Topic: Worked Solutions & Mark Scheme
Core Concepts of Limits & Rates of Change
Featuring Casio fx-CG50 Calculator Instructions

Marks are awarded for Method (M), Accuracy (A), and Reasoning (R). (M1) or (A1) indicates an implied mark.
Note on GDC usage: Solutions below explicitly use Casio fx-CG50 syntax.

1. **[Paper 1 Style, Short Answer, Easy, 4 marks]**

   (a) Substituting $x = 5$ into the function gives $\frac{5^2-25}{5-5} = \frac{0}{0}$. **(M1)**
   Division by zero is mathematically undefined. **R1**

   (b) Use CG50 `MENU 7 (Table)`. Enter $Y1 = (x^2 - 25)/(x - 5)$. Set `SET` or manually type values. **(M1)**
   Values: $f(4.9) = 9.9$, $f(4.99) = 9.99$, $f(5.01) = 10.01$, $f(5.1) = 10.1$. **A1**
   As x approaches 5 from both sides, the value of the function approaches 10.
   Limit $= 10$. **A1**

2. **[Paper 1 Style, Short Answer, Easy, 4 marks]**

   (a) $s(1) = 3(1)^2 - 2(1) + 1 = 2$ m. **A1**
   $s(4) = 3(4)^2 - 2(4) + 1 = 48 - 8 + 1 = 41$ m. **A1**

   (b) Average velocity $= \frac{\Delta s}{\Delta t} = \frac{s(4)-s(1)}{4-1}$. **(M1)**
   $= \frac{41-2}{3} = \frac{39}{3} = 13$ m/s. **A1**

3. **[Paper 1 Style, Short Answer, Easy, 4 marks]**

   (a) Initial population means $t = 0$. $P(0) = 500(1.15)^0 = 500$. **A1**

   (b) Use CG50 `MENU 1 (Run-Matrix)`. Press `OPTN -> F4 (CALC) -> F2 (d/dx)`. **(M1)**
   Enter: $\frac{d}{dx}\left(500 \times 1.15^x\right)\Big|_{x=8}$.
   Result $= 213.785 \cdots \approx 214$. **A1**
   Interpretation: At exactly 8 hours, the population is increasing at a rate of 214 bacteria per hour. **R1**

4. **[Paper 1 Style, Short Answer, Medium, 4 marks]**

   (a) $C(100) = 1500 + 1200 - 500 = 2200$.
   $C(120) = 1500 + 1440 - 720 = 2220$. **(M1)**
   Average rate $= \frac{C(120)-C(100)}{120-100} = \frac{2220-2200}{20} = \frac{20}{20} = \$1$ per item. **A1**

   (b) Use CG50 `d/dx`. Enter $\frac{d}{dx}(1500 + 12x - 0.05x^2)\big|_{x=100}$. **(M1)**
   Result = \$2 per item. **A1**

5. **[Paper 1 Style, Short Answer, Medium, 5 marks]**

   (a) Use CG50 `d/dx`. Enter $\frac{d}{dx}(1500 \ln(x + 1))\big|_{x=6}$. **(M1)**
   Result $= 214.285 \cdots \approx 214$. **A1**

   (b) Dollars per month. **A1**

   (c) $R(6)$ is the **total cumulative revenue** earned over the first 6 months. **R1**
   $R'(6)$ is the **current rate of change**, meaning how fast revenue is flowing in at that exact instant (month 6). **R1**

6. **[Paper 1 Style, Short Answer, Medium, 5 marks]**

   (a) Substitute a large value like $t = 1000$ into the CG50: $25000e^{-150} \approx 0$. **(M1)**
   Limit is \$0. **A1**

   (b) Over a very long time, the car depreciates until it is completely worthless. **R1**

   (c) Use CG50 `d/dx`. Enter $\frac{d}{dx}(25000e^{-0.15x})\big|_{x=3}$. **(M1)**
   Result $= -2391.11 \cdots \approx -2390$. **A1**
   The negative sign indicates the value of the car is **decreasing** (losing \$2390 per year at that exact moment). **R1**

7. **[Paper 1 Style, Short Answer, Medium, 5 marks]**

   (a) Using `MENU 7 (Table)`:
   $x = 3.1 \implies f(x) = 72$; $x = 3.01 \implies f(x) = 702$; $x = 3.001 \implies f(x) = 7002$.
   Approaches $+\infty$. **(M1)A1**

   (b) Using Table:
   $x = 2.9 \implies f(x) = -68$; $x = 2.99 \implies f(x) = -698$; $x = 2.999 \implies f(x) = -6998$.
   Approaches $-\infty$. **A1**

   (c) For a limit to exist, the function must approach the same finite value from both the left and the right side. Since the sides diverge to $+\infty$ and $-\infty$, the limit does not exist (vertical asymptote). **R1R1**

8. **[Paper 1 Style, Short Answer, Medium, 6 marks]**

(a) Use CG50 `d/dx`. Enter $\frac{d}{dx}(2^x - x^2)\big|_{x=3}$. **(M1)**
Result $= -0.454822\cdots \approx -0.455$. **A1**

(b) $h(3) = 8 - 9 = -1$.
For $k = 0.01$: $\frac{h(3.01)-h(3)}{0.01} = \frac{-1.00451-(-1)}{0.01} = -0.451\ldots$. **(M1)A1**
For $k = 0.001$: $\frac{h(3.001)-h(3)}{0.001} = -0.454\ldots$. **A1**

(c) As the interval k gets smaller (approaches 0), the average rate of change fraction approaches the instantaneous rate of change (the derivative) evaluated in part (a). This is the definition of a limit from first principles. **R1**

9. **[Paper 2 Style, Longer Question, Hard, 6 marks]**

(a) $V(0) = 500(1-0)^2 = 500$ L. **A1**
$V(10) = 500(1-0.5)^2 = 500(0.25) = 125$ L. **A1**

(b) Average rate $= \frac{V(10)-V(0)}{10-0} = \frac{125-500}{10} = \frac{-375}{10}$. **(M1)**
$= -37.5$ L/min. **A1**

(c) Use CG50 `d/dx`. Enter $\frac{d}{dx}(500(1-x/20)^2)\big|_{x=10}$. **(M1)**
Result $= -25$ L/min. **A1**

(d) The average rate (-37.5) reflects the overall speed of draining over the entire 10-minute block. The instantaneous rate (-25) shows the exact speed at the 10-minute mark. Because the tank is getting emptier, the pressure drops, so it drains slower at $t = 10$ than it did at the start. **R1**

10. **[Paper 1 Style, Short Answer, Hard, 5 marks]**

(a) Substitute a very large value for t. $e^{-\text{large}} \approx 0$. **(M1)**
Limit $= 20 + 70(0) = 20°$C. **A1**

(b) This is the ambient room temperature that the coffee cools down to over a long period of time (horizontal asymptote). **R1**

(c) We want $T'(t) = -2$.
On CG50 `MENU 5 (Graph)`, enter $Y1 = \frac{d}{dx}(20 + 70e^{-0.08x})$ and $Y2 = -2$. **(M1)**
Use `G-Solv -> ISCT` (Intersection).
$x = 12.869\cdots \implies t \approx 12.9$ minutes. **A1**

11. **[Paper 1 Style, Short Answer, Hard, 5 marks]**

(a) **Ensure calculator is in Radians.** **(M1)**
Use CG50 `d/dx`. Enter $\frac{d}{dx}(x\sin(x))\big|_{x=2}$.
Result $= 0.077003\cdots \approx 0.0770$. **A1**

(b) In `MENU 5 (Graph)`, set $Y1 = \frac{d}{dx}(x\sin x)$. **(M1)**
Set `V-Window` X from 0 to 5. Draw.
Press `G-Solv -> ROOT`.
$x = 2.0287\cdots \approx 2.03$ (and 4.91). **A1**

(c) A gradient of 0 ($f'(x) = 0$) corresponds to a stationary point (local maximum or minimum) on the original curve $y = f(x)$. **R1**

12. **[Paper 1 Style, Short Answer, Hard, 5 marks]**

(a) $N(0) = \frac{1000}{1+19e^0} = \frac{1000}{20} = 50$ users. **(M1)A1**

(b) As $t \to \infty$, $e^{-0.5t} \to 0$.
Limit $= \frac{1000}{1+0} = 1000$ users. **A1**

(c) We need the maximum of the derivative.
In `MENU 5 (Graph)`, enter $Y1 = \frac{d}{dx}\left(\frac{1000}{1+19e^{-0.5x}}\right)$. **(M1)**
Press `G-Solv -> MAX`.
$x = 5.8888\cdots \approx 5.89$ hours.
$y = 125$ users/hour.
Max rate is 125 users/hour, occurring at $t \approx 5.89$ hrs. **A1**

13. **[Paper 1 Style, Short Answer, Hard, 6 marks]**

(a) $P(5) = -0.5(125) + 12(25) - 10(5) + 50 = -62.5 + 300 - 50 + 50 = 237.5$.
$P(10) = -0.5(1000) + 12(100) - 10(10) + 50 = -500 + 1200 - 100 + 50 = 650$.
(M1)
Average rate $= \frac{650-237.5}{10-5} = \frac{412.5}{5} = 82.5$ thousands of dollars. **A1**

(b) Use CG50 `d/dx`. Enter $\frac{d}{dx}(-0.5x^3 + 12x^2 - 10x + 50)\big|_{x=10}$. **(M1)**
Result $= 80$ thousands of dollars. **A1**

(c) In `MENU 5`, enter $Y1 = \frac{d}{dx}(-0.5x^3 + 12x^2 - 10x + 50)$.
Use `G-Solv -> ROOT`. **(M1)**
$x = 15.567\cdots \approx 15.6$. (Reject the $x = 0.432$ root in this context if looking for the upper bound, though both are technically correct roots). **A1**

14. **[Paper 2 Style, Longer Question, Very Hard, 7 marks]**

(a) **Ensure calculator is in Radians.**
$D(2) = 12 + 4\sin(\pi/3) = 12 + 4(0.866) = 15.464$ m.
$D(5) = 12 + 4\sin(5\pi/6) = 12 + 4(0.5) = 14$ m. **(M1)**
Average rate $= \frac{14-15.464}{5-2} = -0.488$ m/hr. **A1**

(b) Use CG50 `d/dx`. Enter $\frac{d}{dx}(12 + 4\sin(\pi x/6))\big|_{x=4}$. **(M1)**
Result $= -1.04719\cdots \approx -1.05$ m/hr. **A1**

(c) In `MENU 5 (Graph)`, enter $Y1 = \frac{d}{dx}(12 + 4\sin(\pi x/6))$.
Set `V-Window` X from 0 to 12. **(M1)**
Use `G-Solv -> MAX`.
$x = 0$ and $x = 12$. **A1**
At midnight ($t = 0$) and noon ($t = 12$), the water level is rising at its fastest rate (2.09 m/hr). **A1**

15. **[Paper 2 Style, Longer Question, Very Hard, 9 marks]**

(a) Substitute a very large number (e.g., $x = 10000$) into the CG50.
Result approaches 0. **(M1)A1**
This implies that as you move infinitely far from the pillar, the stress on the cable diminishes to zero. **R1**

(b) $S(0) = \frac{0}{4} = 0$. $S(2) = \frac{100}{4+4} = 12.5$. **(M1)**
Average rate $= \frac{12.5-0}{2-0} = 6.25$ MPa/m. **A1**

(c) Use CG50 `d/dx`. Enter $\frac{d}{dx}(50x/(x^2+4))\big|_{x=2}$. **(M1)**
Result $= 0$ MPa/m. **A1**

(d) In `MENU 5`, graph $Y1 = 50x/(x^2+4)$.
Press `G-Solv -> MAX`. **(M1)**
$x = 2$ m. **A1**
(The max stress is 12.5 MPa at exactly $x = 2$).

(e) The derivative $S'(x)$ measures the gradient. At $x = 2$, we found in part (c) that the derivative is exactly 0. This perfectly aligns with the fact that local maximums occur at stationary points where the gradient is zero. **R1**