Topic: Differentiation Rules (Polynomials)
IB Math AI SL

Answer all questions. Show all working where appropriate. Total: 80 marks.

1. **[Paper 1 Style, Short Answer, Easy, 3 marks]**
Consider the polynomial function $f(x) = 4x^3 - 5x^2 + 2x - 7$.

   (a) Find the derivative function, $f'(x)$.

   (b) Calculate the exact value of $f'(2)$.

2. **[Paper 1 Style, Short Answer, Easy, 4 marks]**
Let $y = -0.5x^4 + 3x^3 - x$.

   (a) Use the numerical derivative feature (`d/dx`) on your graphic display calculator to find the gradient of the curve at $x = 3$.

   (b) Interpret the geometric meaning of your answer to part (a).

3. **[Paper 1 Style, Short Answer, Easy, 4 marks]**
A function is given by $g(x) = (2x - 3)(x^2 + 1)$.

   (a) Expand the expression for $g(x)$ into the form $ax^3 + bx^2 + cx + d$.

   (b) Hence, find $g'(x)$.

4. **[Paper 1 Style, Short Answer, Medium, 4 marks]**
Consider the function $h(x) = \frac{3x^4 - 2x^2}{x}$ for $x \neq 0$.

   (a) Rewrite $h(x)$ as a polynomial by dividing each term in the numerator by x.

   (b) Find $h'(x)$.

   (c) Determine the value of x for which the gradient of $h(x)$ is 23.

5. **[Paper 1 Style, Short Answer, Medium, 5 marks]**
The curve C has the equation $f(x) = x^3 - 6x^2 + 9x + 2$.

   (a) Find $f'(x)$.

   (b) Stationary points occur where the gradient of the curve is exactly zero. Use your graphic display calculator's polynomial solver to find the x-coordinates of the stationary points of C.

6. **[Paper 1 Style, Short Answer, Medium, 5 marks]**
The daily profit, P in dollars, of a small business is modelled by $P(x) = -0.1x^3 + 12x^2 - 60x - 200$, where x is the number of items sold.

   (a) Find the marginal profit function, $P'(x)$.

   (b) Use your graphic display calculator to find the marginal profit when 50 items are sold, and interpret this value in context.

7. **[Paper 2 Style, Longer Question, Medium, 6 marks]**
The height of a toy rocket, h in metres, t seconds after launch is given by $h(t) = -4.9t^2 + 15t + 2$.

   (a) Find the velocity function, $h'(t)$, of the rocket.

   (b) Determine the initial velocity of the rocket.

   (c) Find the time t when the velocity of the rocket is zero.

   (d) Explain the physical significance of the time found in part (c).

8. **[Paper 1 Style, Short Answer, Medium, 5 marks]**
Let $f(x) = 2x^3 - 3x^2 - 12x$.

   (a) Find the derivative $f'(x)$.

   (b) Find the values of x for which the gradient of $f(x)$ is 24.

9. **[Paper 1 Style, Short Answer, Hard, 5 marks]**
A cubic function has a derivative given by $f'(x) = 6x^2 - 10x + c$, where c is a constant.

   (a) Given that the gradient of the curve at $x = 2$ is 8, find the value of c.

   (b) Using your value of c, find the x-coordinates where the gradient of the curve is zero.

10. **[Paper 2 Style, Longer Question, Hard, 7 marks]**
The function $y = ax^2 + bx$ passes through the point $(1, 5)$. The gradient of the curve at $x = 2$ is 14.

    (a) By substituting the coordinates of the given point, write down an equation in terms of a and b.

    (b) Find an expression for $\frac{dy}{dx}$ in terms of a, b, and x.

    (c) Use the given gradient information to write down a second equation in terms of a and b.

    (d) Solve the system of equations to find the exact values of a and b.

11. **[Paper 1 Style, Short Answer, Hard, 5 marks]**
The second derivative of a function, denoted $f''(x)$, is the derivative of $f'(x)$. Consider the function $f(x) = x^4 - 2x^3 + 5x^2 - x + 10$.

    (a) Find $f'(x)$.

    (b) Hence, find $f''(x)$.

    (c) Evaluate $f''(1)$.

12. **[Paper 2 Style, Longer Question, Hard, 6 marks]**
A function is defined as $f(x) = -x^3 + 3x^2 + 9x - 5$.

    (a) Find the derivative function, $f'(x)$.

    (b) Graph the derivative function $y = f'(x)$ on your graphic display calculator. Sketch the shape of this graph, clearly showing its x-intercepts.

    (c) A function is increasing when its derivative is strictly positive ($f'(x) > 0$). Use your graph from part (b) to state the interval of x for which $f(x)$ is increasing.

13. **[Paper 2 Style, Longer Question, Very Hard, 7 marks]**
A manufacturer creates an open-top box from a rectangular piece of cardboard measuring 15 cm by 10 cm, by cutting identical squares of side length x cm from each corner and folding up the sides.

    (a) Show that the volume of the box, V, is given by $V(x) = 4x^3 - 50x^2 + 150x$.

    (b) Find the derivative, $V'(x)$.

    (c) Use your graphic display calculator's equation solver to find the value of x that makes $V'(x) = 0$. (Note: x must be a valid dimension for the box).

14. **[Paper 1 Style, Short Answer, Very Hard, 6 marks]**
The population of fish in a lake is modelled by $P(t) = -0.2t^3 + 2.4t^2 + 10t + 50$, where t is the number of years since 2020 ($0 \leq t \leq 12$).

    (a) Find the growth rate of the fish population, $P'(t)$.

    (b) The growth rate reaches its maximum when the second derivative, $P''(t)$, is equal to zero. Find $P''(t)$ and hence determine the year in which the fish population is growing the fastest.

15. **[Paper 2 Style, Longer Question, Very Hard, 8 marks]**
Let $f(x) = px^3 + qx^2 - 4x + 1$, where p and q are constants. The graph of $f(x)$ has a local minimum point at $x = 2$, meaning $f'(2) = 0$. The graph also has a point of inflection at $x = 1$, meaning $f''(1) = 0$.

    (a) Find $f'(x)$ in terms of p, q, and x.

    (b) Find $f''(x)$ in terms of p, q, and x.

    (c) Set up a system of two linear equations using the given conditions at $x = 2$ and $x = 1$.

    (d) Solve the system to find the values of p and q.