# IB MATHEMATICS AI HL
# AHL QUESTION BOOKLET

## Matrices, Determinants & Inverses

### Instructions to Candidates

- This extended practice paper contains **15 questions**.
- The paper targets **Advanced Higher Level (AHL)** syllabus components 1.14 and 1.15.
- Note: In Mathematics AI, all papers require a Graphic Display Calculator (GDC). Use it efficiently to evaluate $3 \times 3$ inverses and determinants.
- Answer all questions, showing all step-by-step working clearly in the spaces provided.



### Difficulty Progression

- **Questions 1 - 5 (Easy):** Matrix dimensions, basic addition/subtraction, scalar multiplication, and evaluating $2 \times 2$ determinants and inverses by hand.
- **Questions 6 - 10 (Medium):** Matrix algebra ($AX = B$), setting up matrices from real-world contexts, and finding unknowns in singular matrices.
- **Questions 11 - 15 (Hard):** Complex synthesis. Solving $3 \times 3$ systems using inverse matrices, identity matrix proofs, and scaling properties of determinants.

# SECTION A: EASY (Fundamentals)

## Question 1 (4 Marks)

Consider the matrices $A = \begin{pmatrix} 4 & -2 \\ 1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}$.

**(a)** Calculate exactly the matrix $2A - B$. [**2 marks**]
**(b)** Calculate exactly the matrix product $AB$. [**2 marks**]

## Question 2 (4 Marks)

A matrix $M$ is defined as $M = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$.

**(a)** Calculate the determinant of $M$, denoted as $\det(M)$ or $|M|$. [**2 marks**]
**(b)** Hence, write down the exact inverse matrix, $M^{-1}$. [**2 marks**]

## Question 3 (4 Marks)

Matrix $P$ has dimensions $4 \times 3$. Matrix $Q$ has dimensions $3 \times 5$.
**(a)** State the dimensions of the product matrix $PQ$. [**1 mark**]
**(b)** Explain briefly why it is mathematically impossible to calculate the product $QP$. [**1 mark**]
**(c)** Matrix $R$ is a square identity matrix $I_3$. State the dimensions of $PR$, and describe its relationship to matrix $P$. [**2 marks**]

## Question 4 (4 Marks)

Given the matrix equation:

$$3\begin{pmatrix} x & 4 \\ -2 & y \end{pmatrix} + \begin{pmatrix} 2 & -5 \\ z & 1 \end{pmatrix} = \begin{pmatrix} 14 & 7 \\ -1 & 10 \end{pmatrix}$$

Find the exact values of $x$, $y$, and $z$.

## Question 5 (5 Marks)

Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$.

**(a)** Calculate $A^2$. [**2 marks**]
**(b)** Show that $A^2 - A + 2I = 0$, where $I$ is the $2 \times 2$ identity matrix and $0$ is the $2 \times 2$ zero

matrix. **[3 marks]**

# SECTION B: MEDIUM (Application & Modelling)

## Question 6 (5 Marks)

A company sells two types of bundles: Bundle A contains 3 phones and 2 cases, and costs \$1850. Bundle B contains 4 phones and 5 cases, and costs \$2700.
Let $p$ be the price of one phone and $c$ be the price of one case.

**(a)** Write this information as a system of linear equations in the form $AX = B$, where $X = \begin{pmatrix} p \\ c \end{pmatrix}$. **[2 marks]**

**(b)** Find $A^{-1}$ and use it to calculate the exact price of one phone and one case. **[3 marks]**

## Question 7 (6 Marks)

Consider the matrix $K = \begin{pmatrix} x & 4 \\ 9 & x \end{pmatrix}$.

**(a)** Find an expression for the determinant of $K$ in terms of $x$. **[2 marks]**

**(b)** A matrix is singular if it does not have an inverse. Find the two possible values of $x$ that make matrix $K$ singular. **[4 marks]**

## Question 8 (6 Marks)

Matrix $A = \begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}$ and matrix $B = \begin{pmatrix} 11 \\ 2 \end{pmatrix}$.

Solve the matrix equation $AX = B$ for the column vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

Show your method clearly using the inverse matrix $A^{-1}$.

## Question 9 (6 Marks)

Consider the transformation matrix $T = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}$, which models the yearly transition of populations between a city and the surrounding suburbs.

The initial population vector in year 0 is $V_0 = \begin{pmatrix} 150 \\ 50 \end{pmatrix}$, where the values represent thousands of people.

The population in year $n$ is given by $V_n = T^n V_0$.

**(a)** Calculate the exact population vector for year 1, $V_1$. **[2 marks]**

**(b)** Use your GDC to calculate the population vector for year 10, $V_{10}$. Give your answers correct to three significant figures. **[4 marks]**

## Question 10 (6 Marks)

Given the equation $MA + M = B$, where $A = \begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 9 & -5 \\ 3 & 10 \end{pmatrix}$.

**(a)** Factorise the left-hand side of the equation to isolate $M$. Hint: Use the identity matrix $I$. [**2 marks**]

**(b)** Hence, find matrix $M$. [**4 marks**]

## SECTION C: HARD (Synthesis & Proof)

### Question 11 (7 Marks)

A system of three linear equations is given by:

$$2x - y + 3z = 14$$
$$x + 5y - z = 2$$
$$3x + 2y + 4z = 21$$

**(a)** Express this system in the matrix form $MX = C$. [**2 marks**]
**(b)** Use your GDC to find the inverse matrix $M^{-1}$. Write down all elements to 3 significant figures. [**2 marks**]
**(c)** Hence, use matrix multiplication to find the exact values of $x$, $y$, and $z$. [**3 marks**]

### Question 12 (8 Marks)

Consider a $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
The determinant of $A$ is given as $\det(A) = 15$.
**(a)** Matrix $B$ is created by multiplying every element in $A$ by 3. Find $\det(B)$, showing your algebraic reasoning. [**3 marks**]
**(b)** Matrix $C$ is the inverse of $A$. Given that $\det(A^{-1}) = \frac{1}{\det(A)}$, evaluate exactly the determinant of $4C$. [**5 marks**]

### Question 13 (8 Marks)

A $3 \times 3$ matrix $T$ is used to encrypt a message vector $M = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ into a coded vector $C$ using the rule $C = TM$.
Given $T = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & 0 \end{pmatrix}$ and the received coded vector is $C = \begin{pmatrix} -1 \\ 17 \\ 8 \end{pmatrix}$.
**(a)** Write down an expression for the original message vector $M$ in terms of $T$ and $C$. [**2 marks**]
**(b)** Calculate the determinant of $T$ using your GDC to verify the message can be decoded. [**2 marks**]
**(c)** Find the original message vector $M$. [**4 marks**]

## Question 14 (7 Marks)

Let $A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$.
Find a matrix $X$ such that $AXB = I$, where $I$ is the $2 \times 2$ identity matrix.
Show all your matrix algebra steps clearly before calculating the final values.

## Question 15 (9 Marks)

Consider the system of equations:

$$kx + 2y = 8$$
$$3x + (k-1)y = 12$$

where $k$ is a real constant.
**(a)** Write down the coefficient matrix $A$ for this system. [**1 mark**]
**(b)** Find the values of $k$ for which the system does not have a unique solution (i.e., the coefficient matrix is singular). [**4 marks**]
**(c)** For the positive value of $k$ found in part (b), determine whether the system has infinite solutions or no solution. Justify your answer. [**4 marks**]