IB MATHEMATICS AA HL
AHL TOPIC 1 PRACTICE

Partial Fractions & Applications

Instructions to Candidates

- This practice paper contains **15** questions progressing from Easy to Very Hard.
- Each question indicates whether it is styled for **Paper 1 (No Calculator)** or **Paper 2 (Calculator Allowed)**.
- The paper strictly follows the HL syllabus constraint: *"Decomposing rational functions into partial fractions (limited to a maximum of two distinct linear terms in the denominator)."*
- Answer all questions, showing all your working clearly.
- Total marks available: **66**.

Difficulty Progression

- **SECTION A (Easy):** Basic algebraic decomposition, setting up identities, solving for coefficients using substitutions, and simple repeated roots.
- **SECTION B (Medium):** Improper fractions requiring algebraic long division first, and fundamental applications of partial fractions to indefinite and definite integration.
- **SECTION C (Hard):** Deep synthesis. Using partial fractions to generate infinite Maclaurin/Binomial series, solving non-linear separable differential equations, and telescoping sums.

SECTION A: EASY (Fundamentals)

Question 1 (2 Marks) — Paper 1 (No Calculator Allowed)

Express $\frac{5x+4}{(x-1)(x+2)}$ in the form $\frac{A}{x-1} + \frac{B}{x+2}$, where A and B are integers.

Question 2 (3 Marks) — Paper 1 (No Calculator Allowed)

Decompose the rational function $\frac{3x+5}{(x+1)(x+3)}$ into partial fractions.

Question 3 (3 Marks) — Paper 1 (No Calculator Allowed)

Find the exact values of the constants P and Q such that:

$$\frac{2}{(x-2)(x+4)} \equiv \frac{P}{x-2} + \frac{Q}{x+4}$$

Question 4 (3 Marks) — Paper 1 (No Calculator Allowed)

Express $\frac{7x-4}{(2x-1)(x-2)}$ in partial fractions.

Question 5 (3 Marks) — Paper 1 (No Calculator Allowed)

A rational function contains a repeated linear denominator. Express $\frac{3x+2}{(x+1)^2}$ in the form $\frac{A}{x+1} + \frac{B}{(x+1)^2}$.

SECTION B: MEDIUM (Application & Improper Fractions)

Question 6 (4 Marks) — Paper 1 (No Calculator Allowed)

The rational function $\frac{x^2+1}{x^2-1}$ is an improper fraction because the degree of the numerator is equal to the degree of the denominator.

(a) By using polynomial long division or algebraic manipulation, express the function in the form $C + \frac{Dx+E}{x^2-1}$. **[1 mark]**

(b) Hence, express $\frac{x^2+1}{x^2-1}$ fully in partial fractions. **[3 marks]**

Question 7 (4 Marks) — Paper 2 (Calculator Allowed)

Use your answer from Question 2 to find the indefinite integral:

$$\int \frac{3x+5}{(x+1)(x+3)}\,dx$$

Question 8 (4 Marks) — Paper 1 (No Calculator Allowed)

(a) Express $\frac{x^2+2x+2}{x(x+1)}$ in the form $A + \frac{B}{x} + \frac{C}{x+1}$. **[3 marks]**

(b) State the equation of the horizontal asymptote of the graph of $y = \frac{x^2+2x+2}{x(x+1)}$. **[1 mark]**

Question 9 (5 Marks) — Paper 1 (No Calculator Allowed)

By using partial fractions, show that the exact value of the definite integral:

$$\int_2^3 \frac{2}{(x-1)(x+1)}\,dx$$

can be written in the form $\ln\left(\frac{a}{b}\right)$, where a and b are integers to be found.

Question 10 (5 Marks) — Paper 1 (No Calculator Allowed)

Find the exact area enclosed by the curve $y = \frac{10x-5}{(x-2)(2x+1)}$, the x-axis, and the vertical lines $x = 3$ and $x = 5$. Give your answer in the form $\ln(k)$.

SECTION C: HARD / VERY HARD (Synthesis & Proof)

Question 11 (6 Marks) — Paper 1 (No Calculator Allowed)

Consider the function $f(x) = \frac{x-5}{(x+1)(x-2)}$.

(a) Express $f(x)$ in partial fractions. **[2 marks]**

(b) Hence, find the Maclaurin (binomial) series expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. **[4 marks]**

Question 12 (6 Marks) — Paper 1 (No Calculator Allowed)

A curve $y = f(x)$ passes through the point $(e, 2)$ and satisfies the separable differential equation:

$$\frac{dy}{dx} = \frac{y^2 - 1}{2x} \quad \text{for } x > 0 \text{ and } y > 1$$

(a) Separate the variables and use partial fractions to integrate both sides. **[4 marks]**

(b) Find the particular solution, expressing y explicitly in terms of x. **[2 marks]**

Question 13 (5 Marks) — Paper 1 (No Calculator Allowed)

[Method of Differences / Telescoping Sums]

(a) Express $\frac{1}{r(r+1)}$ in partial fractions. **[1 mark]**

(b) Hence, evaluate the sum:

$$\sum_{r=1}^{n} \frac{1}{r(r+1)}$$

giving your answer as a single algebraic fraction in terms of n. **[3 marks]**

(c) State the value of the sum as $n \to \infty$. **[1 mark]**

Question 14 (7 Marks) — Paper 2 (Calculator Allowed)

A population of animals, P, at time t years, is modelled by the logistic growth differential equation:

$$\frac{dP}{dt} = 0.5P(100 - P)$$

Initially, when $t = 0$, the population is 20.

(a) Express $\frac{1}{P(100-P)}$ in partial fractions. **[2 marks]**

(b) By solving the differential equation, show that the population at time t is given by:

$$P(t) = \frac{100}{1 + 4e^{-50t}}$$

[5 marks]

Question 15 (6 Marks) — Paper 1 (No Calculator Allowed)

Consider the function $g(x) = \frac{3}{(1-x)(1+2x)}$ for $|x| < \frac{1}{2}$.

(a) Express $g(x)$ in partial fractions. **[2 marks]**

(b) Use the infinite binomial expansion to show that the coefficient of x^n in the expansion of $g(x)$ is given by the expression:

$$1 + 2(-2)^n$$

[4 marks]