IB MATHEMATICS AI HL
UNIT 5: CALCULUS

Advanced Differentiation & Related Rates

Instructions to Candidates

- This question booklet contains **15 questions**.
- The paper targets **AHL** syllabus component 5.9.
- Answer all questions, showing all step-by-step working clearly.

Difficulty Progression

- **Questions 1 - 5 (Easy):** The Chain, Product, and Quotient rules, trigonometric derivatives, and tangents.
- **Questions 6 - 10 (Medium):** Implicit/complex combinations, expanding domains, and basic 2D related rates (ladders and circles).
- **Questions 11 - 15 (Hard):** Algebraic derivative proofs, 3D related rates (conical tanks), and rate-of-change modelling with angle elevation.

SECTION A: EASY (Fundamentals)

Question 1 (4 Marks)

Differentiate $y = (3x^2 - 5x)^4$ with respect to x. Express your answer in a fully factorised form.

Question 2 (4 Marks)

Let $f(x) = x^2 \ln(x)$ for $x > 0$.
Use the product rule to find $f'(x)$ and express it in the form $x(a \ln x + b)$ where a and b are integers.

Question 3 (4 Marks)

Let $g(x) = \frac{e^{2x}}{x+1}$.
Use the quotient rule to find $g'(x)$.

Question 4 (4 Marks)

Find the exact gradient of the curve $y = \sin(3x)$ at the point where $x = \frac{\pi}{6}$.

CG50 Tip: Numerical Derivatives

Need to find the gradient of a curve at a specific point without doing the algebra? On the `Run-Matrix` screen, press `MATH (F4)` → `d/dx (F4)`. Type your function into the brackets and the x-value outside to instantly get the gradient!

Question 5 (5 Marks)

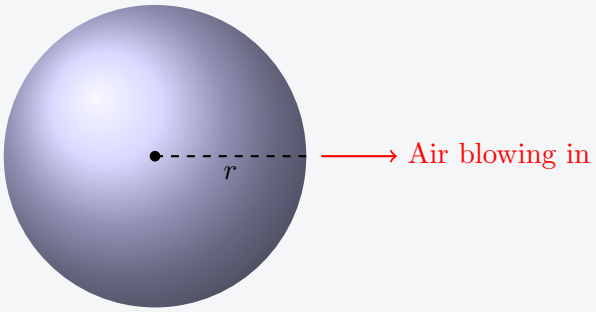
Consider the curve given by $y = e^x \cos x$.
Find the exact equation of the tangent to the curve at the point where $x = 0$. Give your answer in the form $y = mx + c$.

SECTION B: MEDIUM (Application & Modelling)

Question 6 (5 Marks)

Consider the function $y = \ln(\cos x)$.
(a) Find $\frac{dy}{dx}$. **[3 marks]**
(b) State the continuous domain of values for x containing $x = 0$ for which this derivative is valid. **[2 marks]**

Question 7 (6 Marks)



A perfectly spherical balloon is being inflated such that its volume increases at a constant rate of 20 cm^3s^{-1}.
Find the exact rate at which the radius of the balloon is increasing at the instant when the radius is exactly 5 cm.
(The volume of a sphere is $V = \frac{4}{3}\pi r^3$)

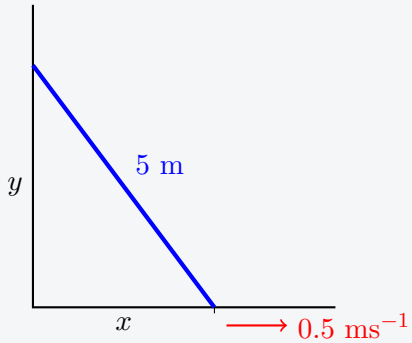
Question 8 (6 Marks)

Find the exact coordinates (x, y) of the stationary point on the curve $f(x) = xe^{-x}$.

Question 9 (6 Marks)

A curve is given by the equation $y = \frac{\ln x}{x}$.
Determine the exact equation of the **normal** to the curve at the point where $x = e$.

Question 10 (6 Marks)



A ladder 5 m long rests against a vertical wall. The bottom of the ladder is pulled away from the wall horizontally at a constant rate of 0.5 ms^{-1}.
Calculate the rate at which the top of the ladder is sliding down the wall at the exact instant the bottom of the ladder is 3 m from the wall.

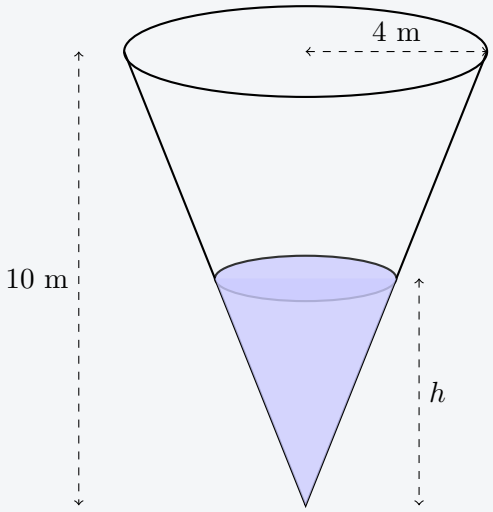
SECTION C: HARD (Synthesis & Proof)

Question 11 (7 Marks)

Consider the curve given by the equation $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.
Use the quotient rule to differentiate this expression and prove algebraically that:

$$\frac{dy}{dx} = 1 - y^2$$

Question 12 (8 Marks)



Water pours into an inverted conical tank at a constant rate of 2 $\text{m}^3\text{min}^{-1}$. The tank has a maximum radius of 4 m at the top and a total depth of 10 m.
Let h be the depth of the water in the tank.
Find the exact rate at which the water level (h) is rising at the instant when the depth of the water is exactly 5 m.
(The volume of a cone is $V = \frac{1}{3}\pi r^2 h$)

Question 13 (8 Marks)

Find the exact x-coordinates of all stationary points of the function $f(x) = e^{\sin x}$ in the domain $0 \leq x \leq 2\pi$.

Question 14 (7 Marks)

The population P of a newly discovered species of bird on an island, t years after discovery, is modelled by:

$$P(t) = 1000 + 500te^{-0.2t}$$

Using calculus, find the exact time t when the population reaches its absolute maximum, and calculate this maximum population correct to the nearest whole bird.

Question 15 (9 Marks)

An observer stands on the ground, exactly 100 m away from a rocket launch pad. A rocket is launched vertically upwards at a constant velocity of 50 ms^{-1}.
Let θ be the angle of elevation from the observer to the rocket, and y be the height of the rocket.
(a) Write down an expression for $\tan\theta$ in terms of y. **[1 mark]**
(b) By using implicit differentiation with respect to time (t) on your equation from part (a), find an expression connecting $\frac{d\theta}{dt}$ and $\frac{dy}{dt}$. **[4 marks]**
(c) Hence, calculate the exact rate of change of the angle of elevation (in rad s^{-1}) at the instant the rocket is exactly 200 m above the ground. **[4 marks]**