IB MATHEMATICS AI HL
UNIT 5: CALCULUS

The Second Derivative & Concavity

Instructions to Candidates

- This question booklet contains **15 questions**.
- The paper targets **AHL** syllabus component 5.10.
- Answer all questions, showing all step-by-step working clearly.

Difficulty Progression

- **Questions 1 - 5 (Easy):** Finding $f''(x)$, basic points of inflexion, the Second Derivative Test, and identifying concavity.
- **Questions 6 - 10 (Medium):** Concavity intervals, interpreting $f'(x)$ graphs, and kinematic acceleration ($s''(t)$).
- **Questions 11 - 15 (Hard):** Algebraic proofs of inflexion points using quotient/product rules, determining unknown parameters, and identifying points of diminishing returns.

SECTION A: EASY (Fundamentals)

Question 1 (4 Marks)

Given the function $f(x) = x^4 - 3x^3 + 2x^2 - 5x + 7$, find an expression for the second derivative, $f''(x)$.

Question 2 (4 Marks)

Consider the function $y = e^{2x} + \sin x$.
Find the exact value of $\frac{d^2y}{dx^2}$ at the point where $x = \frac{\pi}{2}$.

Question 3 (4 Marks)

A curve $y = g(x)$ has a second derivative given by $g''(x) = 3x^2 - 12$.
Determine whether the curve is *concave up* or *concave down* at the point where $x = 1$. Give a mathematical reason for your answer.

CG50 Tip: The Second Derivative Test

You can calculate the numerical second derivative on the CG50 directly! In `Run-Matrix`, press `MATH (F4)` → `d2/dx2 (F5)`. If you have found a stationary point, plug its x-coordinate into this tool. If the result is positive, it's a local minimum. If negative, it's a local maximum!

Question 4 (4 Marks)

A function $f(x)$ has a stationary point at $x = 3$.
You are given that $f'(3) = 0$ and $f''(3) = 14$.
Use the Second Derivative Test to classify this stationary point as a local maximum or a local minimum, and briefly explain why.

Question 5 (5 Marks)

Find the exact coordinates (x, y) of the point of inflexion on the cubic curve $y = x^3 - 6x^2 + 9x + 2$.

SECTION B: MEDIUM (Application & Modelling)

Question 6 (5 Marks)

Consider the function $f(x) = x^4 - 6x^2 + 4$.
Determine the exact intervals for x where the curve is **concave down**.

Question 7 (6 Marks)

Find the second derivative $f''(x)$ of the function $f(x) = \ln(x^2 + 1)$. Express your answer as a single simplified algebraic fraction.

Question 8 (6 Marks)

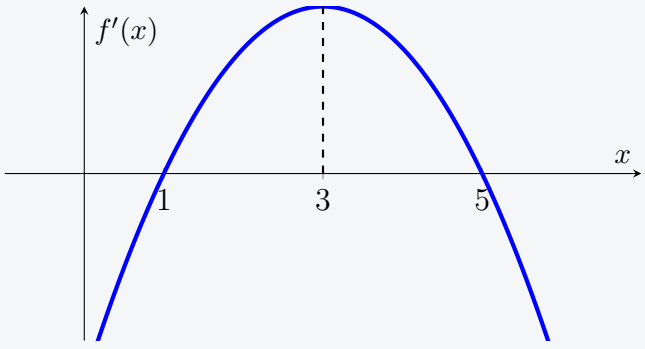
A particle moves along a straight line such that its displacement s (in metres) from a fixed origin O at time t seconds is given by $s(t) = t^3 - 9t^2 + 15t$.
Find the exact time t when the acceleration of the particle is exactly zero.

Question 9 (6 Marks)

Find the exact x-coordinates of the points of inflexion for the function $y = \sin x + \cos x$ in the domain $0 \leq x \leq 2\pi$.

Question 10 (6 Marks)

The diagram below shows the graph of the **first derivative**, $y = f'(x)$, of a function $f(x)$.



State the x-coordinate of the point of inflexion on the original curve $y = f(x)$, and explain how you deduced this from the graph of $f'(x)$.

SECTION C: HARD (Synthesis & Proof)

Question 11 (7 Marks)

Consider the function $f(x) = xe^{-x}$.
Use the product rule to find $f'(x)$ and $f''(x)$, and hence prove algebraically that the curve has exactly one point of inflexion at $x = 2$.

Question 12 (8 Marks)

A cubic function is defined by $f(x) = ax^3 + bx^2 + x$, where a and b are constants.
The curve has a local minimum at $x = 1$, and a point of inflexion at $x = \frac{2}{3}$.
Find the exact values of a and b.

Question 13 (8 Marks)

A function is given by $f(x) = \frac{x}{x^2+1}$.
The first derivative is $f'(x) = \frac{1-x^2}{(x^2+1)^2}$.
Use the quotient rule to find $f''(x)$, and hence determine the exact x-coordinates of all three points of inflexion on the curve.

CG50 Tip: When the Second Derivative Test Fails

If you plug a stationary point into `d2/dx2` and the result is exactly 0, the Second Derivative Test is **inconclusive**! It could be a minimum, maximum, or stationary point of inflexion. You MUST use the First Derivative Test (checking gradients slightly to the left and right) to classify it.

Question 14 (7 Marks)

Consider the function $y = x^4$.
(a) Show that there is a stationary point at $x = 0$, but the Second Derivative Test fails to classify it. **[3 marks]**
(b) Use the First Derivative Test to determine the nature of the stationary point. **[4 marks]**

Question 15 (9 Marks)

A company models its total profit, P (in thousands of dollars), from producing x hundred units of a product using the equation:

$$P(x) = -0.1x^3 + 6x^2 + 400x \quad \text{for } 0 \leq x \leq 50$$

In economics, the "Point of Diminishing Returns" occurs when the *rate of profit increase* reaches its absolute maximum, after which the rate of profit begins to slow down.

(a) Write down expressions for the marginal profit $P'(x)$ and the second derivative $P''(x)$. **[3 marks]**

(b) Find the number of units x that should be produced to reach the point of diminishing returns. **[3 marks]**

(c) By checking the sign of $P''(x)$ before and after this point, mathematically justify why this represents the point of diminishing returns. **[3 marks]**