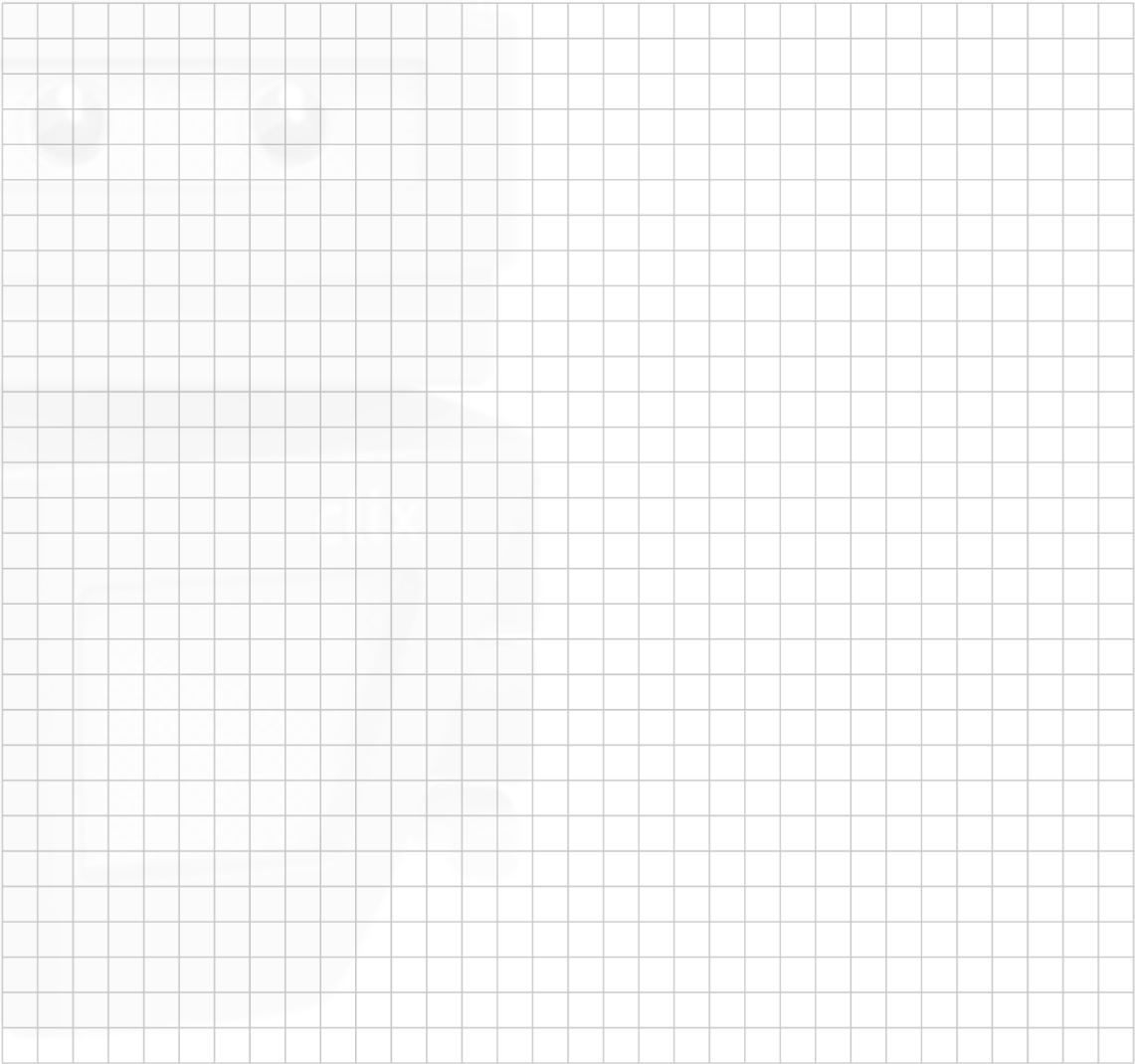


(g) Use the function $q(t) = 3.9e^{-0.05t} \times 10^6$ to find the predicted rate of change of the population in Avalon at the beginning of 2018.

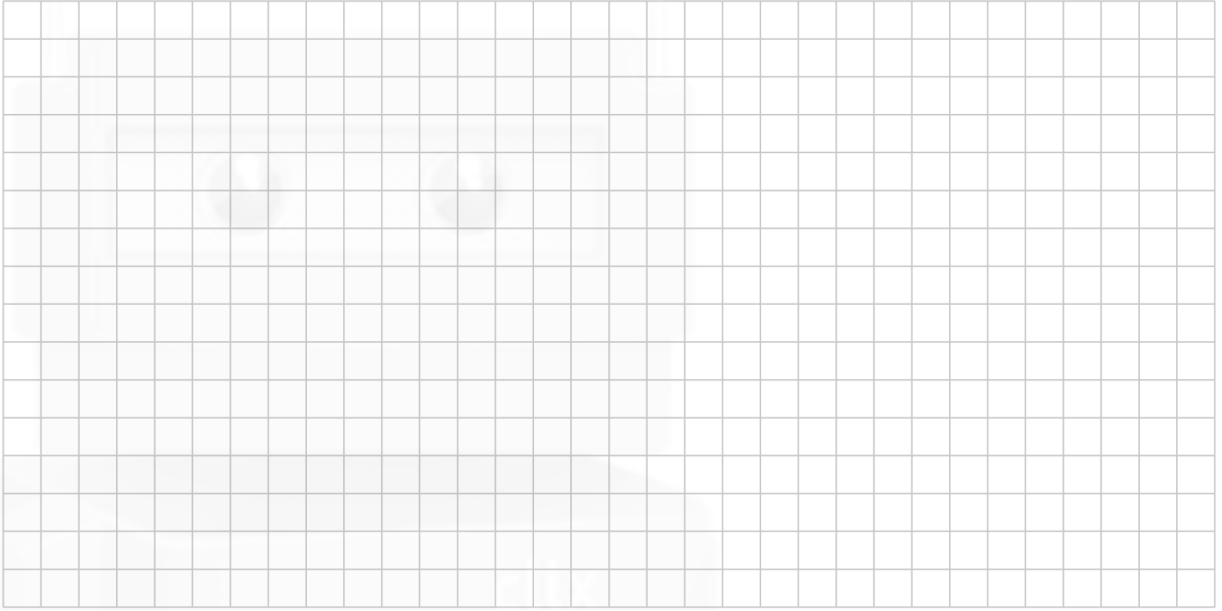


Marking Scheme

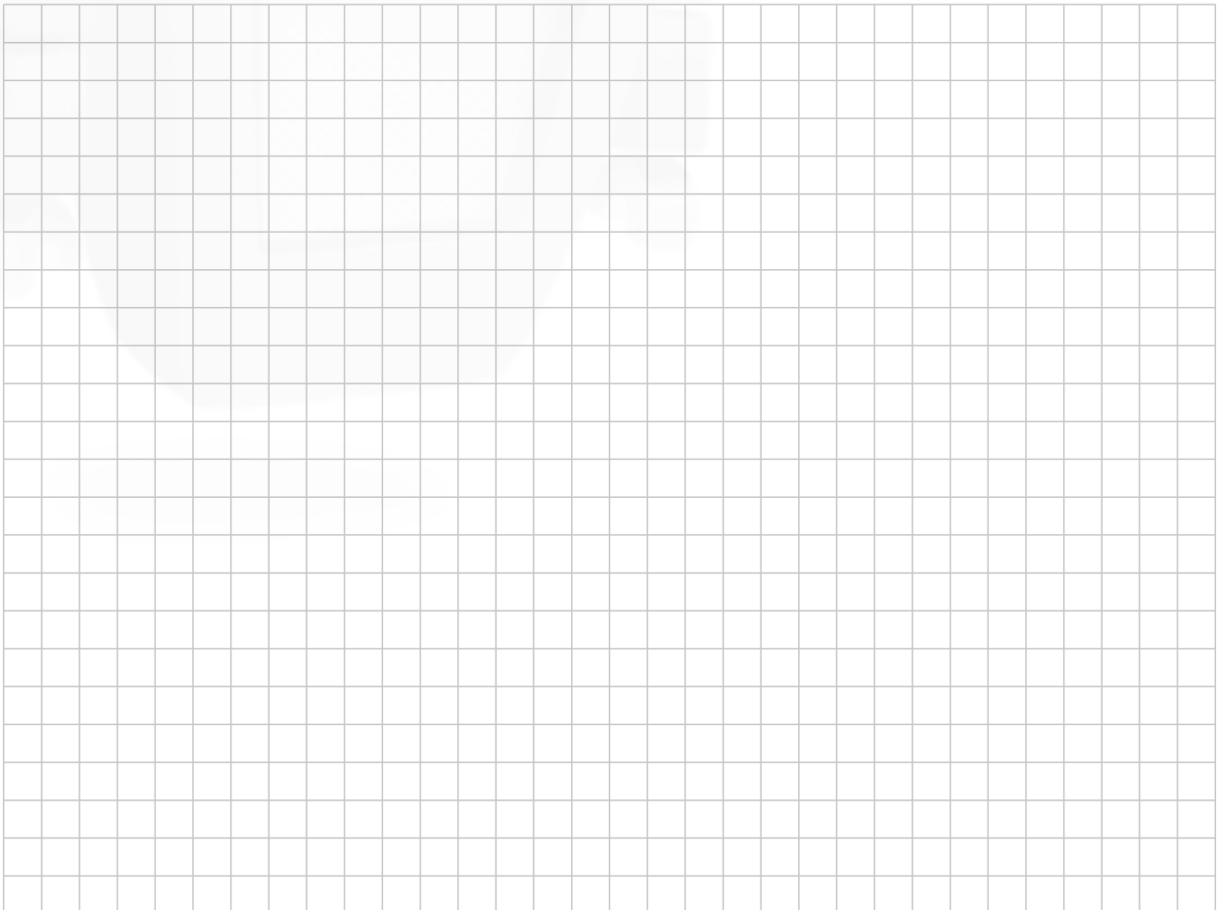
(a)	$Se^{1(0)} \times 10^6 = 1100000$ $S = 1.1$	<p>Scale 10B (0, 4, 10) <i>Partial Credit</i></p> <ul style="list-style-type: none"> equation in S with substitution
(b)	$p(5) = 1.1e^{0.1(5)} \times 10^6$ $= 1.813593 \times 10^6$ $= 1813593$	<p>Scale 10B (0, 4, 10) <i>Partial Credit</i></p> <ul style="list-style-type: none"> substitution into formula for $p(5)$
(c)	$p(6) = 1.1e^{0.6} \times 10^6$ $p(5) = 1.1e^{0.5} \times 10^6$ $p(6) - p(5) = (1.1e^{0.6} - 1.1e^{0.5}) \times 10^6$ $= 0.1907372 \times 10^6$ $= 190737$	<p>Scale 5C (0, 3, 4, 5) <i>Low Partial Credit:</i></p> <ul style="list-style-type: none"> substitution into formula for $p(6)$ use of $p(5)$ from previous part $p(6) - p(5)$ written or implied <p><i>High partial Credit</i></p> <ul style="list-style-type: none"> Formulates $p(6) - p(5)$ with some substitution

(d)	$q(t) = 3.9e^{kt} \times 10^6$ $3709795 = 3.9e^k \times 10^6$ $\frac{3.709795}{3.9} = e^k$ $\log_e \frac{3.709795}{3.9} = k$ $k = -0.0499 = -0.05$	<p>Scale 15C (0, 5, 10, 15) <i>Low Partial Credit</i></p> <ul style="list-style-type: none"> • Either substitution into formula for k • Verifies k value only. <p><i>High Partial Credit</i></p> <ul style="list-style-type: none"> • relevant equation in k
(e)	$p(t) = q(t)$ $1.1e^{0.1t} \times 10^6 = 3.9e^{-0.05t} \times 10^6$ $1.1e^{0.1t} = 3.9e^{-0.05t}$ $\frac{e^{0.1t}}{e^{-0.05t}} = \frac{3.9}{1.1}$ $e^{0.15t} = \frac{39}{11}$ $\ln \frac{39}{11} = 0.15t$ <p>$t = 8.44$ years</p> <p>In 2018 both populations equal</p>	<p>Scale 5C (0, 3, 4, 5) <i>Low Partial Credit</i></p> <ul style="list-style-type: none"> • $p(t) = q(t)$ written or implied <p><i>High Partial Credit</i></p> <ul style="list-style-type: none"> • relevant equation in t
(f)	$\frac{1}{15} \int_0^{15} 3.9e^{-0.05t} \times 10^6 dt$ $\frac{1}{15} \left[\frac{3.9}{-0.05} e^{-0.05(15)} - \frac{3.9}{-0.05} e^{-0.05(0)} \right]$ $\times 10^6$ 2.743694×10^6 2743694	<p>Scale 5C (0, 3, 4, 5) <i>Low Partial Credit:</i></p> <ul style="list-style-type: none"> • integral formulated (with limits) <p><i>High Partial Credit:</i></p> <ul style="list-style-type: none"> • integration with full substitution
(g)	$q(t) = 3.9e^{-0.05t} \times 10^6$ $q'(t) = -0.05(3.9e^{-0.05t} \times 10^6)$ $q'(8) = -0.05(3.9e^{-0.05(8)} \times 10^6)$ $= -130712$	<p>Scale 5C (0, 3, 4, 5) <i>Low Partial Credit</i></p> <ul style="list-style-type: none"> • $q'(t)$ <p><i>High Partial Credit</i></p> <ul style="list-style-type: none"> • $q'(t)$ fully substituted

- (a) Show that $\frac{\cos 7A + \cos A}{\sin 7A - \sin A} = \cot 3A$.



- (b) Given that $\cos 2\theta = \frac{1}{9}$, find $\cos \theta$ in the form $\pm \frac{\sqrt{a}}{b}$, where $a, b \in \mathbb{N}$.



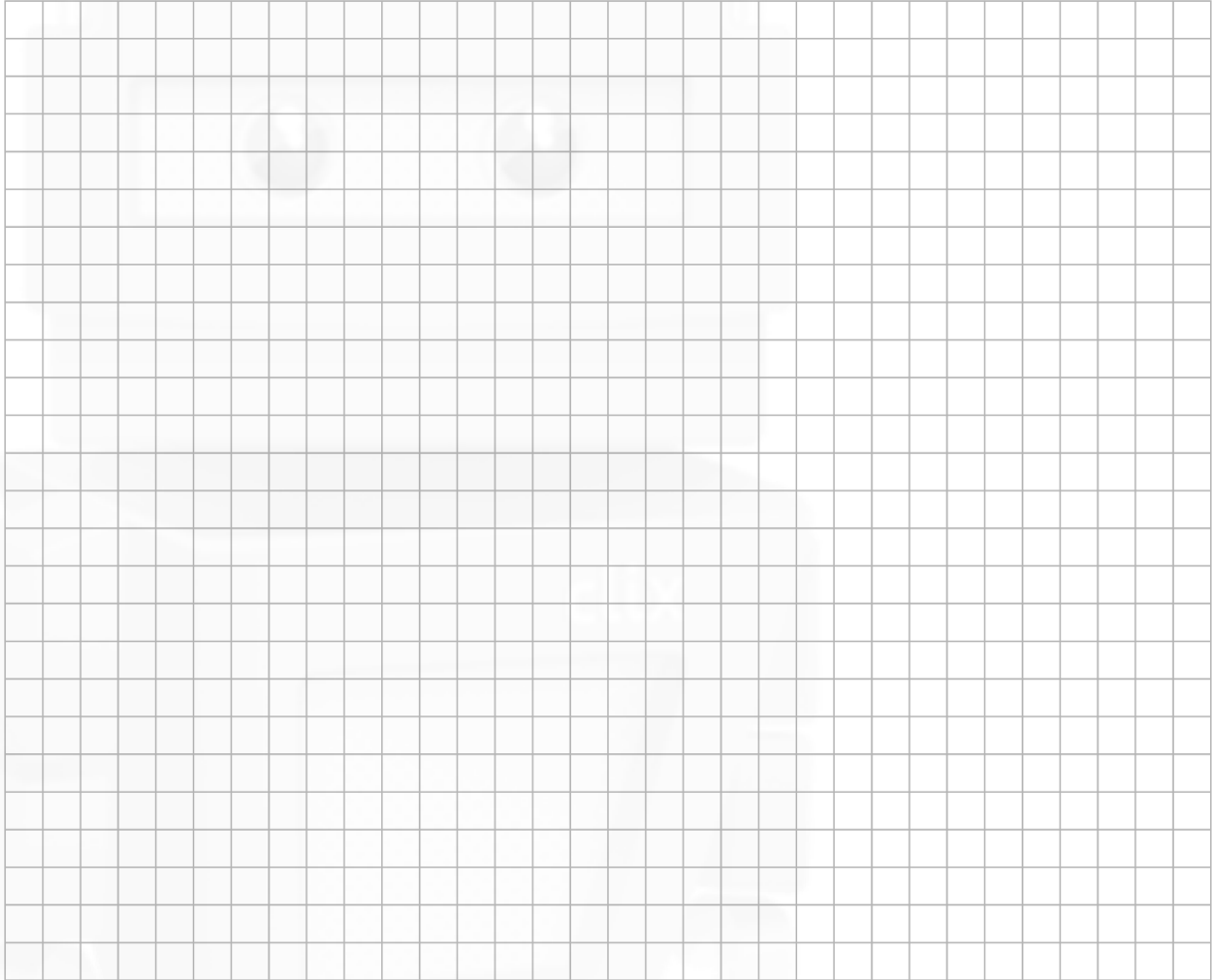
Q3	Model Solution – 25 Marks	Marking Notes
(a)	$\frac{2 \cos \frac{7A + A}{2} \cos \frac{7A - A}{2}}{2 \cos \frac{7A + A}{2} \sin \frac{7A - A}{2}}$ $\frac{2 \cos 4A \cos 3A}{2 \cos 4A \sin 3A}$ $= \frac{\cos 3A}{\sin 3A}$ $= \cot 3A$	<p>Scale 15C (0, 5, 10, 15)</p> <p><i>Low Partial Credit</i></p> <ul style="list-style-type: none"> sum to product formula with some substitution <p><i>High Partial Credit</i></p> <ul style="list-style-type: none"> sum to product formula fully substituted
(b)	<p>Method 1:</p> $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ $= \frac{1}{2}\left(1 + \frac{1}{9}\right) = \frac{5}{9}$ $\cos \theta = \pm \frac{\sqrt{5}}{3}$ <p style="text-align: center;">or</p> <p>Method 2:</p> $\cos 2\theta = 1 - 2\sin^2 \theta = \frac{1}{9}$ $9 - 18 \sin^2 \theta = 1$ $\sin^2 \theta = \frac{4}{9} \Rightarrow \sin \theta = \pm \frac{2}{3} \Rightarrow \cos \theta = \pm \frac{\sqrt{5}}{3}$ <p style="text-align: center;">or</p> <p>Method 3:</p> $\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1}{9}$ $9 - 9 \tan^2 \theta = 1 + \tan^2 \theta$ $\tan^2 \theta = \frac{4}{5}$ $\Rightarrow \tan \theta = \pm \frac{2}{\sqrt{5}} \Rightarrow \cos \theta = \pm \frac{\sqrt{5}}{3}$	<p>Scale 10D (0, 3, 5, 8, 10)</p> <p><i>Low Partial Credit</i></p> <ul style="list-style-type: none"> Use of a relevant formula in $\cos 2\theta$ $\cos^{-1}\left(\frac{1}{9}\right) = 83.62^\circ$ $\theta = 41.8^\circ$ <p><i>Mid Partial Credit</i></p> <ul style="list-style-type: none"> correct substitution (method 1) expression in $\sin^2 \theta$ (method 2) expression in $\tan^2 \theta$ (method 3) expression in $\cos^2 \theta$ (method 4) $\theta = 41.8^\circ$ and $\theta = 132.2^\circ$ or $\theta = 221.8^\circ$ <p><i>High Partial Credit</i></p> <ul style="list-style-type: none"> one value only (e.g. $+\frac{\sqrt{5}}{3}$) values found for $\cos 41.8^\circ$ and $\cos 138.2^\circ$ or $\cos 221.8^\circ$

Marking Scheme

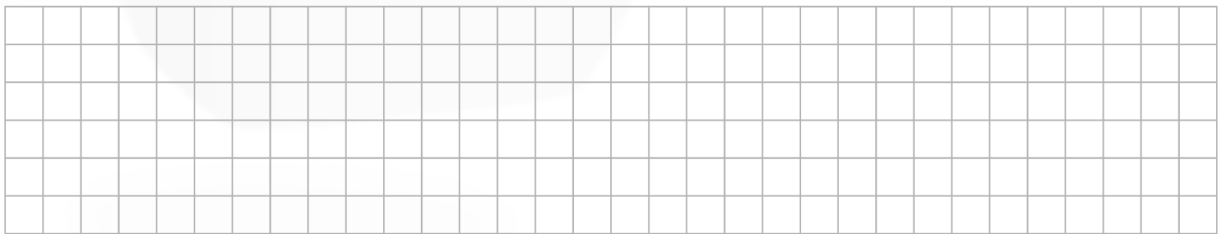
<p>(b) (i)</p>	$p = \log_a 2, \quad q = \log_a 3$ $\log_a \frac{8}{3} = \log_a 8 - \log_a 3$ $= \log_a (2)^3 - \log_a 3$ $= 3 \log_a 2 - \log_a 3$ $= 3p - q$	<p>Scale 5C (0, 2, 4, 5)</p> <p><i>Low Partial Credit</i></p> <ul style="list-style-type: none"> • $\log_a 8 - \log_a 3$ <p><i>High Partial Credit</i></p> <ul style="list-style-type: none"> • $\log_a 8 = 3 \log_a 2$ (and/or $= 3p$)
<p>(ii)</p>	$\log_a \frac{9a^2}{16} = \log_a (3a)^2 - \log_a (2)^4$ $= 2 \log_a 3 + 2 \log_a a - 4 \log_a 2$ $= 2q + 2(1) - 4p$ $= 2q + 2 - 4p$	<p>Scale 5D (0, 2, 3, 4, 5)</p> <p><i>Low Partial Credit</i></p> <ul style="list-style-type: none"> • $\log_a 9a^2 - \log_a 16$ <p><i>Mid Partial Credit</i></p> <ul style="list-style-type: none"> • $2 \log_a 3$ • $2 \log_a a$ • $4 \log_a 2$ • $4p$ or $2q$ or 2 <p><i>High Partial Credit</i></p> <ul style="list-style-type: none"> • $2(\log_a 3 + \log_a a) - 4 \log_a 2$ or equivalent

Question 2**(25 marks)**

- (a) (i) Prove by induction that, for any n , the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.



- (ii) Find the sum of all the natural numbers from 51 to 100, inclusive.



- (b) Given that $p = \log_c x$, express $\log_c \sqrt{x} + \log_c (cx)$ in terms of p .



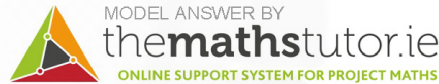
First we check that the statement is true for $n = 1$. The sum of the first 1 natural numbers is 1, and when $n = 1$ we have $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$. So the statement is true for $n = 1$. Now suppose that the statement is true for some $n \geq 1$. So

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}.$$

Now, add $n + 1$ to both sides and we get

$$\begin{aligned} 1 + 2 + \dots + n + (n + 1) &= \frac{n(n+1)}{2} + (n + 1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

So the sum of the first $n + 1$ natural numbers is $\frac{(n+1)((n+1)+1)}{2}$, which completes the induction step. Therefore, by induction, the statement is true for all natural numbers n .



(ii) Find the sum of all the natural numbers from 51 to 100, inclusive.

By part (i), we know that

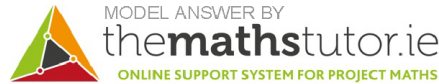
$$1 + 2 + \dots + 100 = \frac{100(101)}{2} = 5050.$$

We also know that

$$1 + 2 + \dots + 50 = \frac{50(51)}{2} = 1275.$$

Subtracting the second equation from the first yields

$$51 + 52 + \dots + 100 = 5050 - 1275 = 3775.$$



(b) Given that $p = \log_c x$, express $\log_c \sqrt{x} + \log_c(cx)$ in terms of p .

We know that

$$\log_c \sqrt{x} = \log_c x^{\frac{1}{2}} = \frac{1}{2} \log_c x = \frac{1}{2}p$$

using the power law for logarithms.

Also,

$$\log_c(cx) = \log_c c + \log_c x = \log_c c + p$$

using the product rule for logarithms.

But $\log_c c = 1$ since $c^1 = c$. Therefore

$$\log_c \sqrt{x} + \log_c(cx) = \frac{1}{2}p + 1 + p = \frac{3p}{2} + 1.$$

Scientists can estimate the age of certain ancient items by measuring the proportion of carbon-14, relative to the total carbon content in the item. The formula used is $Q = e^{-\frac{0.693t}{5730}}$, where Q is the proportion of carbon-14 remaining and t is the age, in years, of the item.

- (a) An item is 2000 years old. Use the formula to find the proportion of carbon-14 in the item.

- (b) The proportion of carbon-14 in an item found at Lough Boora, County Offaly, was 0.3402. Estimate, correct to two significant figures, the age of the item.

