IB MATHEMATICS AI HL
UNIT 3: GEOMETRY

Scalar Products, Matrices & Voronoi Diagrams

Instructions to Candidates

- This question booklet contains **15 questions**.
- The paper targets **AHL** syllabus components 3.6, 3.13, and 3.14.
- Answer all questions, showing all step-by-step working clearly.

Difficulty Progression

- **Questions 1 - 5 (Easy):** Basic 2D dot products, perpendicular vectors, Voronoi midpoints, and area scale factors.
- **Questions 6 - 10 (Medium):** Angles in 3D, Voronoi vertices, composite matrix transformations, and inverse matrices.
- **Questions 11 - 15 (Hard):** Eigenvalues and invariant lines (eigenvectors), transforming straight line equations, and vector proofs.

SECTION A: EASY (Fundamentals)

CG50 Tip: Scalar Products and Angles

In `MENU 1 (Run-Matrix)`, go to `OPTN` → `MAT/VCT` → `F6` → `DotP(`. Type `DotP([A], [B])` to instantly find the scalar product. To find the angle between them directly, use the `Angle([A], [B])` command found right next to it!

Question 1 (4 Marks)

Calculate the exact scalar (dot) product of the vectors $a = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

Question 2 (4 Marks)

The vectors $p = \begin{pmatrix} k \\ 3 \end{pmatrix}$ and $q = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$ are orthogonal (perpendicular). Find the exact value of the constant k.

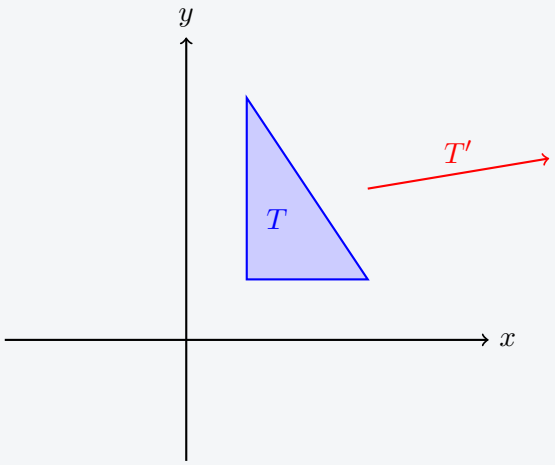
Question 3 (4 Marks)

Two sites on a Voronoi diagram are located at $A(-2, 4)$ and $B(4, -2)$. Find the coordinates of the midpoint of $[AB]$ and calculate the gradient of the perpendicular bisector separating the two sites.

Question 4 (4 Marks)

A triangle has an area of 10 cm^2. It is transformed by the matrix $M = \begin{pmatrix} 4 & -1 \\ 2 & 3 \end{pmatrix}$. Calculate the area of the transformed image triangle.

Question 5 (5 Marks)



The triangle T has vertices at $(1, 1)$, $(3, 1)$, and $(1, 4)$. It is mapped to T' by the transformation matrix $W = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Write down the coordinates of the vertices of T' and describe the geometric transformation represented by W.

SECTION B: MEDIUM (Application & Modelling)

Question 6 (5 Marks)

Find the angle between the 3D vectors $u = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $v = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$.
Give your answer in degrees correct to 1 decimal place.

Question 7 (6 Marks)

A Voronoi diagram is constructed for three towns, A, B, and C. The perpendicular bisector separating towns A and B is given by $y = x + 1$. The perpendicular bisector separating towns B and C is given by $2x + y = 13$.
Find the exact coordinates of the Voronoi vertex formed by these towns.

Question 8 (6 Marks)

A polygon is transformed by a reflection in the y-axis, represented by matrix R. It is then subsequently stretched by a scale factor of 2 in the x-direction, represented by matrix S.
Find the 2×2 matrices for R and S, and hence find the single composite matrix C that represents the combined transformation of R followed by S.

Question 9 (6 Marks)

A point P is transformed by the matrix $M = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ to produce an image point P' with coordinates $(5, -2)$.
By finding the inverse matrix M^{-1}, calculate the exact original coordinates of point P.

Question 10 (6 Marks)

A transformation matrix is given by $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
Determine the image of the point $(3, 4)$ under this transformation. State the precise geometric description of the transformation represented by T.

SECTION C: HARD (Synthesis & Proof)

Question 11 (7 Marks)

A transformation matrix is given by $A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$.

In matrix transformations, *invariant lines* (lines passing through the origin that map onto themselves) correspond to the **eigenvectors** of the matrix.

By solving the characteristic equation $\det(A - \lambda I) = 0$, find the two eigenvalues of matrix A.

Question 12 (8 Marks)

Using your smaller eigenvalue from Question 11, find the corresponding eigenvector. Hence, state the Cartesian equation of this invariant line in the form $y = mx$.

Question 13 (8 Marks)

Three schools are located at $S_1(1, 1)$, $S_2(5, 1)$, and $S_3(3, 5)$. The local council wishes to build a toxic waste facility as far away from the schools as possible, within the triangle formed by the three schools.

This optimal point is the Voronoi vertex, which is located at $(3, 2)$.

Calculate the radius of the largest empty circle that can be drawn around the waste facility, and explain what this distance represents in the context of the problem.

Question 14 (7 Marks)

The straight line L has the Cartesian equation $y = 2x + 1$.

The line L undergoes a geometric transformation represented by the matrix $M = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$ to form a new line L'.

Find the Cartesian equation of the newly transformed line L' in the form $y = mx + c$.

Question 15 (9 Marks)

Consider a rhombus $ABCD$. Let the vectors representing two adjacent sides be $\vec{AB} = \mathbf{a}$ and $\vec{AD} = \mathbf{b}$.

Since the shape is a rhombus, the magnitudes of these sides are equal ($|\mathbf{a}| = |\mathbf{b}|$).

Express the diagonals $\vec{AC}$ and $\vec{BD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Hence, use the scalar (dot) product to prove algebraically that the diagonals of any rhombus intersect at right angles.