Question 7 **(55 marks)**

Sometimes it is possible to predict the future population in a city using a function.
The population in Sapphire City, over time, can be predicted using the following function:

$$p(t) = Se^{0\cdot 1t} \times 10^6.$$

The population in Avalon, over time, can be predicted using the following function:

$$q(t) = 3\cdot 9e^{kt} \times 10^6.$$

In the functions above, t is time, in years; $t = 0$ is the beginning of 2010; and both S and k are constants.

(a) The population in Sapphire City at the beginning of 2010 is 1 100 000 people.
Find the value of S.

(b) Find the predicted population in Sapphire City at the beginning of 2015.

(c) Find the predicted change in the population in Sapphire City during 2015.

(d) The predicted population in Avalon at the beginning of 2011 is 3 709 795 people.
Write down and solve an equation in k to show that $k = -0{\cdot}05$, correct to 2 decimal places.

(e) Find the year during which the populations in both cities will be equal.

(f) Find the predicted average population in Avalon from the beginning of 2010 to the beginning of 2025.

This question is continued on the next page

(g) Use the function $q(t) = 3{\cdot}9e^{-0{\cdot}05t} \times 10^6$ to find the predicted rate of change of the population in Avalon at the beginning of 2018.

(a)	$Se^{\cdot 1(0)} \times 10^6 = 1100000$ $S = 1{\cdot}1$	**Scale 10B (0, 4, 10)** *Partial Credit* • equation in S with substitution
(b)	$p(5) = 1{\cdot}1e^{0{\cdot}1(5)} \times 10^6$ $= 1{\cdot}813593 \times 10^6$ $= 1813593$	**Scale 10B (0, 4, 10)** *Partial Credit* • substitution into formula for $p(5)$
(c)	$p(6) = 1{\cdot}1e^{0{\cdot}6} \times 10^6$ $p(5) = 1{\cdot}1e^{0{\cdot}5} \times 10^6$ $p(6) - p(5) = (1{\cdot}1e^{0{\cdot}6} - 1{\cdot}1e^{0{\cdot}5}) \times 10^6$ $= 0{\cdot}1907372 \times 10^6$ $= 190737$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* • substitution into formula for $p(6)$ • use of $p(5)$ from previous part • $p(6) - p(5)$ written or implied *High partial Credit* • Formulates $p(6) - p(5)$ with some substitution

(d)	$$q(t) = 3{\cdot}9e^{kt} \times 10^6$$ $$3709795 = 3{\cdot}9e^{k} \times 10^6$$ $$\frac{3{\cdot}709795}{3{\cdot}9} = e^k$$ $$\log_e \frac{3{\cdot}709795}{3{\cdot}9} = k$$ $$k = -0{\cdot}0499 = -0{\cdot}05$$	**Scale 15C (0, 5, 10, 15)** *Low Partial Credit* • Either substitution into formula for k • Verifies k value only. *High Partial Credit* • relevant equation in k
(e)	$$p(t) = q(t)$$ $$1{\cdot}1e^{0{\cdot}1t} \times 10^6 = 3{\cdot}9e^{-0{\cdot}05t} \times 10^6$$ $$1{\cdot}1e^{0{\cdot}1t} = 3{\cdot}9e^{-0{\cdot}05t}$$ $$\frac{e^{0{\cdot}1t}}{e^{-0{\cdot}05t}} = \frac{3{\cdot}9}{1{\cdot}1}$$ $$e^{0{\cdot}15t} = \frac{39}{11}$$ $$\ln\frac{39}{11} = 0{\cdot}15t$$ $t = 8{\cdot}44$ years In 2018 both populations equal	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit* • $p(t) = q(t)$ written or implied *High Partial Credit* • relevant equation in t
(f)	$$\frac{1}{15}\int_0^{15} 3{\cdot}9e^{-0{\cdot}05t} \times 10^6 dt$$ $$\frac{1}{15}\left[\frac{3{\cdot}9}{-0{\cdot}05}e^{-0{\cdot}05(15)} - \frac{3{\cdot}9}{-0{\cdot}05}e^{-0{\cdot}05(0)}\right]$$ $$\times 10^6$$ $$2{\cdot}743694 \times 10^6$$ $$2743694$$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* • integral formulated (with limits) *High Partial Credit:* • integration with full substitution
(g)	$$q(t) = 3.9e^{-0.05t} \times 10^6$$ $$q'(t) = -0{\cdot}05(3{\cdot}9e^{-0{\cdot}05t} \times 10^6)$$ $$q'(8) = -0{\cdot}05\left(3{\cdot}9e^{-0{\cdot}05(8)} \times 10^6\right)$$ $$= -130712$$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit* • $q'(t)$ *High Partial Credit* • $q'(t)$ fully substituted

(a) Show that $\dfrac{\cos 7A + \cos A}{\sin 7A - \sin A} = \cot 3A$.

(b) Given that $\cos 2\theta = \dfrac{1}{9}$, find $\cos\theta$ in the form $\pm\dfrac{\sqrt{a}}{b}$, where $a, b \in \mathbb{N}$.

Q3	Model Solution – 25 Marks	Marking Notes
(a)	$$\frac{2\cos\frac{7A+A}{2}\cos\frac{7A-A}{2}}{2\cos\frac{7A+A}{2}\sin\frac{7A-A}{2}}$$ $$\frac{2\cos 4A\cos 3A}{2\cos 4A\sin 3A}$$ $$=\frac{\cos 3A}{\sin 3A}$$ $$=\cot 3A$$	Scale 15C (0, 5, 10, 15) *Low Partial Credit* • sum to product formula with some substitution *High Partial Credit* • sum to product formula fully substituted
(b)	Method 1: $$\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$$ $$=\frac{1}{2}\left(1+\frac{1}{9}\right)=\frac{5}{9}$$ $$\cos\theta=\pm\frac{\sqrt{5}}{3}$$ **or** Method 2: $$\cos 2\theta=1-2\sin\theta=\frac{1}{9}$$ $$9-18\sin^2\theta=1$$ $$\sin^2\theta=\frac{4}{9}\Rightarrow\sin\theta=\pm\frac{2}{3}\Rightarrow\cos\theta=\pm\frac{\sqrt{5}}{3}$$ **or** Method 3: $$\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{1}{9}$$ $$9-9\tan^2\theta=1+\tan^2\theta$$ $$\tan^2\theta=\frac{4}{5}$$ $$\Rightarrow\tan\theta=\pm\frac{2}{\sqrt{5}}\Rightarrow\cos\theta=\pm\frac{\sqrt{5}}{3}$$	Scale 10D (0, 3, 5, 8, 10) *Low Partial Credit* • Use of a relevant formula in $\cos 2\theta$ • $\cos^{-1}\left(\frac{1}{9}\right)=83{\cdot}62°$ • $\theta=41{\cdot}8°$ *Mid Partial Credit* • correct substitution (method 1) • expression in $\sin^2\theta$ (method 2) • expression in $\tan^2\theta$ (method 3) • expression in $\cos^2\theta$ (method 4) • $\theta=41{\cdot}8°$ and $\theta=132{\cdot}2°$ or $\theta=221{\cdot}8°$ *High Partial Credit* • one value only (e.g. $+\frac{\sqrt{5}}{3}$) • values found for $\cos 41{\cdot}8°$ and $\cos 138{\cdot}2°$ or $\cos 221{\cdot}8°$

or

Method 4:

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$1-\cos^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$2-2\cos^2\theta = 1-\cos 2\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} = \frac{1+\frac{1}{9}}{2}$$

$$\cos^2\theta = \frac{5}{9}$$

$$\cos\theta = \pm\frac{\sqrt{5}}{3}$$

2016

(b) Given $\log_a 2 = p$ and $\log_a 3 = q$, where $a > 0$, write each of the following in terms of p and q:

(i) $\log_a \frac{8}{3}$

(ii) $\log_a \frac{9a^2}{16}$.

(b) **(i)**	$p = \log_a 2$, $q = \log_a 3$ $\log_a \frac{8}{3} = \log_a 8 - \log_a 3$ $= \log_a (2)^3 - \log_a 3$ $= 3\log_a 2 - \log_a 3$ $= 3p - q$	Scale 5C (0, 2, 4, 5) *low Partial Credit* • $\log_a 8 - \log_a 3$ *High Partial Credit* • $\log_a 8 = 3\log_a 2$ (and/or = 3p)
(ii)	$\log_a \frac{9a^2}{16} = \log_a (3a)^2 - \log_a (2)^4$ $= 2\log_a 3 + 2\log_a a - 4\log_a 2$ $= 2q + 2(1) - 4p$ $= 2q + 2 - 4p$	Scale 5D (0, 2, 3, 4, 5) *Low Partial Credit* • $\log_a 9a^2 - \log_a 16$ *Mid Partial Credit* • $2\log_a 3$ • $2\log_a a$ • $4\log_a 2$ • $4p$ or $2q$ or 2 *High Partial Credit* • $2(\log_a 3 + \log_a a) - 4\log_a 2$ or equivalent

Question 2 **(25 marks)**

(a) **(i)** Prove by induction that, for any n, the sum of the first n natural numbers is $\dfrac{n(n+1)}{2}$.

(ii) Find the sum of all the natural numbers from 51 to 100, inclusive.

(b) Given that $p = \log_c x$, express $\log_c \sqrt{x} + \log_c (cx)$ in terms of p.

First we check that the statement is true for $n = 1$. The sum of the first 1 natural numbers is 1, and when $n = 1$ we have $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$. So the statement is true for $n = 1$. Now suppose that the statement is true for some $n \geq 1$. So

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Now, add $n + 1$ to both sides and we get

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

So the sum of the first $n+1$ natural numbers is $\frac{(n+1)((n+1)+1)}{2}$, which completes the induction step. Therefore, by induction, the statement is true for all natural numbers n.

(ii) Find the sum of all the natural numbers from 51 to 100, inclusive.

By part (i), we know that

$$1 + 2 + \cdots + 100 = \frac{100(101)}{2} = 5050.$$

We also know that

$$1 + 2 + \cdots + 50 = \frac{50(51)}{2} = 1275.$$

Subtracting the second equation from the first yields

$$51 + 52 + \cdots + 100 = 5050 - 1275 = 3775.$$

(b) Given that $p = \log_c x$, express $\log_c \sqrt{x} + \log_c(cx)$ in terms of p.

We know that

$$\log_c \sqrt{x} = \log_c x^{\frac{1}{2}} = \frac{1}{2} \log_c x = \frac{1}{2} p$$

using the power law for logarithms.
Also,

$$\log_c(cx) = \log_c c + \log_c x = \log_c c + p$$

using the product rule for logarithms.
But $\log_c c = 1$ since $c^1 = c$. Therefore

$$\log_c \sqrt{x} + \log_c(cx) = \frac{1}{2} p + 1 + p = \frac{3p}{2} + 1.$$

Scientists can estimate the age of certain ancient items by measuring the proportion of carbon−14, relative to the total carbon content in the item. The formula used is $Q = e^{-\frac{0{\cdot}693t}{5730}}$, where Q is the proportion of carbon−14 remaining and t is the age, in years, of the item.

(a) An item is 2000 years old. Use the formula to find the proportion of carbon−14 in the item.

(b) The proportion of carbon−14 in an item found at Lough Boora, County Offaly, was 0·3402. Estimate, correct to two significant figures, the age of the item.

$$Q = e^{-\frac{0\cdot693t}{5730}} = e^{-\frac{0\cdot693\times2000}{5730}} = 0\cdot7851$$

$$Q = e^{-\frac{0\cdot693t}{5730}} = 0\cdot3402$$

$$\Rightarrow -\frac{0\cdot693t}{5730} = \ln 0\cdot3402$$

$$\Rightarrow t = -\frac{5730\times\ln 0\cdot3402}{0\cdot693} \approx 8915 \approx 8900 \text{ years}$$

2012

(b) Given that $p = \log_c x$, express $\log_c \sqrt{x} + \log_c (cx)$ in terms of p.

Marking Scheme