IB MATHEMATICS AA HL
AHL TOPIC 5 PRACTICE

Differential Equations and Maclaurin Series

Instructions to Candidates

- This practice paper contains **20** questions progressing from Easy to Very Hard.
- Each question indicates whether it is styled for **Paper 1 (No Calculator)** or **Paper 2 (Calculator Allowed)**.
- The paper tests syllabus topics AHL 5.18 and 5.19: First-order DEs, separable variables, homogeneous DEs ($y = vx$), integrating factors, Euler's numerical method, and Maclaurin series expansions.
- Answer all questions, showing all your working clearly.
- Total marks available: **103**.

Difficulty Progression

- **SECTION A (Easy):** Basic separation of variables, verifying integrating factors, standard Maclaurin expansions, and single-step Euler's method approximations.
- **SECTION B (Medium):** Full IF solutions, deriving Maclaurin series via substitution or differentiation, logistic growth equations, and Newton's law of cooling.
- **SECTION C (Hard):** Complex homogeneous DEs, solving coupled mixture tank models, formal proofs of Maclaurin series by repeatedly differentiating a DE, and multi-step trigonometric substitutions.

SECTION A: EASY (Fundamentals)

Question 1 (3 Marks) — Paper 1 (No Calculator Allowed)

Find the general solution to the differential equation:

$$\frac{dy}{dx} = 4xe^{-y}$$

Express your answer in the form $y = f(x)$.

Question 2 (3 Marks) — Paper 1 (No Calculator Allowed)

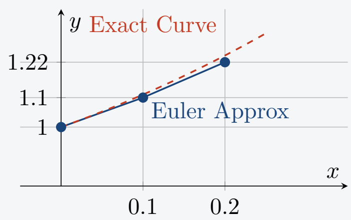
Consider the linear differential equation:

$$\frac{dy}{dx} + \frac{2}{x}y = 5x$$

Find the integrating factor required to solve this differential equation.

Question 3 (4 Marks) — Paper 2 (Calculator Allowed)

Consider the differential equation $\frac{dy}{dx} = x + y$, with the initial condition $y(0) = 1$. Use Euler's method with a step length of $h = 0.1$ to find an approximation for $y(0.2)$.



Question 4 (3 Marks) — Paper 1 (No Calculator Allowed)

Using the standard Maclaurin series expansion for e^x, find the first four non-zero terms of the Maclaurin series for $f(x) = e^{2x}$.

Question 5 (4 Marks) — Paper 1 (No Calculator Allowed)

A curve satisfies the differential equation $\frac{dy}{dx} = \frac{x}{y}$ and passes through the point $(0, 3)$. Solve the differential equation to find the exact equation of the curve in the form $y^2 - x^2 = k$.

Question 6 (3 Marks) — Paper 1 (No Calculator Allowed)

Show algebraically that the differential equation $\frac{dy}{dx} = \frac{y^2+xy}{x^2}$ is a homogeneous differential equation.

Question 7 (4 Marks) — Paper 1 (No Calculator Allowed)

The Maclaurin series expansion for $\ln(1+x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$
Use a suitable substitution to find the Maclaurin series expansion for $y = x^2 \ln(1-x^2)$ up to and including the term in x^6.

SECTION B: MEDIUM (Application & Algebraic Methods)

Question 8 (5 Marks) — Paper 1 (No Calculator Allowed)

Using your integrating factor from Question 2, find the general solution to the linear differential equation:

$$\frac{dy}{dx} + \frac{2}{x}y = 5x$$

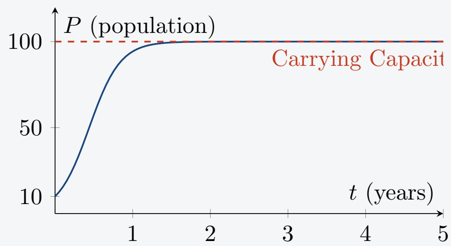
Give your answer in the form $y = f(x)$.

Question 9 (6 Marks) — Paper 2 (Calculator Allowed)

The population P of a colony of endangered birds in a nature reserve is modelled by the logistic differential equation:

$$\frac{dP}{dt} = 0.05P(100 - P)$$

where t is the time in years.



(a) Given that the initial population is 10 birds, solve the differential equation to find an expression for P in terms of t. **[5 marks]**

(b) State the limiting value of the population as $t \to \infty$. **[1 mark]**

Question 10 (6 Marks) — Paper 1 (No Calculator Allowed)

Use the substitution $y = vx$ to solve the homogeneous differential equation from Question 6:

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2}$$

Give your answer in the form $y = f(x)$.

Question 11 (5 Marks) — Paper 1 (No Calculator Allowed)

By finding the derivatives of $f(x) = \tan x$ evaluated at $x = 0$, determine the first three non-zero terms of the Maclaurin series expansion for $\tan x$.

Question 12 (6 Marks) — Paper 2 (Calculator Allowed)

According to Newton's Law of Cooling, the rate at which an object cools is proportional to the difference between its temperature T and the ambient room temperature T_A. This is modelled by $\frac{dT}{dt} = -k(T - T_A)$.
A cup of coffee is initially 100°C in a room kept at a constant 20°C. After 10 minutes, the coffee has cooled to 60°C.
Calculate the exact temperature of the coffee after 20 minutes.

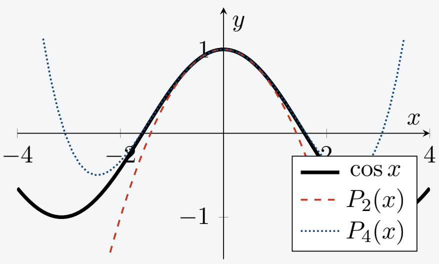
Question 13 (5 Marks) — Paper 1 (No Calculator Allowed)

Find the integrating factor for the differential equation:

$$\frac{dy}{dx} + (\cot x)y = \sin x$$

Question 14 (6 Marks) — Paper 2 (Calculator Allowed)

The graph below shows $y = \cos x$ alongside its first three Maclaurin polynomial approximations (P_2, P_4, P_6).



Using the Maclaurin series for $\cos x$, estimate the value of $\int_0^1 \cos(x^2)\,dx$ using the first three non-zero terms of the series. Give your answer correct to 4 decimal places.

SECTION C: HARD / VERY HARD (Synthesis & Proof)

Question 15 (6 Marks) — Paper 1 (No Calculator Allowed)

A function $y = f(x)$ satisfies the differential equation:

$$(1-x)\frac{dy}{dx} = 2y$$

Given that $y(0) = 1$, find the Maclaurin series expansion for y up to and including the term in x^3 by repeatedly differentiating the differential equation.

Question 16 (7 Marks) — Paper 1 (No Calculator Allowed)

Consider the non-linear differential equation:

$$\frac{dy}{dx} = (x+y)^2$$

Use the substitution $v = x + y$ to transform this equation into a separable differential equation in terms of v and x, and hence find the general solution for y in terms of x.

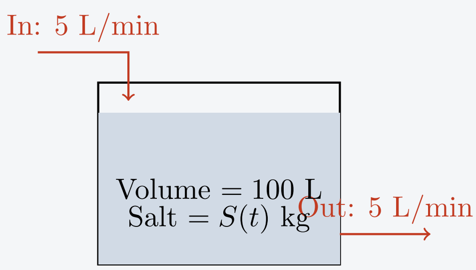
Question 17 (7 Marks) — Paper 1 (No Calculator Allowed)

(a) Using the binomial expansion, find the Maclaurin series for $(1-x^2)^{-1/2}$ up to the term in x^4. **[4 marks]**

(b) Hence, by integrating your answer to part (a), deduce the Maclaurin series for $\arcsin x$ up to the term in x^5. **[3 marks]**

Question 18 (7 Marks) — Paper 2 (Calculator Allowed)

A mixing tank initially contains 100 L of pure water. A brine solution containing 0.2 kg of salt per litre flows into the tank at a rate of 5 L/min. The well-mixed solution is pumped out of the tank at the exact same rate of 5 L/min. Let $S(t)$ be the amount of salt (in kg) in the tank at time t (in minutes).



Set up a differential equation for $\frac{dS}{dt}$ and solve it to find the amount of salt in the tank after exactly 30 minutes.

Question 19 (7 Marks) — Paper 1 (No Calculator Allowed)

Solve the non-linear homogeneous differential equation:

$$x\frac{dy}{dx} = y + \sqrt{x^2 + y^2} \quad \text{for } x > 0$$

Use the substitution $y = vx$ to find the general solution in the form $y = f(x)$.

Question 20 (7 Marks) — Paper 1 (No Calculator Allowed)

Find the exact particular solution to the differential equation:

$$\frac{dy}{dx}\cos x + y\sin x = \cos^3 x$$

given the initial boundary condition $y\left(\frac{\pi}{4}\right) = 1$.