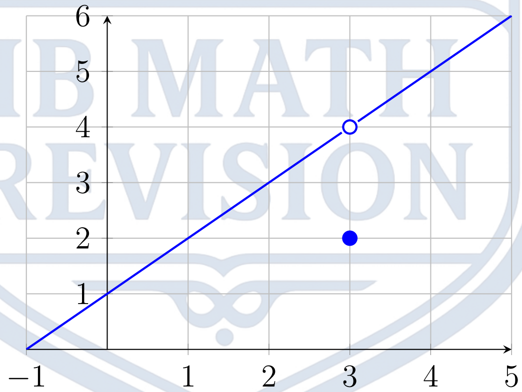
Unit 5: Limits & Rates of Change
IB Math AA SL

Answer all 15 questions. Show all working. For Paper 1 questions, use analytical methods and conceptual reasoning. For Paper 2 questions, use your graphic display calculator (GDC) efficiently.

1. **[Paper 1 Style, Non-Calculator, Easy, 4 marks]**
   The graph of a function $y = f(x)$ is shown below for $-1 \leq x \leq 5$. There is a hollow circle (a "hole") at the point $(3, 4)$.

   Use the graph to write down the value of:

   (a) $f(3)$.

   (b) $\lim_{x\to 3} f(x)$.

2. **[Paper 2 Style, Calculator Required, Easy, 4 marks]**
A student is investigating the limit of a function $g(x)$ as x approaches 2. They create the following table of values using their calculator.

x	1.9	1.99	1.999	2.001	2.01	2.1
$g(x)$	6.710	6.970	6.997	7.003	7.030	7.310

(a) Estimate the value of $\lim_{x\to 2} g(x)$.

(b) State why calculating $g(2)$ directly might result in a "Math Error" on a calculator, even though the limit exists.

3. **[Paper 1 Style, Non-Calculator, Easy, 4 marks]**
The derivative of a function $f(x)$ is known as its gradient function, denoted by $f'(x)$. Given that $f(x) = -x^2 + 4x + 5$ and its derivative is $f'(x) = -2x + 4$:

(a) Find the exact gradient of the curve at $x = 3$.

(b) State whether the function $f(x)$ is increasing or decreasing at $x = 3$. Give a reason for your answer.

4. **[Paper 2 Style, Calculator Required, Easy, 4 marks]**
The volume of a sphere is given by the formula $V(r) = \frac{4}{3}\pi r^3$, where r is the radius in cm. The derivative of the volume with respect to the radius is denoted by $\frac{dV}{dr}$.

(a) Use the numerical derivative feature on your GDC to evaluate $\frac{dV}{dr}$ when $r = 5$. Give your answer to 3 significant figures.

(b) Explain the physical meaning of your answer to part (a) in the context of the sphere.

5. **[Paper 1 Style, Non-Calculator, Medium, 5 marks]**
The mathematical definition of the derivative $f'(x)$ is given by the limit:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Let $f(x) = x^2$.

(a) Expand and simplify the expression $\frac{f(x+h)-f(x)}{h}$.

(b) Hence, evaluate the limit as $h \to 0$ to find the gradient function $f'(x)$.

6. **[Paper 2 Style, Calculator Required, Medium, 5 marks]**
Consider the function $f(x) = \frac{4}{x} + x^2 - 4$, for $x > 0$.

(a) Calculate the average rate of change of $f(x)$ between $x = 2$ and $x = 2.25$.

(b) Explain what would happen to your average rate of change calculation if you moved the second x-value closer and closer to 2 (e.g., $x = 2.1$, then $x = 2.01$, then $x = 2.001$).

(c) Use your GDC to find the exact instantaneous rate of change of $f(x)$ at $x = 2$.

7. **[Paper 1 Style, Non-Calculator, Medium, 5 marks]**
A particle moves in a straight line such that its displacement, s in metres, from a fixed origin at time t seconds is given by $s(t) = 2t^2 - 8t + 9$ for $t \geq 0$. The rate of change of displacement with respect to time is the velocity, $v(t)$.

(a) Find the average velocity of the particle between $t = 1$ and $t = 4$.

(b) Given that $v(t) = s'(t) = 4t - 8$, find the instantaneous velocity of the particle at $t = 1$.

(c) Describe the direction the particle is moving at $t = 1$, justifying your answer using the sign of the velocity.

8. **[Paper 2 Style, Calculator Required, Medium, 5 marks]**
The population of rabbits on an island, P, at time t months after the beginning of a study can be modelled by the function:

$$P(t) = \frac{3000}{1 + 99e^{-0.5t}}$$

(a) Find the initial population of rabbits on the island.

(b) Use the numerical derivative feature on your GDC to find the value of $P'(4)$.

(c) Interpret the meaning of $P'(4)$ in the context of the rabbit population.

9. **[Paper 1 Style, Non-Calculator, Hard, 5 marks]**
Consider the rational function $f(x) = \frac{3x+1}{x-2}$, where $x \neq 2$.

(a) By considering the behaviour of the function as x becomes very large, evaluate $\lim_{x\to\infty} f(x)$.

(b) State the geometric feature of the graph of $f(x)$ that corresponds to your answer in part (a).

(c) Evaluate $\lim_{x\to 2^+} f(x)$, describing what happens to the curve as x approaches 2 from the positive side.

10. **[Paper 2 Style, Calculator Required, Hard, 5 marks]**
A cup of coffee is left to cool in a room. The temperature of the coffee, T in degrees Celsius (°C), is measured every 5 minutes (t).

Time (t minutes)	5	10	15	20
Temperature (T°C)	72.5	60.1	51.2	44.8

An estimate for the instantaneous rate of change of temperature at $t = 15$ can be found using the central difference quotient:

$$T'(15) \approx \frac{T(20) - T(10)}{20 - 10}$$

(a) Calculate this estimate for $T'(15)$.

(b) Explain the physical significance of the negative sign in your answer.

11. **[Paper 1 Style, Non-Calculator, Hard, 5 marks]**
An economics student is modelling the total cost, $C(x)$ in dollars, of producing x units of a product. The function is $C(x) = 150 + 4x - 0.02x^2$. The marginal cost is defined as the instantaneous rate of change of the total cost with respect to the number of units produced ($C'(x)$).

(a) Given $C'(x) = 4 - 0.04x$, calculate $C'(50)$.

(b) Calculate the exact actual cost of producing the 51st unit, given by $C(51) - C(50)$.

(c) Compare your answers to (a) and (b), and briefly explain why the marginal cost $C'(x)$ is frequently used by economists to estimate the cost of producing one additional unit.

12. **[Paper 2 Style, Calculator Required, Hard, 6 marks]**
Let $f(x) = x^3 - 3x^2 - 9x + 2$.

(a) Graph the function $y = f(x)$ on your GDC for $-4 \leq x \leq 6$.

(b) Use your GDC to draw the tangent to the curve at $x = -2$ and write down its gradient.

(c) A function is said to be "decreasing" when its rate of change is strictly negative ($f'(x) < 0$). Use your GDC to find the exact interval of x-values for which $f(x)$ is decreasing.

13. **[Paper 1 Style, Non-Calculator, Very Hard, 6 marks]**
The diagram shows the graph of a function f with a local maximum point at A and a local minimum point at B. The derivative of f is $f'(x) = \frac{1}{3}x^2 + \frac{1}{2}x - 2$.

   (a) The concept of a derivative tells us that the gradient of a curve is exactly zero at its local maximum and minimum points. Set $f'(x) = 0$ to find the exact x-coordinates of points A and B.

   (b) Explain analytically how you can determine which x-coordinate corresponds to the maximum (Point A) and which corresponds to the minimum (Point B) by examining the sign of $f'(x)$ around those points.

14. **[Paper 2 Style, Calculator Required, Very Hard, 6 marks]**
A company determines that the daily profit, P in thousands of dollars, from selling a specific software package depends on the price, x in dollars, set for the package. The relationship is modelled by $P(x) = (x - 15) \times N(x)$, where $N(x) = 500 - 4x$ is the number of packages sold per day.

   (a) Write down an expanded expression for $P(x)$ in terms of x.

   (b) Use the numerical derivative feature on your GDC to find $P'(50)$.

   (c) Interpret the value of $P'(50)$. What advice would you give the company regarding the current price of $50?

15. **[Paper 1 Style, Non-Calculator, Very Hard, 6 marks]**
A piecewise function $h(x)$ is defined as:

$$h(x) = \begin{cases} x^2 + 1 & \text{for } x < 2 \\ kx - 1 & \text{for } x \geq 2 \end{cases}$$

where k is a constant.

   (a) Find $\lim_{x \to 2^-} h(x)$ (the limit as x approaches 2 from the left).

   (b) For the overall limit $\lim_{x \to 2} h(x)$ to exist, the left-hand limit must equal the right-hand limit. Find the value of k that guarantees this limit exists.

   (c) Using your value of k, sketch the graph of $y = h(x)$ around $x = 2$, and state whether the function has a well-defined gradient (derivative) at exactly $x = 2$. Give a reason.