

IB MATHEMATICS AA HL

AHL TOPIC 5 PRACTICE

Limits, First Principles, and L'Hôpital's Rule

Instructions to Candidates

- This practice paper contains **20** questions progressing from Easy to Very Hard.
- Each question indicates whether it is styled for **Paper 1 (No Calculator)** or **Paper 2 (Calculator Allowed)**.
- The paper tests syllabus topics AHL 5.12 and 5.13: Informal understanding of continuity and differentiability, evaluating limits, finding derivatives from first principles, higher derivatives, and L'Hôpital's Rule for indeterminate forms.
- Answer all questions, showing all your working clearly.
- Total marks available: **95**.

Difficulty Progression

- **SECTION A (Easy):** Setting up first principles formulas, evaluating simple limits (direct substitution and basic L'Hôpital), checking continuity on graphs, and 2nd derivatives.
- **SECTION B (Medium):** First principles derivations for quadratic/cubic polynomials, applying L'Hôpital to e^x and $\ln x$, handling $0 \cdot \infty$ indeterminate forms, and algebraically proving differentiability for piecewise functions.
- **SECTION C (Hard):** Repeated L'Hôpital's rule applications, 1^∞ indeterminate forms using logarithms, formal proofs of differentiation for $\frac{1}{x}$ and $\sin x$, and limits evaluating Maclaurin series coefficients.

SECTION A: EASY (Fundamentals)**Question 1 (3 Marks) — Paper 1 (No Calculator Allowed)**

Evaluate the following limit, showing your working clearly:

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$$

Question 2 (3 Marks) — Paper 1 (No Calculator Allowed)

Using L'Hôpital's rule, evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(4x)}{x}$$

Question 3 (3 Marks) — Paper 1 (No Calculator Allowed)

State the formal limit definition of the derivative $f'(x)$ from first principles. Use this definition to set up (but do not evaluate) the expression for the derivative of $f(x) = 2x + 5$.

Question 4 (4 Marks) — Paper 2 (Calculator Allowed)

Find the second derivative, $f''(x)$, of the function $f(x) = x^4 \ln x$ for $x > 0$.

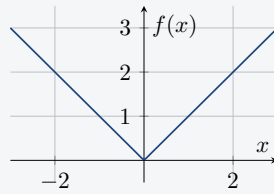
Question 5 (3 Marks) — Paper 1 (No Calculator Allowed)

Evaluate the limit as x approaches infinity:

$$\lim_{x \rightarrow \infty} \frac{5x^3 - 2x + 1}{2x^3 + 4}$$

Question 6 (4 Marks) — Paper 1 (No Calculator Allowed)

Consider the function $f(x) = |x|$.



- (a) State whether $f(x)$ is continuous at $x = 0$. [1 mark]
- (b) Determine whether $f(x)$ is differentiable at $x = 0$. Justify your answer using left and right limits. [3 marks]

Question 7 (3 Marks) — Paper 2 (Calculator Allowed)

Use L'Hôpital's rule to evaluate:

$$\lim_{x \rightarrow 0} \frac{e^{3x} - 1}{2x}$$

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SECTION B: MEDIUM (Application & Algebraic Methods)**Question 8 (5 Marks) — Paper 1 (No Calculator Allowed)**

Prove from first principles that the derivative of $f(x) = x^2 + 3x$ is $f'(x) = 2x + 3$.

Question 9 (5 Marks) — Paper 2 (Calculator Allowed)

A function $f(x)$ is defined as:

$$f(x) = \frac{\ln x}{x^2 - 1}$$

The function is undefined at $x = 1$. Determine the value that $f(1)$ must be assigned in order to make the function continuous at $x = 1$.

Question 10 (7 Marks) — Paper 1 (No Calculator Allowed)

A piecewise function $g(x)$ is defined as:

$$g(x) = \begin{cases} px^2 + q & \text{for } x \leq 1 \\ 4x - 1 & \text{for } x > 1 \end{cases}$$

- (a) Given that $g(x)$ is differentiable at $x = 1$, find the value of p . **[3 marks]**
- (b) Hence, find the value of q to ensure the function is continuous everywhere. **[4 marks]**

Question 11 (5 Marks) — Paper 1 (No Calculator Allowed)

Evaluate the following limit, which initially presents the indeterminate form $0 \cdot \infty$:

$$\lim_{x \rightarrow 0^+} x^2 \ln x$$

(Hint: Rewrite the expression as a fraction before applying L'Hôpital's rule).

Question 12 (5 Marks) — Paper 2 (Calculator Allowed)

Given the function $y = e^{-2x} \cos x$, find the exact value of the third derivative, $\frac{d^3y}{dx^3}$, at $x = 0$.

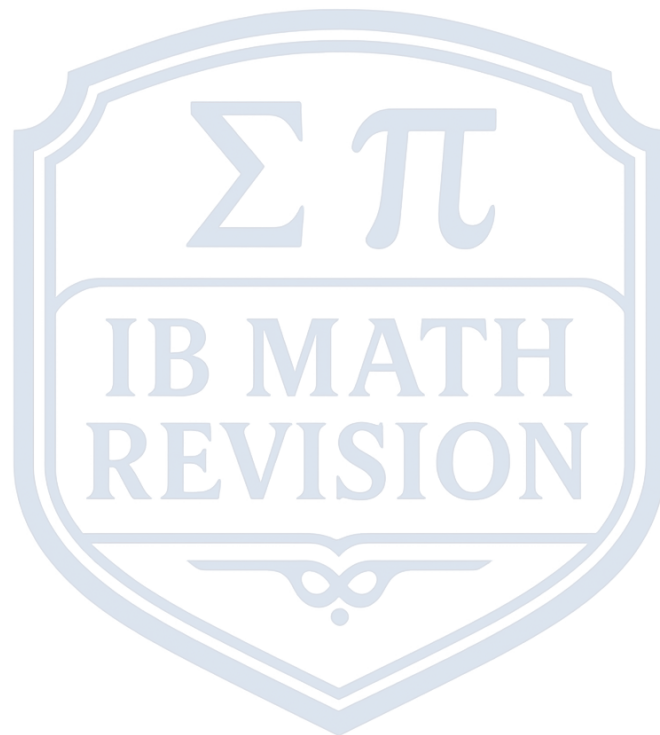
Question 13 (5 Marks) — Paper 1 (No Calculator Allowed)

Evaluate the limit:

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^x}$$

Question 14 (6 Marks) — Paper 1 (No Calculator Allowed)

Prove from first principles that the derivative of $f(x) = \frac{1}{x}$ is $f'(x) = -\frac{1}{x^2}$.



SECTION C: HARD / VERY HARD (Synthesis & Proof)**Question 15 (6 Marks) — Paper 1 (No Calculator Allowed)**

Evaluate the following limit requiring repeated applications of L'Hôpital's rule:

$$\lim_{x \rightarrow 0} \frac{x - \tan x}{x^3}$$

Question 16 (7 Marks) — Paper 2 (Calculator Allowed)

Evaluate the limit:

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$$

Question 17 (7 Marks) — Paper 1 (No Calculator Allowed)

Evaluate the following limit involving the indeterminate form 1^∞ :

$$\lim_{x \rightarrow \infty} \left(1 + \frac{3}{x} \right)^x$$

(Hint: Let y equal the limit expression and take the natural logarithm of both sides).

Question 18 (7 Marks) — Paper 2 (Calculator Allowed)

The Maclaurin series expansion for $\cos x$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

Evaluate the following limit:

$$\lim_{x \rightarrow 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$$

Question 19 (5 Marks) — Paper 1 (No Calculator Allowed)

Using the formal definition of the derivative from first principles, prove that for $f(x) = \sqrt{x}$, the derivative is $f'(x) = \frac{1}{2\sqrt{x}}$.

(Hint: You will need to multiply the numerator and denominator by the conjugate).

Question 20 (7 Marks) — Paper 1 (No Calculator Allowed)

Find the exact values of the constants a and b such that:

$$\lim_{x \rightarrow 0} \frac{\sin(2x) + ax + bx^3}{x^3} = 0$$