IB MATHEMATICS AI HL
UNIT 4: STATISTICS

Chi-Square Goodness of Fit & Type I/II Errors

Instructions to Candidates

- This question booklet contains **15 questions**.
- The paper targets **AHL** syllabus components 4.17 and 4.18.
- Answer all questions, showing all step-by-step working clearly.

Difficulty Progression

- **Questions 1 - 5 (Easy):** Degrees of freedom, uniform goodness of fit, definitions of Type I and Type II errors.
- **Questions 6 - 10 (Medium):** Full χ^2 independence tests, combining bins ($f_e < 5$), and calculating Type I error probability (α).
- **Questions 11 - 15 (Hard):** Fitting data to Poisson and Normal distributions, reducing degrees of freedom, and calculating Type II error probabilities (β) from overlapping curves.

SECTION A: EASY (Fundamentals)

Question 1 (4 Marks)

A researcher conducts a Chi-Square test for independence to see if eye color (Blue, Green, Brown, Hazel) is independent of hair color (Blonde, Brown, Black, Red, Other).
Calculate the exact number of degrees of freedom (df) for this test.

Question 2 (4 Marks)

A 6-sided die is rolled 120 times to test if it is fair (a uniform distribution).
Write down the null hypothesis (H_0), and state the expected frequency (f_e) for the die landing on a "4".

Question 3 (4 Marks)

In hypothesis testing, define what is meant by a **Type I error** in the context of the null hypothesis (H_0). State the symbol used to represent the probability of making a Type I error.

Question 4 (4 Marks)

A medical test screens patients for a rare disease. The null hypothesis H_0 is that the patient is **healthy**. The alternative hypothesis H_1 is that the patient has the **disease**.
Explain the practical consequence of the test making a **Type II error** for a patient.

CG50 Tip: The Goodness of Fit Test (GOF)

To run a Goodness of Fit test on your CG50, go to `MENU 2 (STAT)`. Enter your Observed frequencies in `List 1` and Expected frequencies in `List 2`. Then press `TEST (F3)` → `CHI (F3)` → `GOF (F1)`. Remember to manually enter your Degrees of Freedom (df)!

Question 5 (5 Marks)

A local café claims that customer drink preferences are distributed as follows: 50% Coffee, 30% Tea, and 20% Hot Chocolate. A sample of 80 customers yields the following observed frequencies: - Coffee: 35 - Tea: 30 - Hot Chocolate: 15
Using the formula $\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e}$, calculate the exact χ^2 test statistic by hand. Show your working.

SECTION B: MEDIUM (Application & Modelling)

Question 6 (5 Marks)

A survey of 200 adults asked for their preferred mode of transport and their age group.

	Car	Bus	Bicycle
18 - 35	45	25	20
Over 35	65	35	10

Perform a Chi-Square test for independence at the 5% significance level. State the p-value and your conclusion.

Question 7 (6 Marks)

A geneticist expects a ratio of 9 : 3 : 3 : 1 for four phenotypes (A, B, C, D) in a certain plant. She grows 160 plants and observes the following: - Phenotype A: 85 - Phenotype B: 34 - Phenotype C: 35 - Phenotype D: 6
Perform a Goodness of Fit test at the 10% significance level to determine if the plants follow the expected genetic ratio.

Question 8 (6 Marks)

A researcher is testing if a dataset follows a Binomial distribution $B(3, 0.4)$. The total sample size is 50. The expected frequencies for outcomes $x = 0, 1, 2, 3$ are calculated as:

$$f_e(0) = 10.8, \quad f_e(1) = 21.6, \quad f_e(2) = 14.4, \quad f_e(3) = 3.2$$

Explain why a standard Goodness of Fit test cannot be performed directly on these 4 categories. Demonstrate the necessary adjustment, and state the new degrees of freedom (df) for the test.

Question 9 (6 Marks)

A quality control machine rejects metal bolts if their length is greater than 52.5 mm. The null hypothesis (H_0) is that the machine is functioning properly and the bolt lengths are normally distributed with $\mu = 50$ mm and $\sigma = 1.2$ mm. Calculate the exact probability of a **Type I error** for a perfectly manufactured bolt.

Question 10 (6 Marks)

A town council wants to check if the number of accidents per week at a junction follows a Poisson distribution. They record data for 100 weeks.

Accidents	0	1	2	3	≥ 4
Weeks (Obs)	32	40	18	7	3

The council uses the sample data to estimate the Poisson mean, calculating $\lambda = 1.1$ accidents per week.

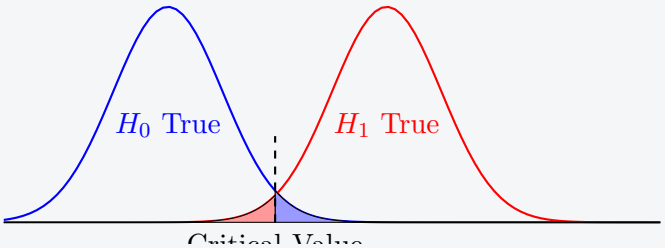
State the number of degrees of freedom that should be used when executing the χ^2 Goodness of Fit test for this data, explaining your mathematical reasoning.

SECTION C: HARD (Synthesis & Proof)

CG50 Tip: Type II Errors

Type II Errors (β) are about the *Alternative Hypothesis*. It is the probability of falling into the H_0 **Acceptance Region**, but calculated using the mean and standard deviation from the **Alternative** distribution!

Question 11 (7 Marks)



The diagram shows the distributions for a 1-tailed hypothesis test. $H_0 : \mu = 50$, against $H_1 : \mu = 55$. The population standard deviation is $\sigma = 10$, and sample size $n = 25$. The decision rule is to reject H_0 if the sample mean $\bar{x} > 53.29$.
Calculate the exact probability of making a **Type II error** (β).

Question 12 (8 Marks)

A researcher claims that the heights of plants in a greenhouse follow a Normal distribution with $\mu = 120$ cm and $\sigma = 15$ cm. He categorizes 200 plants into three intervals: - Group 1: < 105 cm - Group 2: 105 cm to 135 cm - Group 3: > 135 cm
Calculate the expected frequency (f_e) for each of the three groups, correct to 1 decimal place.

Question 13 (8 Marks)

A 3×2 contingency table is tested for independence. The calculated test statistic is $\chi^2_{calc} = 5.81$. Using the χ^2 distribution on your GDC, find the exact p-value for this test. If the test was conducted at the 10% significance level, state whether the variables are independent.

Question 14 (7 Marks)

Using the scenario from Question 11, the researcher wants to reduce the probability of making a Type I error (α), so they change the critical value from 53.29 to 54.65.
Explain theoretically and visually what changing this critical value will do to the probability of making a **Type II error** (β).

Question 15 (9 Marks)

A school analyzes the grades (1 to 7) of 150 students in Math AI HL to see if they fit a binomial distribution model $B(6, 0.6) + 1$. The expected frequencies are generated, but the expected count for Grade 1 is 0.614 and for Grade 2 is 5.53. To perform the χ^2 Goodness of Fit test, the statistician combines Grades 1 and 2 into a single category. Grades 3, 4, 5, 6, and 7 remain as separate categories.

(a) State the number of categories that will be inputted into the GDC for the GOF test. **[1 mark]**

(b) The parameters $n = 6$ and $p = 0.6$ were given theoretically, not estimated from the sample mean. Deduce the correct degrees of freedom (df) for this specific test. **[3 marks]**

(c) The observed frequencies (after combining Grades 1 and 2) are: $10, 25, 45, 40, 20, 10$. The expected frequencies are: $6.14, 20.7, 41.5, 46.1, 27.6, 7.96$. Calculate the χ^2 statistic and state if the data fits the binomial model at the 5% significance level. **[5 marks]**