Unit 4: Discrete Random Variables & Binomial Distribution
IB Math AA SL

Answer all 15 questions. Show all working. For Paper 1 questions, use analytical fraction methods. For Paper 2 questions, use your graphic display calculator (GDC) efficiently.

1. **[Paper 1 Style, Non-Calculator, Easy, 4 marks]**
A biased four-sided die is rolled. Let X be the score obtained. The probability distribution for X is given in the following table.

x	1	2	3	4
$P(X = x)$	p	p	p	$\frac{1}{2}$

   (a) Find the exact value of p.

   (b) Hence, find the expected value, $E(X)$.

2. **[Paper 2 Style, Calculator Required, Easy, 4 marks]**
A fair coin is tossed 20 times and the number of times it lands heads up is recorded as the random variable X.

   (a) Find the expected number of times that the coin will land heads up.

   (b) Use your graphic display calculator to find the probability that the coin lands heads up exactly 15 times.

3. **[Paper 1 Style, Non-Calculator, Easy, 4 marks]**
The random variable Y follows a binomial distribution such that $Y \sim B(n, p)$. Given that the expected value is $E(Y) = 4.5$ and the variance is $\text{Var}(Y) = 3.15$:

   (a) Find the value of p.

   (b) Find the value of n.

4. **[Paper 2 Style, Calculator Required, Easy, 4 marks]**
The random variable X is binomially distributed such that $X \sim B(40, 0.15)$. Find the values of:

   (a) $P(3 \leq X < 14)$

   (b) $\text{Var}(X)$

5. **[Paper 1 Style, Non-Calculator, Medium, 5 marks]**
The probability distribution of a discrete random variable X is given by the formula:

$$P(X = x) = \frac{x}{k} \quad \text{for } x \in \{1, 2, 3, 4\}$$

where k is a positive constant.

   (a) Find the exact value of k.

   (b) Find the expected value, $E(X)$.

6. **[Paper 2 Style, Calculator Required, Medium, 5 marks]**
Zara is a gymnast. It is known from her past performance that she has a 20% chance of making a critical mistake in any given routine. Zara performs exactly ten routines in a competition.

   (a) Find the standard deviation of the number of routines in which Zara makes a mistake.

   (b) Find the probability that Zara makes a mistake in no more than two of her routines.

7. **[Paper 1 Style, Non-Calculator, Medium, 4 marks]**
At a school fair, students pay \$3 to play a game. They spin a spinner and win a cash prize, W, depending on where the spinner lands. The probability distribution of the prize money W is shown below.

Prize, w (\$)	0	2	5	A
$P(W = w)$	0.4	0.3	0.2	0.1

The game is considered mathematically "fair" if the expected profit for the player is exactly zero (meaning the expected prize equals the cost to play). Find the value of the top prize, A, that makes this a fair game.

8. **[Paper 2 Style, Calculator Required, Medium, 6 marks]**
For cans of a particular brand of soft drink, the probability that a can contains less than 320 ml of liquid is 0.0296. Tilly buys a pack of 24 cans. It may be assumed that these 24 cans represent a random sample. Let L represent the number of cans that contain less than 320 ml of soft drink.

   (a) State two conditions that must be satisfied for L to be modelled by a binomial distribution.

   (b) Find the probability that exactly two of the cans contain less than 320 ml.

   (c) Find the probability that at least two of the cans contain less than 320 ml.

9. **[Paper 1 Style, Non-Calculator, Hard, 5 marks]**
The discrete random variable X has the following probability distribution:

x	1	2	3
$P(X = x)$	a	b	0.2

Given that the expected value of X is $E(X) = 1.7$, formulate a system of simultaneous equations and solve it to find the exact values of a and b.

10. **[Paper 2 Style, Calculator Required, Hard, 5 marks]**
A company manufactures computer chips. It is known that 12% of the chips produced are defective. A quality control inspector takes a random sample of 50 chips. Let the random variable D represent the number of defective chips in the sample.

    (a) Find the probability that the inspector finds fewer than 8 defective chips.

    (b) Given that the inspector finds fewer than 8 defective chips, find the conditional probability that they found exactly 5 defective chips.

11. **[Paper 1 Style, Non-Calculator, Hard, 5 marks]**
A biased coin has a probability p of landing on heads, where $0 < p < 1$. The coin is tossed exactly 3 times. The probability of obtaining exactly two heads is equal to the probability of obtaining exactly one head. By utilizing the binomial combination formula $\binom{n}{r}$, formulate an equation in terms of p and solve it algebraically to find the exact value of p.

12. **[Paper 2 Style, Calculator Required, Hard, 5 marks]**
An archer has a constant probability of 0.25 of hitting the bullseye on any given shot. The shots are independent of each other. Let n be the number of shots the archer takes. Find the smallest value of n such that the probability of hitting the bullseye at least once is greater than or equal to 0.99. Show your working clearly.

13. **[Paper 1 Style, Non-Calculator, Very Hard, 6 marks]**
A discrete random variable X follows a binomial distribution $X \sim B(n, p)$, where $n > 3$ and $0 < p < 1$. Given that $P(X = 2) = P(X = 3)$, prove algebraically that:

$$p = \frac{3}{n+1}$$

14. **[Paper 2 Style, Calculator Required, Very Hard, 7 marks]**
A discrete random variable X has the following probability distribution:

x	0	1	2	3
$P(X = x)$	q	$4p^2$	p	$0.7 - 4p^2$

where p and q are positive constants.

(a) Use the property of probability distributions to find an expression for q in terms of p.

(b) Find an expression for $E(X)$ in the form of a quadratic equation in p.

(c) Hence, using calculus or your GDC, find the value of p which gives the largest possible value of $E(X)$, and state this maximum expected value.

15. **[Paper 1 Style, Non-Calculator, Very Hard, 6 marks]**
Two independent discrete random variables, X and Y, have the following probability distributions:

x	1	2
$P(X = x)$	$\frac{1}{3}$	$\frac{2}{3}$

y	1	2	3
$P(Y = y)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

(a) Show that $E(X) = E(Y)$.

(b) Find the exact probability $P(X < Y)$.