

# IB MATHEMATICS AI HL

## UNIT 4: STATISTICS

### Markov Chains & Transition Matrices

#### Instructions to Candidates

- This question booklet contains **15 questions**.
- The paper targets **AHL** syllabus component 4.19.
- Note: In the IB, transition matrices operate on column vectors ( $s_n = T^n s_0$ ). Therefore, the **columns** of a transition matrix must sum to 1.
- Answer all questions, showing all step-by-step working clearly.

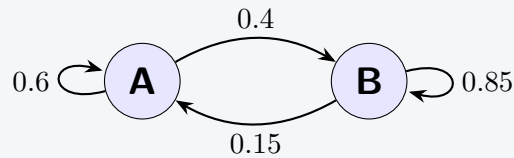
#### Difficulty Progression

- **Questions 1 - 5 (Easy):** Reading state transition diagrams, constructing basic matrices, and calculating  $s_1 = T s_0$ .
- **Questions 6 - 10 (Medium):** Powers of matrices  $T^n$ , GDC steady-state calculations, and 3-state systems.
- **Questions 11 - 15 (Hard):** Algebraic steady-state proofs ( $Ts = s$ ), finding eigenvalues of  $T$ , absorbing states, and reverse-engineering unknown probabilities.

## SECTION A: EASY (Fundamentals)

## Question 1 (4 Marks)

The transition between two states, A and B, is shown in the directed network diagram below.



Write down the  $2 \times 2$  transition matrix  $T$  for this Markov chain. Ensure that the states A and B map to the columns (initial state) and rows (final state) in alphabetical order.

## Question 2 (4 Marks)

A system is modelled by the transition matrix  $T = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$ .

The initial state of the system is given by the column vector  $s_0 = \begin{pmatrix} 100 \\ 50 \end{pmatrix}$ .

Calculate the exact state vector after one transition,  $s_1$ .

## Question 3 (4 Marks)

A weather model categorizes days as either Sunny (S) or Rainy (R). - If it is Sunny today, there is an 80% chance it will be Sunny tomorrow. - If it is Rainy today, there is a 60% chance it will be Rainy tomorrow.

Construct the transition matrix  $T$  for this model, using the order (S, R).

## Question 4 (4 Marks)

Using the weather model from Question 3, assume that today (Day 0) is a Rainy day. This can be represented by the initial state vector  $s_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .

Calculate the exact probability that it will be Sunny on Day 2.

## Question 5 (5 Marks)

Consider the transition matrix  $M = \begin{pmatrix} 0.5 & 0 & 0.3 \\ 0.3 & 1 & 0.4 \\ 0.2 & 0 & 0.3 \end{pmatrix}$  representing states 1, 2, and 3.

(a) Identify which state is an *absorbing state*, and explain how you know. [2 marks]

(b) Determine the values of the column vector resulting from  $M \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ . [3 marks]

## SECTION B: MEDIUM (Application & Modelling)

### CG50 Tip: Long-Term Steady States

To find the long-term steady state (equilibrium) of a regular Markov chain on your GDC without doing algebra, define your transition matrix as **Mat A**. Then, calculate **Mat A**<sup>50</sup>. The matrix will converge, and all of its columns will become identical! Any one of these columns is your steady-state vector  $s$ .

### Question 6 (5 Marks)

A car rental company operates in two cities, City X and City Y. - A car rented in City X has a 70% chance of being returned to City X, and a 30% chance of being returned to City Y. - A car rented in City Y has a 90% chance of being returned to City Y, and a 10% chance of being returned to City X.

Currently, there are 400 cars in City X and 200 cars in City Y.

Using your GDC, find the number of cars expected to be in City X after 5 rental cycles (transitions). Round your answer to the nearest whole car.

### Question 7 (6 Marks)

Using the same car rental company from Question 6, determine the *long-term steady-state* distribution of the 600 cars. In the long run, how many cars will be located in City Y?

### Question 8 (6 Marks)

A transition matrix is given as  $T = \begin{pmatrix} 0.6 & x \\ y & 0.2 \end{pmatrix}$ .

(a) Find the values of  $x$  and  $y$ . [2 marks]

(b) The steady state vector is  $s = \begin{pmatrix} a \\ b \end{pmatrix}$ . Use the algebraic property  $Ts = s$  and the fact that  $a + b = 1$  to find the exact fractions for  $a$  and  $b$ . [4 marks]

### Question 9 (6 Marks)

Three brands of coffee (A, B, C) compete for market share. Their monthly transition matrix is:

$$T = \begin{pmatrix} 0.5 & 0.2 & 0.1 \\ 0.3 & 0.6 & 0.4 \\ 0.2 & 0.2 & 0.5 \end{pmatrix}$$

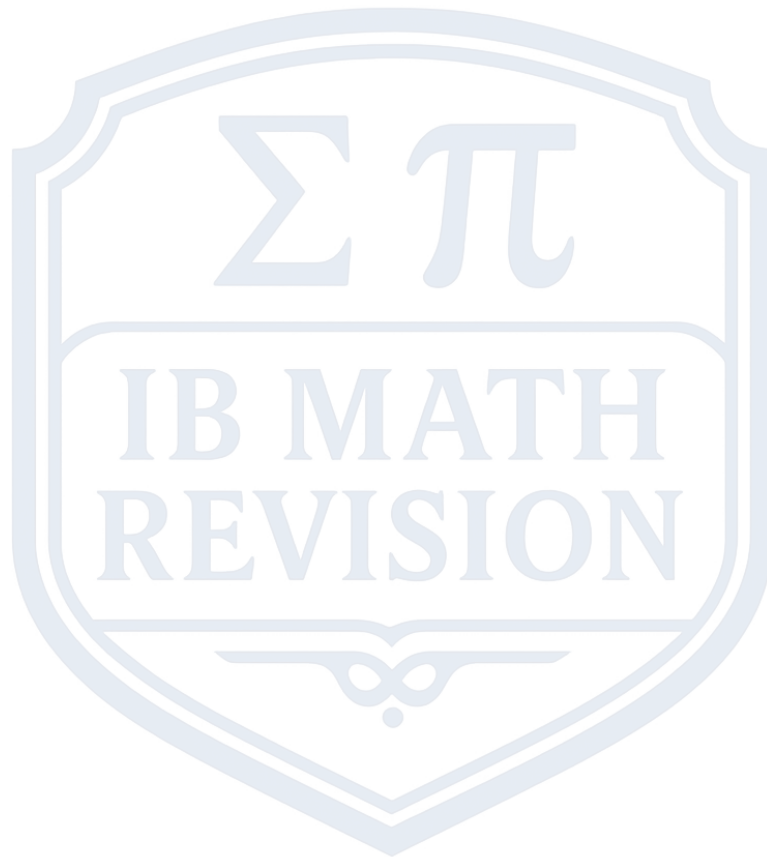
Initially, Brand A has 50%, Brand B has 30%, and Brand C has 20% of the market.

Calculate the market share of Brand B after 2 months. Give your answer as a percentage.

**Question 10 (6 Marks)**

A machine has two states: Working (W) and Broken (B). If it is working on day  $n$ , there is a 5% chance it will be broken on day  $n + 1$ . If it is broken on day  $n$ , it costs \$200 to repair, and there is an 80% chance it will be working on day  $n + 1$ .

Find the long-term steady-state probability that the machine is broken, and hence estimate the expected repair cost per day over a very long period.



## SECTION C: HARD (Synthesis & Proof)

### Question 11 (7 Marks)

Consider the transition matrix  $T = \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix}$ .

In linear algebra, a scalar  $\lambda$  is an *eigenvalue* of a matrix  $T$  if  $\det(T - \lambda I) = 0$ , where  $I$  is the identity matrix.

Prove algebraically that  $\lambda = 1$  is an eigenvalue of  $T$ , and find the other eigenvalue.

### Question 12 (8 Marks)

A subscription service offers three tiers: Basic (B), Premium (P), and VIP (V). The yearly transition probabilities between the tiers are given by:

$$T = \begin{pmatrix} 0.5 & 0.2 & 0.1 \\ 0.4 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}$$

Let the steady-state column vector be  $s = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ .

By setting up a system of linear equations from  $Ts = s$  and  $x + y + z = 1$ , use your GDC's simultaneous equation solver to find the exact steady-state vector  $s$ .

### Question 13 (8 Marks)

A transition matrix has an unknown probability  $k$ :

$$T = \begin{pmatrix} 0.4 & k \\ 0.6 & 1 - k \end{pmatrix}$$

The long-term steady state probability of being in state 1 is exactly 0.25.  
Find the exact value of  $k$ .

### Question 14 (7 Marks)

Phoebe opens a coffee shop near an established Apollo coffee shop. - 15% of customers who preferred Apollo one month will prefer Phoebe's the next month. - 35% of customers who preferred Phoebe's one month will prefer Apollo the next month.

Let  $T$  be the transition matrix with order (Apollo, Phoebe).

(a) Write down the transition matrix  $T$ . [2 marks]

(b) If there is a total fixed population of 5000 coffee drinkers, determine how many customers Phoebe's coffee shop will have in the long-term equilibrium state. [5 marks]

**Question 15 (9 Marks)**

A rat is placed in a maze with 3 rooms: Room 1, Room 2, and a Trap (Room 3). If the rat is in Room 1, it has a 0.5 probability of staying in Room 1, a 0.3 probability of moving to Room 2, and a 0.2 probability of moving to the Trap. If the rat is in Room 2, it has a 0.4 probability of staying in Room 2, a 0.4 probability of moving to Room 1, and a 0.2 probability of moving to the Trap. If the rat enters the Trap (Room 3), it cannot leave.

- (a) Write down the  $3 \times 3$  transition matrix  $T$ . [3 marks]  
(b) The rat starts in Room 1. Calculate the probability that the rat is still alive (not in the Trap) after exactly 3 moves. [4 marks]  
(c) State the long-term steady state vector for this system. [2 marks]

