IB MATHEMATICS AA HL
AHL TOPIC 1 PRACTICE

Advanced Formal Proof (Induction, Contradiction, Counterexamples)

Instructions to Candidates

- This practice paper contains **15** questions progressing from Easy to Very Hard.
- Each question indicates whether it is styled for **Paper 1 (No Calculator)** or **Paper 2 (Calculator Allowed)**.
- The paper tests syllabus topic AHL 1.15: Proof by Mathematical Induction, Proof by Contradiction, and disproving mathematical statements using Counterexamples.
- Answer all questions, showing all your working clearly.
- Total marks available: **81**.

Difficulty Progression

- **SECTION A (Easy):** Basic counterexamples, setting up contradiction assumptions, and foundational induction (simple sums and divisibility).
- **SECTION B (Medium):** Standard induction for series, classic contradictions (e.g., irrationality proofs), and finding counterexamples for algebraic/trigonometric identities.
- **SECTION C (Hard):** Complex induction involving differentiation, recursive sequences, De Moivre's theorem, inequalities, and advanced contradiction (e.g., infinitude of primes).

SECTION A: EASY (Fundamentals)

Question 1 (2 Marks) — Paper 2 (Calculator Allowed)

A student claims that "If n is a prime number, then $2^n - 1$ is also a prime number." Provide a specific counterexample to prove that this statement is false.

Question 2 (2 Marks) — Paper 1 (No Calculator Allowed)

To prove the statement "If x^2 is an even integer, then x is an even integer" by contradiction, write down the exact assumption you must make at the beginning of the proof.

Question 3 (5 Marks) — Paper 1 (No Calculator Allowed)

Use mathematical induction to prove that for all $n \in \mathbb{Z}^+$:

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

Question 4 (3 Marks) — Paper 2 (Calculator Allowed)

Disprove the following trigonometric identity by finding a suitable counterexample:

$$\sin(x+y) = \sin x + \sin y$$

Question 5 (5 Marks) — Paper 1 (No Calculator Allowed)

Use mathematical induction to prove that $3^n - 1$ is an even number (divisible by 2) for all positive integers n.

SECTION B: MEDIUM (Application & Standard Proofs)

Question 6 (6 Marks) — Paper 1 (No Calculator Allowed)

Prove by contradiction that $\sqrt{3}$ is an irrational number.

Question 7 (7 Marks) — Paper 1 (No Calculator Allowed)

Use mathematical induction to prove that for all $n \in \mathbb{Z}^+$:

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

Question 8 (4 Marks) — Paper 2 (Calculator Allowed)

A mathematician suggests that the polynomial $P(x) = x^2 + x + 41$ generates a prime number for every positive integer x. By finding a counterexample, show that this conjecture is false.

Question 9 (5 Marks) — Paper 1 (No Calculator Allowed)

Prove by contradiction that there are no rational roots to the equation $x^2 - 2 = 0$.

Question 10 (6 Marks) — Paper 1 (No Calculator Allowed)

Use mathematical induction to prove that $8^n - 3^n$ is divisible by 5 for all integers $n \geq 1$.

SECTION C: HARD / VERY HARD (Synthesis & Complex Induction)

Question 11 (8 Marks) — Paper 1 (No Calculator Allowed)

[Calculus & Induction Synthesis]
Let $f(x) = xe^x$. Use mathematical induction to prove that the n-th derivative of $f(x)$ is given by:

$$f^{(n)}(x) = (x+n)e^x \quad \text{for all } n \in \mathbb{Z}^+$$

Question 12 (7 Marks) — Paper 1 (No Calculator Allowed)

[Complex Numbers & Induction Synthesis]
Use mathematical induction to prove De Moivre's Theorem for positive integers. That is, prove:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \quad \text{for all } n \in \mathbb{Z}^+$$

Question 13 (6 Marks) — Paper 1 (No Calculator Allowed)

Prove by contradiction that there are an infinite number of prime numbers.

Question 14 (7 Marks) — Paper 1 (No Calculator Allowed)

A sequence is defined recursively by $u_1 = 2$ and $u_{k+1} = 3u_k - 2$ for $k \geq 1$.
Use mathematical induction to prove that the explicit formula for the sequence is:

$$u_n = 3^{n-1} + 1 \quad \text{for all } n \in \mathbb{Z}^+$$

Question 15 (8 Marks) — Paper 1 (No Calculator Allowed)

[Inequalities & Induction]
Use mathematical induction to prove that $2^n > n^2$ for all integers $n \geq 5$.