* 10×10

(b) Find a, b, c, and d, if $\frac{(n+3)!\,(n+2)!}{(n+1)!(n+1)!} = an^3 + bn^2 + cn + d$, where a, b, c, and $d \in \mathbb{N}$.

$a =$	$b =$	$c =$	$d =$

Marking Scheme

(b)		
	$\frac{(n+3)!\ (n+2)!}{(n+1)!\,(n+1)!} =$ $(n+3)(n+2)(n+2) =$ $n^3 + 7n^2 + 16n + 12$ **Or** $\frac{(n+3)!\,(n+2)!}{(n+1)!\,(n+1)!} = an^3 + bn^2 + cn + d$ $n = 0 \rightarrow \frac{3!.2!}{1!\,1!} = 12 = d$ $n = 1 \rightarrow a + b + c + d = 36$ $n = 2 \rightarrow 8a + 4b + 2c + d = 80$ $n = 3 \rightarrow 27a + 9b + 3c + d = 150$ Solving the simultaneous equations $a = 1, b = 7, c = 16, d = 12$	**Scale 5C (0, 2, 4, 5)** *Low Partial Credit:* Factorial expansion (e.g. $(n+3)! = (n+3)(n+2)(n+1) \ldots \ldots \ldots .1)$ Effort at a numerical value for *n* on both LHS and RHS (method 2) *High Partial Credit:* $(n+3)(n+2)(n+2)$ Four simultaneous equations

Question 1 **(25 marks)**

When Conor rings Ciara's house, the probability that Ciara answers the phone is $\frac{1}{5}$.

(a) Conor rings Ciara's house once every day for 7 consecutive days. Find the probability that she will answer the phone on the 2^{nd}, 4^{th}, and 6^{th} days but not on the other days.

(b) Find the probability that she will answer the phone for the 4^{th} time on the 7^{th} day.

(c) Conor rings her house once every day for *n* days. Write, in terms of *n*, the probability that Ciara will answer the phone at least once.

(d) Find the minimum value of *n* for which the probability that Ciara will answer the phone at least once is greater than 99%.

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(a)	$\frac{4}{5}\times\frac{1}{5}\times\frac{4}{5}\times\frac{1}{5}\times\frac{4}{5}\times\frac{1}{5}\times\frac{4}{5}=\frac{256}{78125}$ **or** $=0{\cdot}0032768$	**Scale 10C (0, 4, 5, 10)** *Low Partial Credit:* • $\frac{4}{5}$ • $(\frac{1}{5})^3$ *High Partial Credit:* • $\frac{4}{5}\times\frac{1}{5}\times\frac{4}{5}\times\frac{1}{5}\times\frac{4}{5}\times\frac{1}{5}\times\frac{4}{5}$ in any order
(b)	$\binom{6}{3}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)$ $=\frac{1280}{78125}$ **or** $\frac{256}{15625}$ **or** $0{\cdot}016384$	**Scale 5D (0, 2, 3, 4, 5)** *Low Partial Credit:* • $\binom{6}{3}$ or $\left(\frac{1}{5}\right)^3$ or $\left(\frac{4}{5}\right)^3$ • $\frac{1}{5}$ for last day *Mid Partial Credit:* • $\binom{6}{3}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^3$ and stops or continues • $\binom{7}{4}\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^3$ and continues *High Partial Credit:* • $\binom{6}{3}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)$
(c)	$1-\left(\frac{4}{5}\right)^n$	**Scale 5B (0, 3, 5)** *Partial Credit:* • 1 or $\left(\frac{4}{5}\right)^n$ • any correct term from the expansion
(d)	$1-\left(\frac{4}{5}\right)^n>0{\cdot}99$ $\left(\frac{4}{5}\right)^n<0{\cdot}01$ $\left(\frac{4}{5}\right)^{20{\cdot}6377}\approx 0{\cdot}01000000517$ $n=21$	**Scale 5C (0, 2, 4, 5)** *Low Partial Credit:* • Ans **(c)** $>0{\cdot}99$ *High Partial Credit:* • viable solution to inequality • n = 20·6377 and stops

(b) In Galway, rain falls in the morning on $\frac{1}{3}$ of the school days in the year.

When it is raining the probability of heavy traffic is $\frac{1}{2}$.

When it is not raining the probability of heavy traffic is $\frac{1}{4}$.

When it is raining and there is heavy traffic, the probability of being late for school is $\frac{1}{2}$.

When it is not raining and there is no heavy traffic, the probability of being late for school is $\frac{1}{8}$.

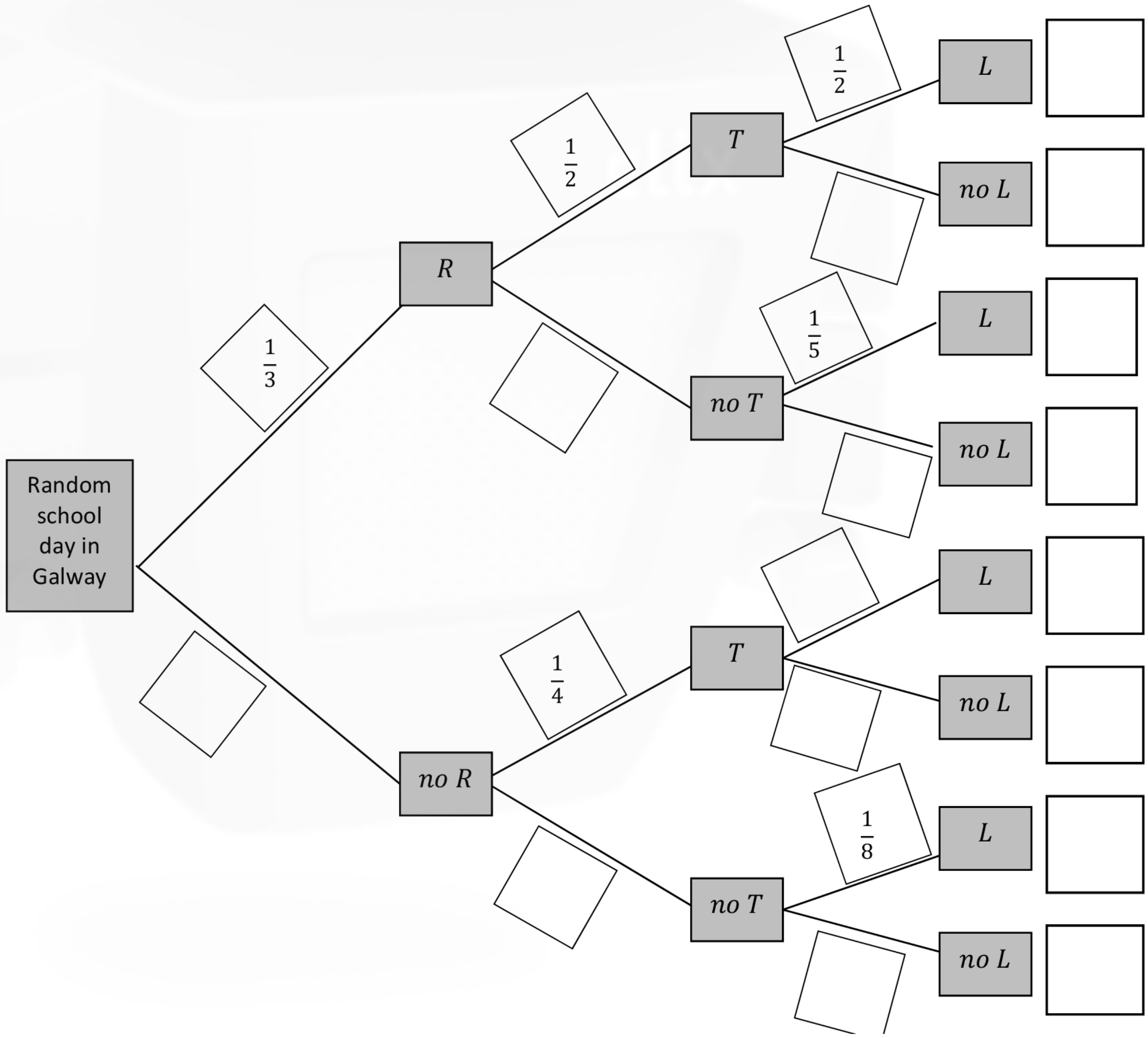
In any other situation the probability of being late for school is $\frac{1}{5}$.

Some of this information is shown in the tree diagram below.

(i) Write the probability associated with each branch of the tree diagram **and** the probability of each outcome into the blank boxes provided.

Give each answer in the form $\frac{a}{b}$, where $a, b \in \mathbb{N}$.

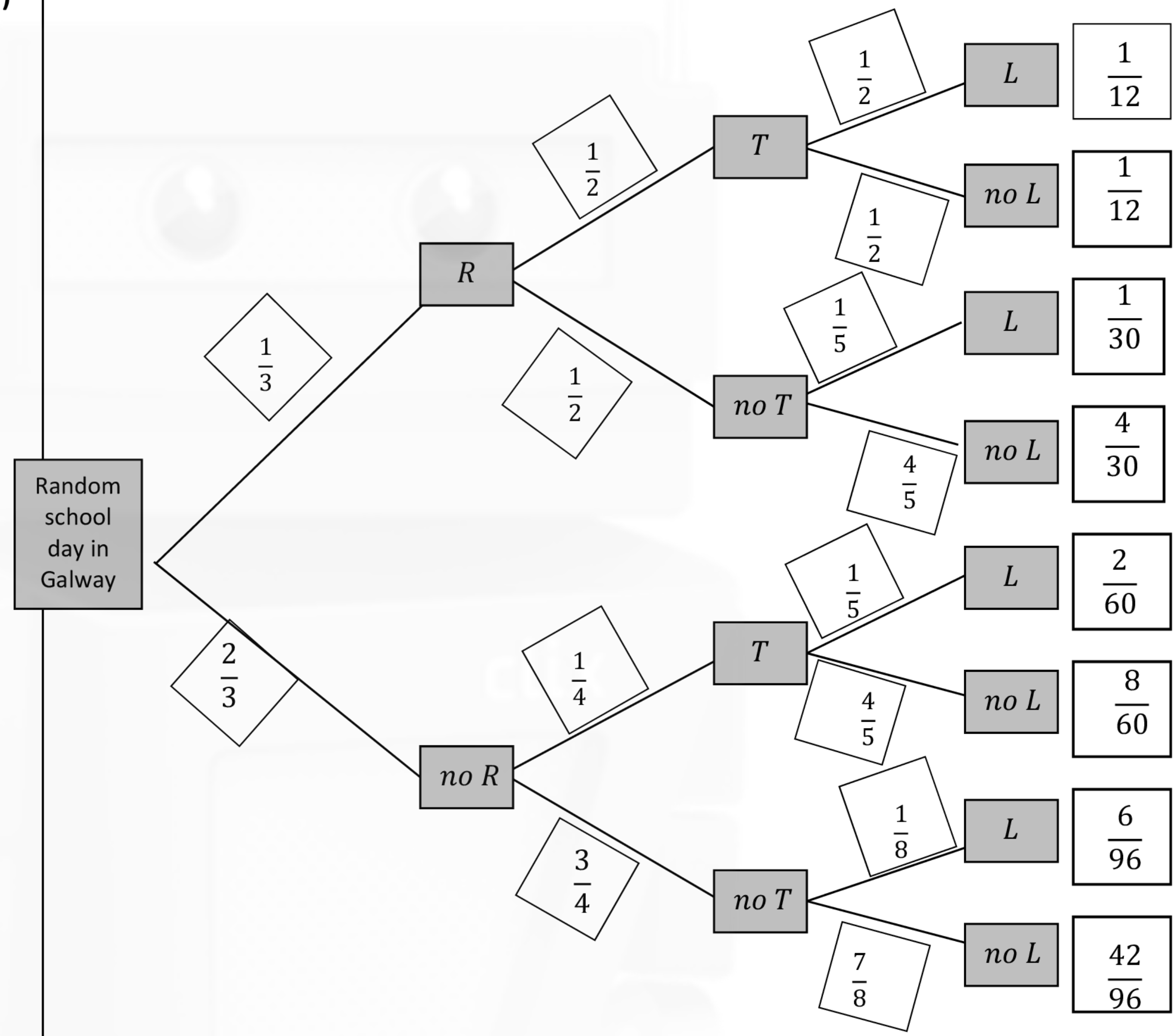
Key	Rain $= R$	Heavy traffic $= T$	Late $= L$
	No rain $= no\ R$	Not heavy traffic $= no\ T$	Not late $= no\ L$



(ii) On a random school day in Galway, find the probability of being late for school.

(iii) On a random school day in Galway, find the probability that it rained in the morning, given that you were late for school.

(b)
(i)



(b)
(i)

Scale 15C (0, 6, 9, 15)

Low Partial Credit:

- 3 boxes filled correctly

High Partial Credit:

- 10 boxes filled correctly

(b) (ii)	$\frac{1}{12}+\frac{1}{30}+\frac{2}{60}+\frac{6}{96}=\frac{17}{80}$ **or** $0{\cdot}2125$	**Scale 10C (0, 4, 5, 10)** *Low Partial Credit:* • 2 relevant fractions transferred *High Partial Credit:* • 4 relevant fractions identified but fails to complete
(b) (iii)	$P(R\|L)=\frac{P(R\cap L)}{P(L)}=\frac{\frac{1}{12}+\frac{1}{30}}{\frac{17}{80}}$ $=\frac{28}{51}$ **or** $0{\cdot}5490$	**Scale 5C (0, 2, 4, 5)** *Low Partial Credit:* • $P(L)$ • $P(R\cap L)$ *High Partial Credit:* • Formula fully substituted

2016

(a) **(i)** In an archery competition, the team consisting of John, David, and Mike will win 1st prize if at least two of them hit the bullseye with their last arrows. From past experience, they know that the probability that John, David, and Mike will hit the bullseye on their last arrow is $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{4}$ respectively.
Complete the table below to show all the ways in which they could win 1st prize.

	Way 1	Way 2	Way 3	Way 4
John	✓			
David	✓			
Mike	×			

✓ = Hit
× = Miss

(ii) Hence or otherwise find the probability that they will win the competition.

<table>
<tr><th>Q5</th><th>Model Solution – 25 Marks</th><th>Marking Notes</th></tr>
<tr><td>(a)
(i)</td><td><table><tr><td>John</td><td>✓</td><td>✓</td><td>×</td><td>✓</td></tr><tr><td>David</td><td>✓</td><td>×</td><td>✓</td><td>✓</td></tr><tr><td>Mike</td><td>×</td><td>✓</td><td>✓</td><td>✓</td></tr></table></td><td>Scale 5B (0, 2, 5)
Partial Credit
• 1 correct column</td></tr>
<tr><td>(a)
(ii)</td><td>$$P(win) = \left(\frac{1}{5}\times\frac{1}{6}\times\frac{3}{4}\right)+\left(\frac{1}{5}\times\frac{5}{6}\times\frac{1}{4}\right)$$ $$+\left(\frac{4}{5}\times\frac{1}{6}\times\frac{1}{4}\right)+\left(\frac{1}{5}\times\frac{1}{6}\times\frac{1}{4}\right)$$ $$=\frac{13}{120}$$</td><td>Scale 10C (0, 3, 7, 10)
Low Partial Credit
• one correct triple (numerical or descriptive)
• probability of any one Miss

High Partial Credit
• 4 correct triples (numerical)</td></tr>
</table>

A local sports club is planning to run a weekly lotto. To win the Jackpot of €1000, contestants must match one letter chosen from the 26 letters in the alphabet and two numbers chosen, in the correct order, from the numbers 0 to 9. In this lotto, repetition of numbers is allowed (e.g. M, 3, 3 is an outcome).

(a) Calculate the probability that M, 3, 3 would be the winning outcome in a particular week.

(b) If a contestant matches the letter only, or the letter and one number (but not both numbers), they will win €50. Using the table below, or otherwise, find how much the club should expect to make or lose on each play, correct to the nearest cent, if they charge €2 per play.

Event	**Payout** (x) €	**Probability** (P(x))	x.P(x)
Win Jackpot			
Match letter and first number only			
Match letter and second number only			
Match letter and neither number			
Fail to win			

(c) The club estimates that the average number of plays per week will be 845. If the club wants to make an average profit of €600 per week from the lotto, how much should the club charge per play, correct to the nearest cent?

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<table>
<tr><th>Q6</th><th>Model Solution – 25 Marks</th><th>Marking Notes</th></tr>
<tr><td>(a)</td><td>$$P(M,3,3) = \frac{1}{26} \times \frac{1}{10} \times \frac{1}{10} = \frac{1}{2600}$$</td><td>Scale 10C (0, 3, 7, 10)
Low Partial Credit
• any correct relevant probability

High Partial credit
• correct probabilities but not expressed as single fraction or equivalent

Note: Accept correct answer without supporting work</td></tr>
<tr><td>(b)</td><td>

Event	Payout	Prob (P(*x*))	*x*.P(*x*)
Win	1000	$\frac{1}{2600}$	$\frac{1000}{2600}$
letter 1 No.	50	$\frac{9}{2600}$	$\frac{450}{2600}$
letter 2^{nd} No	50	$\frac{9}{2600}$	$\frac{450}{2600}$
letter only	50	$\frac{81}{2600}$	$\frac{4050}{2600}$
Fail to win	0		0

$$\sum x.P(x) = \frac{5950}{2600} = 2{\cdot}29$$

Club loses 29 cent per play

Or

Event	Pay out	Prob (P(*x*)	*x*.P(*x*)
Win	-998	$^{1}/_{2600}$	$^{-998}/_{2600}$
letter + 1^{st} No.	-48	$^{9}/_{2600}$	$^{-432}/_{2600}$
Letter + 2^{nd} No	-48	$^{9}/_{2600}$	$^{-432}/_{2600}$
letter only	-48	$^{81}/_{2600}$	$^{-3888}/_{2600}$
Fail to Win	$+2$	$^{2500}/_{2600}$	$^{5000}/_{2600}$

$$\sum x.P(x) = -\frac{750}{2600} = -29 \text{ cent}$$

</td><td>Scale 10C (0, 3, 7, 10)
Low Partial Credit
• 1 correct entry to table
High Partial Credit
• all entries correct but fails to finish or finishes incorrectly
• no conclusion</td></tr>
</table>

(c)	Profit = Revenue – Pay-out $600 = 845(x - 2{\cdot}29)$ $$x = \frac{600 + 845(2{\cdot}29)}{845}$$ $$x = 3$$ **or** $$\frac{600}{845} = 0{\cdot}71$$ $$0{\cdot}71 + 2{\cdot}29 = 3$$	Scale 5C (0, 2, 4, 5) *Low Partial Credit* • links profit, revenue and payout *High partial Credit* • formula fully substituted

Question 1 **(25 marks)**

An experiment consists of throwing two fair, standard, six-sided dice and noting the sum of the two numbers thrown. If the sum is 9 or greater it is recorded as a "win" (W). If the sum is 8 or less it is recorded as a "loss" (L).

(a) Complete the table below to show all possible outcomes of the experiment.

		Die 2					
		1	2	3	4	5	6
Die 1	1		L				
	2						
	3						
	4						
	5						W
	6						

(b) **(i)** Find the probability of a win on one throw of the two dice.

(ii) Find the probability that each of 3 successive throws of the two dice results in a loss. Give your answer correct to four decimal places.

(c) The experiment is repeated until a total of 3 wins occur. Find the probability that the third win occurs on the tenth throw of the two dice. Give your answer correct to four decimal places.

(a) Complete the table below to show all possible outcomes of the experiment.

		Die 2					
		1	2	3	4	5	6
Die 1	1	L	L	L	L	L	L
	2	L	L	L	L	L	L
	3	L	L	L	L	L	W
	4	L	L	L	L	W	W
	5	L	L	L	W	W	W
	6	L	L	W	W	W	W

(b) **(i)** Find the probability of a win on one throw of the two dice.

$$P(W) = \frac{10}{36} = \frac{5}{18}$$

(ii) Find the probability that each of 3 successive throws of the two dice results in a loss. Give your answer correct to four decimal places.

$$P(L, L, L) = \left(\frac{13}{18}\right)^3 = 0 \cdot 3767$$

(c) The experiment is repeated until a total of 3 wins occur. Find the probability that the third win occurs on the tenth throw of the two dice. Give your answer correct to four decimal places.

$$P(2 \text{ wins in } 9) = \binom{9}{2}\left(\frac{5}{18}\right)^2\left(\frac{13}{18}\right)^7$$

$$P(3 \text{ wins, 3rd on 10th throw}) = \binom{9}{2}\left(\frac{5}{18}\right)^2\left(\frac{13}{18}\right)^7\left(\frac{5}{18}\right) = 0 \cdot 0791$$

Question 8 **(65 marks)**

In basketball, players often have to take free throws. When Michael takes his first free throw in any game, the probability that he is successful is 0·7.
For all subsequent free throws in the game, the probability that he is successful is:

- 0·8 if he has been successful on the previous throw
- 0·6 if he has been unsuccessful on the previous throw.

(a) Find the probability that Michael is successful (S) with all three of his first three free throws in a game.

P(S, S, S) =

(b) Find the probability that Michael is unsuccessful (U) with his first two free throws and successful with the third.

P(U, U, S) =

(c) List all the ways that Michael could be successful with his third free throw in a game and hence find the probability that Michael is successful with his third free throw.

(d) **(i)** Let p_n be the probability that Michael is successful with his n^{th} free throw in the game (and hence $(1 - p_n)$ is the probability that Michael is unsuccessful with his n^{th} free throw). Show that $p_{n+1} = 0{\cdot}6 + 0{\cdot}2 p_n$.

(ii) Assume that p is Michael's success rate in the long run; that is, for large values of n, we have $p_{n+1} \approx p_n \approx p$.
Using the result from part **(d) (i)** above, or otherwise, show that $p = 0{\cdot}75$.

(e) For all positive integers n, let $a_n = p - p_n$, where $p = 0{\cdot}75$ as above.

(i) Use the ratio $\frac{a_{n+1}}{a_n}$ to show that a_n is a geometric sequence with common ratio $\frac{1}{5}$.

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(ii) Find the smallest value of n for which $p - p_n < 0{\cdot}00001$.

(f) You arrive at a game in which Michael is playing. You know that he has already taken many free throws, but you do not know what pattern of success he has had.

(i) Based on this knowledge, what is your estimate of the probability that Michael will be successful with his next free throw in the game?

Answer: ________________

(ii) Why would it **not** be appropriate to consider Michael's subsequent free throws in the game as a sequence of Bernoulli trials?

(a) Find the probability that Michael is successful (S) with all three of his first three free throws in a game.

$$P(S, S, S) = 0 \cdot 7 \times 0 \cdot 8 \times 0 \cdot 8 = 0 \cdot 448$$

(b) Find the probability that Michael is unsuccessful (U) with his first two free throws and successful with the third.

$$P(U, U, S) = 0 \cdot 3 \times 0 \cdot 4 \times 0 \cdot 6 = 0 \cdot 072$$

(c) List all the ways that Michael could be successful with his third free throw in a game and hence find the probability that Michael is successful with his third free throw.

S, S, S   U, U, S   S, U, S   U, S, S

$$P(S, S, S) = 0 \cdot 7 \times 0 \cdot 8 \times 0 \cdot 8 = 0 \cdot 448$$
$$P(U, U, S) = 0 \cdot 3 \times 0 \cdot 4 \times 0 \cdot 6 = 0 \cdot 072$$
$$P(S, U, S) = 0 \cdot 7 \times 0 \cdot 2 \times 0 \cdot 6 = 0 \cdot 084$$
$$P(U, S, S) = 0 \cdot 3 \times 0 \cdot 6 \times 0 \cdot 8 = 0 \cdot 144$$

$$P = 0 \cdot 448 + 0 \cdot 072 + 0 \cdot 084 + 0 \cdot 144 = 0 \cdot 748$$

(d) **(i)** Let p_n be the probability that Michael is successful with his n^{th} free throw in the game (and hence $(1 - p_n)$ is the probability that Michael is unsuccessful with his n^{th} free throw). Show that $p_{n+1} = 0\cdot6 + 0\cdot2p_n$.

$$\begin{aligned} p_{n+1} &= \mathrm{P}(\mathrm{S,S}) + \mathrm{P}(\mathrm{U,S}) \\ &= p_n \times 0\cdot8 + (1 - p_n)0\cdot6 \\ &= 0\cdot6 + 0\cdot2p_n \end{aligned}$$

(ii) Assume that p is Michael's success rate in the long run; that is, for large values of n, we have $p_{n+1} \approx p_n \approx p$.
Using the result from part **(d)** **(i)** above, or otherwise, show that $p = 0\cdot75$.

$$p \approx p_n \approx p_{n+1} = 0\cdot6 + 0\cdot2p_n$$
$$\Rightarrow 0\cdot8p_n = 0\cdot6$$
$$\Rightarrow p_n = \frac{0\cdot6}{0\cdot8} = 0\cdot75 = p$$

(e) For all positive integers n, let $a_n = p - p_n$, where $p = 0\cdot75$ as above.

(i) Use the ratio $\frac{a_{n+1}}{a_n}$ to show that a_n is a geometric sequence with common ratio $\frac{1}{5}$.

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{p - p_{n+1}}{p - p_n} \\ &= \frac{0\cdot75 - (0\cdot6 + 0\cdot2p_n)}{0\cdot75 - p_n} \\ &= \frac{0\cdot15 - 0\cdot2p_n}{5(0\cdot15 - 0\cdot2p_n)} = \frac{1}{5} \end{aligned}$$

(ii) Find the smallest value of n for which $p - p_n < 0{\cdot}00001$.

$$a_n = p - p_n$$
$$a_1 = p - p_1 = 0\cdot75 - 0\cdot7 = 0\cdot05$$

$$ar^{n-1} = 0\cdot05(0\cdot2)^{n-1} < 0\cdot00001$$
$$(n-1)\ln 0\cdot2 < \ln 0\cdot0002$$
$$\Rightarrow n - 1 > \frac{\ln 0\cdot0002}{\ln 0\cdot2} = 5\cdot29$$
$$\Rightarrow n > 6\cdot29$$

$$n = 7$$

(f) You arrive at a game in which Michael is playing. You know that he has already taken many free throws, but you do not know what pattern of success he has had.

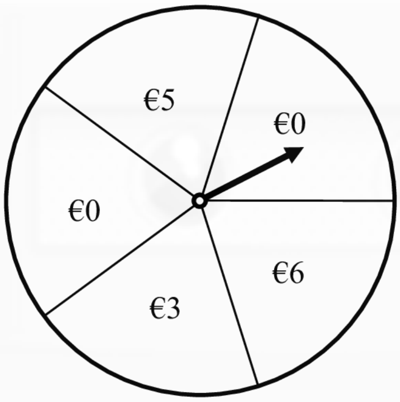
(i) Based on this knowledge, what is your estimate of the probability that Michael will be successful with his next free throw in the game?

Answer: 0·75 **or** p

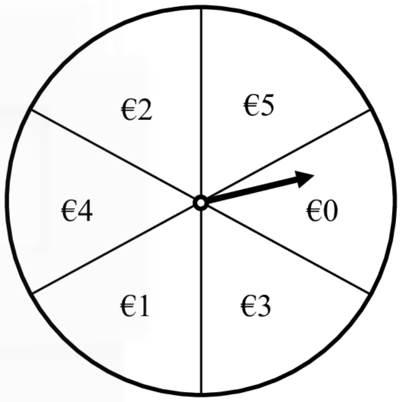
(ii) Why would it **not** be appropriate to consider Michael's subsequent free throws in the game as a sequence of Bernoulli trials?

Events not independent

Two different games of chance, shown below, can be played at a charity fundraiser. In each game, the player spins an arrow on a wheel and wins the amount shown on the sector that the arrow stops in. Each game is fair in that the arrow is just as likely to stop in one sector as in any other sector on that wheel.



Game *A*



Game *B*

(a) John played Game *A* four times and tells us that he has won a total of €8. In how many different ways could he have done this?

(b) To spin either arrow once, the player pays €3. Which game of chance would you expect to be more successful in raising funds for the charity? Give a reason for your answer.

(c) Mary plays Game *B* six times. Find the probability that the arrow stops in the €4 sector exactly twice.

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(a) John played Game A four times and tells us that he has won a total of €8. In how many different ways could he have done this?

5, 3, 0, 0;	3, 5, 0, 0;	0, 5, 3, 0;	0, 3, 5, 0;	
5, 0, 3, 0;	3, 0, 5, 0;	0, 5, 0, 3;	0, 3, 0, 5;	12 ways
5, 0, 0, 3;	3, 0, 0, 5;	0, 0, 5, 3;	0, 0, 3, 5.	

(b) To spin either arrow once, the player pays €3. Which game of chance would you expect to be more successful in raising funds for the charity? Give a reason for your answer.

Expected outcome $E(X) = \sum x.P(x)$

Game A: $E(X) = 0\left(\frac{2}{5}\right) + 3\left(\frac{1}{5}\right) + 5\left(\frac{1}{5}\right) + 6\left(\frac{1}{5}\right) = 2\frac{4}{5}$

Game B: $E(X) = 0\left(\frac{1}{6}\right) + 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) = 2\frac{3}{6} = 2\frac{1}{2}$

Game B - it pays out less money.

Or

On average, over the long term:
Game A pays out €14 for every €15 taken in.
Game B pays out €15 for every €18 taken in.
Game B – it pays out a smaller proportion on the money taken in.

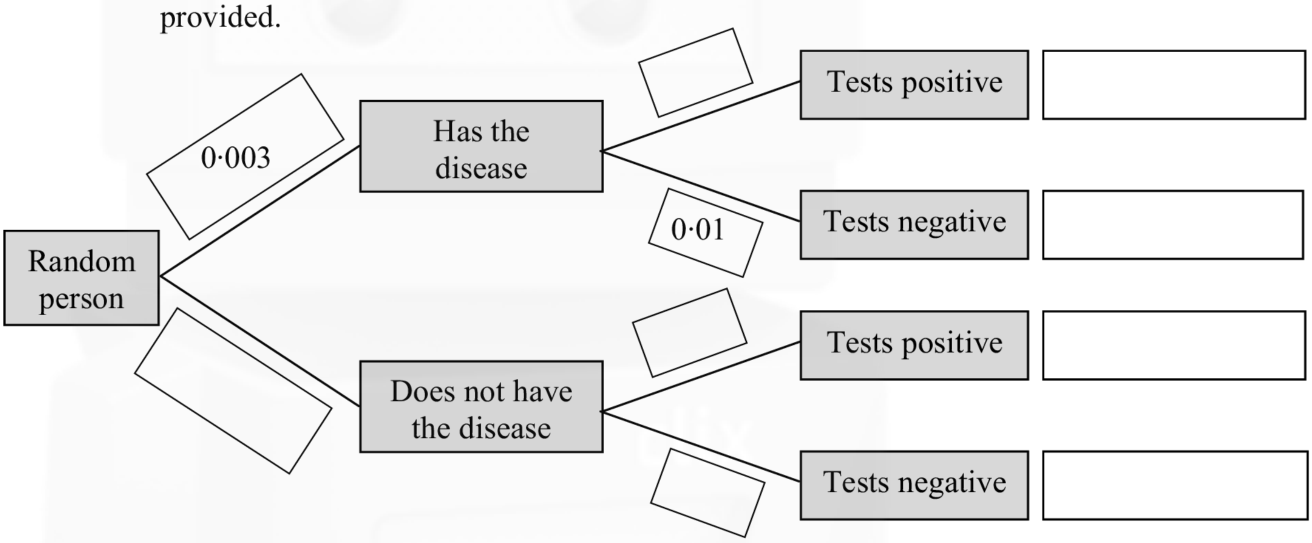
(c) Mary plays Game B six times. Find the probability that the arrow stops in the €4 sector exactly twice.

P(stops in €4 sector exactly twice) $= \binom{6}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^4 = 0\cdot 2$

Blood tests are sometimes used to indicate if a person has a particular disease. Sometimes such tests give an incorrect result, either indicating the person has the disease when they do not (called a false positive) or indicating that they do not have the disease when they do (called a false negative).

It is estimated that 0·3% of a large population have a particular disease. A test developed to detect the disease gives a false positive in 4% of tests and a false negative in 1% of tests. A person picked at random is tested for the disease.

(a) **(i)** Write the probability associated with each branch of the tree diagram in the blank boxes provided.



(ii) Hence, or otherwise, calculate the probability that a person selected at random from the population tests positive for the disease.

(iii) A person tests positive for the disease. What is the probability that the person actually has the disease? Give your answer correct to three significant figures.

(iv) The health authority is considering using a test on the general population with a view to treatment of the disease. Based on your results, do you think that the above test would be an effective way to do this? Give a reason for your answer.

(b) A generic drug used to treat a particular condition has a success rate of 51%. A company is developing two new drugs, A and B, to treat the condition. They carried out clinical trials on two groups of 500 patients suffering from the condition. The results showed that Drug A was successful in the case of 296 patients. The company claims that Drug A is more successful in treating the condition than the generic drug.

(i) Use a hypothesis test at the 5% level of significance to decide whether there is sufficient evidence to justify the company's claim. State the null hypothesis and state your conclusion clearly.

(ii) The null hypothesis was accepted for Drug B. Estimate the greatest number of patients in that trial who could have been successfully treated with Drug B.

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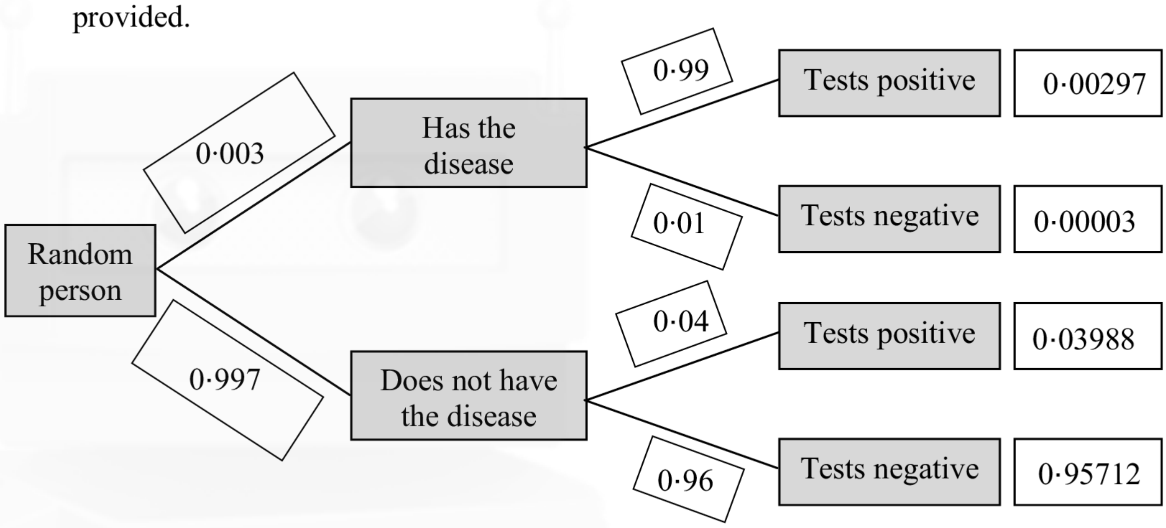
(a) **(i)** Write the probability associated with each branch of the tree diagram in the blank boxes provided.



(ii) Hence, or otherwise, calculate the probability that a person selected at random from the population tests positive for the disease.

$$\text{P(Positive test)} = 0\cdot00297 + 0\cdot03988 = 0\cdot04285$$

(iii) A person tests positive for the disease. What is the probability that the person actually has the disease. Give your answer correct to three significant figures.

$$\text{P(Has disease}|\text{positive test)} = \frac{0\cdot00297}{0\cdot04285} = 0\cdot0693$$

(iv) The health authority is considering using a test on the general population with a view to treatment of the disease. Based on your results, do you think that the above test would be an effective way to do this? Give a reason for your answer.

Test is not very useful.
A person who tests positive has the disease only 7% of the time.

(i) Use a hypothesis test at the 5% level of significance to decide whether there is sufficient evidence to justify the company's claim. State the null hypothesis and state your conclusion clearly.

H_0: The new drug is not more successful than the generic drug.
$p = 0\cdot 51$

95% margin of error $= \dfrac{1}{\sqrt{500}} = 0\cdot 045$

The success rate for the new drug is $\dfrac{296}{500} = 0\cdot 592$.
This is outside the interval $[0\cdot 51 - 0\cdot 045, 0\cdot 51 + 0\cdot 045] = [0\cdot 465, 0\cdot 555]$
Result is significant, reject the null hypothesis.
There is evidence to conclude that the new drug is more successful than the generic.

Or

H_0: The new drug is not more successful than the generic drug.
H_1: The new drug is more successful than the generic drug.
$p = 0\cdot 51$

95% margin of error $= \dfrac{1}{\sqrt{500}} = 0\cdot 045$

The success rate for the new drug is $\dfrac{296}{500} = 0\cdot 592$.
The 95% confidence interval for the population is
$0\cdot 592 - 0\cdot 045 < p < 0\cdot 592 + 0\cdot 045 = 0\cdot 547 < p < 0\cdot 637$
$p = 0\cdot 51$ is outside this interval.
Result is significant, reject the null hypothesis.
There is evidence to conclude that the new drug is more successful than the generic

(ii) The null hypothesis was accepted for Drug *B*. Estimate the greatest number of patients in that trial who could have been successfully treated with Drug *B*.

The result must lie in the interval $[0\cdot 465, 0\cdot 555]$

Thus, $\dfrac{n}{500} < 0\cdot 555 \Rightarrow n < 277\cdot 5$
Hence, 277 patients.

Or

$k - 0\cdot 045 < 0\cdot 51 < k + 0\cdot 045$

$\Rightarrow k - 0\cdot 045 < 0\cdot 51$

$\Rightarrow k < 0\cdot 555$
Number of patients $< 0\cdot 555 \times 500 = 277\cdot 5$
Hence, 277 patients.

Question 1 **(25 marks)**

The random variable X has a discrete distribution. The probability that it takes a value other than 13, 14, 15 or 16 is negligible.

(a) Complete the probability distribution table below and hence calculate $E(X)$, the expected value of X.

x	13	14	15	16
$P(X = x)$	0·383	0·575		0·004

(b) If X is the age, in complete years, on 1 January 2013 of a student selected at random from among all second-year students in Irish schools, explain what $E(X)$ represents.

(c) If ten students are selected at random from this population, find the probability that exactly six of them were 14 years old on 1 January 2013. Give your answer correct to three significant figures.

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(a) Complete the probability table below and hence calculate $E(X)$, the expected value of X.

x	13	14	15	16
$P(X=x)$	0.383	0.575	**0.038**	0.004

Note that the sum of the probabilities must be 1. Therefore $P(15) = 1 - 0.383 - 0.575 - 0.004 = 0.038$.
$E(X) = 13(0.383) + 14(0.575) + 15(0.038) + 16(0.004) = 13.663$.

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(b) If X is the age, in complete years, on 1 January 2013 of a student selected at random from among all second-year students in Irish schools, explain what $E(X)$ represents.

$E(X)$ represents the mean of the ages of all second-year students in Irish schools on 1 January 2013.

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(c) If ten students are selected at random from this population, find the probability that exactly six of them were 14 years old on 1 January 2013. Give your answer correct to three significant figures.

We have 10 Bernoulli trials with $p = P(X = 14) = 0.575$. So the probability of exactly six successes is given by

$$\begin{aligned}\binom{10}{6} p^6 (1-p)^4 &= \frac{10!}{6!4!}(0.575)^6(1-0.575)^4 \\ &= 210(0.575)^6(0.425)^4 \\ &= 0.248\end{aligned}$$

correct to three significant places.

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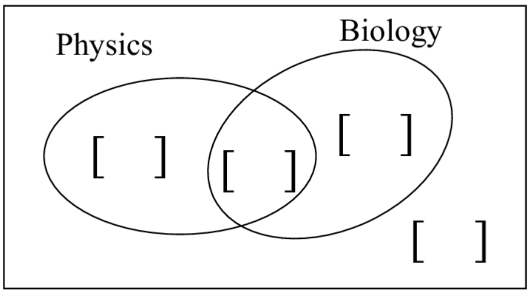
(b) In a class of 30 students, 20 study Physics, 6 study Biology and 4 study both Physics and Biology.

(i) Represent the information on the Venn Diagram.



A student is selected at random from this class.
The events E and F are:

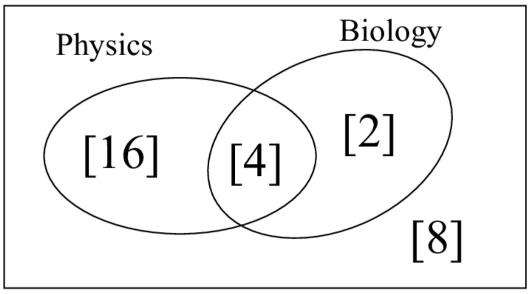
E: The student studies Physics
F: The student studies Biology.

(ii) By calculating probabilities, investigate if the events E and F are independent.

Marking Scheme

(b) In a class of 30 students, 20 study Physics, 6 study Biology and 4 study both Physics and Biology.

(i) Represent the information on the Venn Diagram.



A student is selected at random from this class.
The events E and F are:

E: The student studies Physics
F: The student studies Biology.

(ii) By calculating probabilities, investigate if the events E and F are independent.

$P(E \cap F) = \frac{4}{30}$

$P(E) \times P(F) = \frac{20}{30} \times \frac{6}{30} = \frac{4}{30}$

$P(E \cap F) = P(E) \times P(F) \quad \Rightarrow$ E and F are independent events

Question 4 **(25 marks)**

A certain basketball player scores 60% of the free-throw shots she attempts. During a particular game, she gets six free throws.

(a) What assumption(s) must be made in order to regard this as a sequence of Bernoulli trials?

(b) Based on such assumption(s), find, correct to three decimal places, the probability that:

(i) she scores on exactly four of the six shots

(ii) she scores for the second time on the fifth shot.

Trials are independent of each other.
Probability of success is the same each time.

[Only two outcomes .... (Given)]
[Finite number of throws...... (Given)]

(b) Based on such assumption(s), find, correct to three decimal places, the probability that:

(i) she scores on exactly four of the six shots

$$P(X=4) = {}^{6}C_{4}(0{\cdot}6)^{4}(0{\cdot}4)^{2} = 0{\cdot}31104$$
$$= 0{\cdot}311 \text{ to three decimal places.}$$

(ii) she scores for the second time on the fifth shot.

Exactly one success among first four throws, followed by success on fifth:

$$\left({}^{4}C_{1}(0{\cdot}6)(0{\cdot}4)^{3}\right)(0{\cdot}6) = 0{\cdot}09216$$
$$= 0{\cdot}092 \text{ to three decimal places.}$$

Question 1 **(25 marks)**

The events A and B are such that $P(A) = 0{\cdot}7$, $P(B) = 0{\cdot}5$ and $P(A \cap B) = 0{\cdot}3$.

(a) Find $P(A \cup B)$

(b) Find $P(A | B)$

(c) State whether A and B are independent events, and justify your answer.

Marking Scheme

Question 1 **(25 marks)**

The events A and B are such that $P(A) = 0.7, P(B) = 0.5$ and $P(A \cap B) = 0.3$.

(a) Find $P(A \cup B)$.

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
&= 0.7 + 0.5 - 0.3 \\
&= 0.9
\end{aligned}
$$

(b) Find $P(A|B)$.

$$
\begin{aligned}
P(A|B) &= \frac{P(A \cap B)}{P(B)} \\
&= \frac{0.3}{0.5} \\
&= 0.6
\end{aligned}
$$

(c) State whether A and B are independent events and justify your answer

If A and B are independent events then $P(A \cap B) = P(A)P(B)$.
Here, $P(A \cap B) = 0.3$ but $P(A)P(B) = (0.7)(0.9) = 0.63$.
So A and B are NOT independent events.

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(b) There are 16 girls and 8 boys in a class. Half of these 24 students study French. The probability that a randomly selected girl studies French is 1·5 times the probability that a randomly selected boy studies French. How many of the boys in the class study French?

Let x = number of boys, who study French.

$\therefore 12 - x$ = number of girls who study French.

$$\frac{12-x}{16} = 1{\cdot}5\left(\frac{x}{8}\right)$$

$$96 - 8x = 24x$$

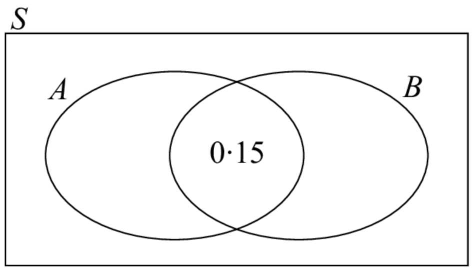
$$32x = 96$$

$$x = 3$$

Three boys study French.

Two events A and B are such that $P(A) = 0{\cdot}2$, $P(A \cap B) = 0{\cdot}15$ and $P(A' \cap B) = 0{\cdot}6$.

(a) Complete this Venn diagram.

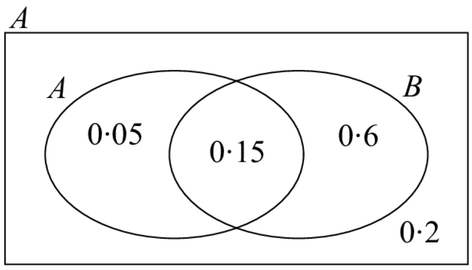


(b) Find the probability that neither A nor B happens.

(c) Find the conditional probability $P(A \mid B)$.

(d) State whether A and B are independent events and justify your answer.

(a) Complete this Venn diagram.



(b) Find the probability that neither A nor B happens.

$$0{\cdot}2.$$

or

$$P(A \cup B)' = 1 - P(A \cup B) = 1 - (0{\cdot}05 + 0{\cdot}15 + 0{\cdot}6) = 0{\cdot}2$$

(c) Find the conditional probability $P(A \mid B)$.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \mid B) = \frac{0{\cdot}15}{0{\cdot}75} = 0{\cdot}2.$$

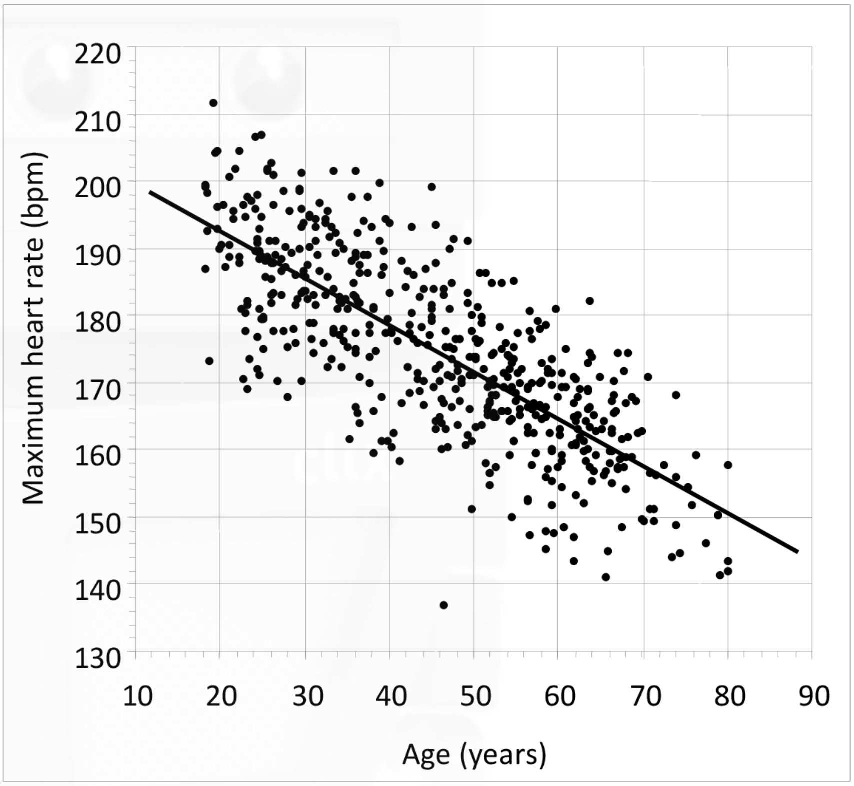
(d) State whether A and B are independent events and justify your answer.

A and B are independent events as, $P(A \mid B) = P(A) = 0{\cdot}2$.

or

A and B are independent events as, $P(A)P(B) = (0{\cdot}2)(0{\cdot}75) = 0{\cdot}15 = P(A \cap B)$.

A person's *maximum heart rate* is the highest rate at which their heart beats during certain extreme kinds of exercise. It is measured in beats per minute (bpm). It can be measured under controlled conditions. As part of a study in 2001, researchers measured the maximum heart rate of 514 adults and compared it to each person's age. The results were like those shown in the scatter plot below.



Source: Simulated data based on:Tanaka H, Monaghan KD, and Seals DR. *Age-predicted maximal heart rate revisited*, J. Am. Coll. Cardiol. 2001;37;153-156.

(a) From the diagram, estimate the correlation coefficient.

Answer:

(b) Circle the *outlier* on the diagram and write down the person's age and maximum heart rate.

Age =

Max. heart rate =

(c) The line of best fit is shown on the diagram. Use the line of best fit to estimate the maximum heart rate of a 44-year-old person.

Answer:

(d) By taking suitable readings from the diagram, calculate the slope of the line of best fit.

(e) Find the equation of the line of best fit and write it in the form: $MHR = a - b \times (\text{age})$, where *MHR* is the maximum heart rate.

(f) The researchers compared their new rule for estimating maximum heart rate to an older rule. The older rule is: $MHR = 220 - \text{age}$. The two rules can give different estimates of a person's maximum heart rate. Describe how the level of agreement between the two rules varies according to the age of the person. Illustrate your answer with two examples.

(g) A particular exercise programme is based on the idea that a person will get most benefit by exercising at 75% of their estimated *MHR*. A 65-year-old man has been following this programme, using the old rule for estimating *MHR*. If he learns about the researchers' new rule for estimating *MHR*, how should he change what he is doing?

(a) From the diagram, estimate the correlation coefficient.

Answer: – 0·75

(b) Circle the *outlier* on the diagram and write down the person's age and maximum heart rate.

Age = 47 years

Max. heart rate = 137 bpm

(c) The line of best fit is shown on the diagram. Use the line of best fit to estimate the maximum heart rate of a 44-year-old person.

Answer: 176 bpm

(d) By taking suitable readings from the diagram, calculate the slope of the line of best fit.

Possible Readings

$(10, 200)$ and $(90, 144)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{144 - 200}{90 - 10} = -\frac{-56}{80} = -\frac{7}{10} \text{ or } m = -0{\cdot}7.$$

(e) Find the equation of the line of best fit and write it in the form: $MHR = a - b \times (\text{age})$, where MHR is the maximum heart rate.

$$y - y_1 = m(x - x_1)$$
$$y - 200 = -0{\cdot}7(x - 10)$$
$$y = -0{\cdot}7x + 207$$
$$MHR = 207 - 0{\cdot}7 \times (age)$$

(f) The researchers compared their new rule for estimating maximum heart rate to an older rule. The older rule is: $MHR = 220 - \text{age}$. The two rules can give different estimates of a person's maximum heart rate. Describe how the level of agreement between the two rules varies according to the age of the person. Illustrate your answer with two examples.

For young adults the old rule gives a greater *MHR* than the new rule.

Adult aged 20

$MHR = 220 - 20 = 200$ bpm (Old rule)

$MHR = 207 - 0{\cdot}7(20) = 193$ bpm (New Rule)

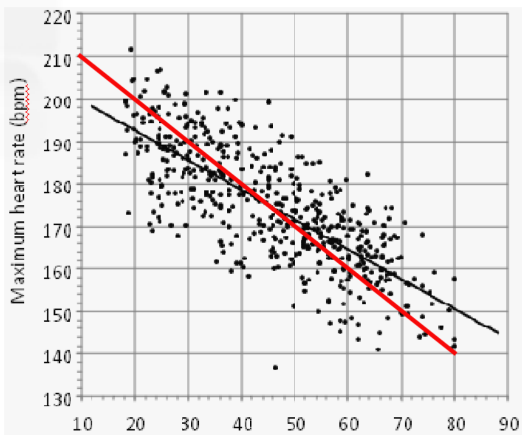
Towards middle age there is a greater agreement between the rules.

For older people the new rule gives a greater *MHR* than the old rule.

Adult aged 70

$MHR = 220 - 70 = 150$ bpm

$MHR = 207 - 0{\cdot}7(70) = 158$ bpm



(g) A particular exercise programme is based on the idea that a person will get most benefit by exercising at 75% of their estimated *MHR*. A 65-year-old man has been following this programme, using the old rule for estimating *MHR*. If he learns about the researchers' new rule for estimating *MHR*, how should he change what he is doing?

He should exercise a bit more intensely.

Using the old rule he exercises to 75% of $(220 - 65) = 116$ bpm.

Using the new rule he can exercise to 75% of $(207 - 0{\cdot}7 \times 65) = 121$ bpm.

A factory manufactures aluminium rods. One of its machines can be set to produce rods of a specified length. The lengths of these rods are normally distributed with mean equal to the specified length and standard deviation equal to 0·2 mm.

The machine has been set to produce rods of length 40 mm.

(a) What is the probability that a randomly selected rod will be less than 39·7 mm in length?

(b) Five rods are selected at random. What is the probability that at least two of them are less than 39·7 mm in length?

(c) The operators want to check whether the setting on the machine is still accurate. They take a random sample of ten rods and measure their lengths. The lengths in millimetres are:

39·5	40·0	39·7	40·2	39·8
39·7	40·2	39·9	40·1	39·6

Conduct a hypothesis test at the 5% level of significance to decide whether the machine's setting has become inaccurate. You should start by clearly stating the null hypothesis and the alternative hypothesis, and finish by clearly stating what you conclude about the machine.

(a) What is the probability that a randomly selected rod will be less than 39·7 mm in length?

$$P(X < 39{\cdot}7) = P\left(Z < \frac{39{\cdot}7 - 40}{0{\cdot}2}\right) = P(Z < -1{\cdot}5)$$
$$= P(z > 1{\cdot}5)$$
$$= 1 - P(Z \le 1{\cdot}5)$$
$$= 1 - 0{\cdot}9332$$
$$= 0{\cdot}0668$$

(b) Five rods are selected at random. What is the probability that at least two of them are less than 39·7 mm in length?

Binomial distribution with $n = 5$, $p = 0{\cdot}0668$, $q = 0{\cdot}9332$.

$$P(X \ge 2) = 1 - P(X < 2) = 1 - [P(X = 1) + P(X = 0)]$$
$$= 1 - \left[\binom{5}{1}(0{\cdot}0668)(0{\cdot}9332)^4 + \binom{5}{0}(0{\cdot}9332)^5\right]$$
$$= 0{\cdot}03895.$$

Or

$$P(X \ge 2) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$$
$$= \binom{5}{2}(0{\cdot}0668)^2(0{\cdot}9332)^3 + \binom{5}{3}(0{\cdot}0668)^3(0{\cdot}9332)^2 + \binom{5}{4}(0{\cdot}0668)^4(.9332) + \binom{5}{5}(0{\cdot}0668)^5$$
$$= 0{\cdot}03895$$

$H_0 : \mu = 40$ mm (null hypothesis)
$H_1 : \mu \ne 40$ mm (alternative hypothesis)

$$\sigma_{\bar{x}} = \frac{0{\cdot}2}{\sqrt{10}} = 0{\cdot}0632456$$

Observed value of $\bar{x} = 39{\cdot}87$

$$\therefore \text{ Observed } z = \frac{39{\cdot}87 - 40}{0{\cdot}0632456} = -2{\cdot}055$$

The critical values for the test are $\pm 1{\cdot}96$

As $-2{\cdot}055 < -1{\cdot}96$, we reject the null hypothesis at the 5% level of significance and we conclude that the machine setting has become inaccurate.