



Unit 3: Worked Solutions & Mark Scheme
The Unit Circle & Exact Values
IB Math AA SL

Marks are awarded for Method (M), Accuracy (A), and Reasoning (R). (M1) or (A1) indicates an implied mark.

Note on GDC usage: Solutions for Paper 2 explicitly map out Casio fx-CG50 steps for the equation solver and visual confirmation.

1. [Paper 1 Style, Non-Calculator, Easy, 5 marks]

Recall standard unit circle coordinates.

- (a) $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. A1
- (b) $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ (or $-\frac{1}{\sqrt{2}}$). A1
- (c) $\tan\left(\frac{4\pi}{3}\right) = \sqrt{3}$. A1
- (d) $\cos\left(\frac{3\pi}{2}\right) = 0$. A1
- (e) $\sin\left(\frac{11\pi}{6}\right) = -\frac{1}{2}$. A1

2. [Paper 1 Style, Non-Calculator, Easy, 4 marks]

Since $\frac{\pi}{2} < \theta < \pi$, the angle θ is in the second quadrant. (R1)

In Quadrant 2, sine is positive, but cosine and tangent are negative.

Using the Pythagorean identity: $\cos^2 \theta = 1 - \sin^2 \theta = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$. (M1)

Therefore, $\cos \theta = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$. A1

Using $\tan \theta = \frac{\sin \theta}{\cos \theta}$: $\tan \theta = \frac{3/5}{-4/5} = -\frac{3}{4}$. A1

3. [Paper 1 Style, Non-Calculator, Easy, 4 marks]

$\sqrt{2} \cos x + 1 = 0 \implies \sqrt{2} \cos x = -1 \implies \cos x = -\frac{1}{\sqrt{2}}$ (or $-\frac{\sqrt{2}}{2}$). (M1)A1

Cosine is negative in Quadrants 2 and 3. The principal reference angle is $\frac{\pi}{4}$. (M1)

Quadrant 2: $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. A1

Quadrant 3: $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$. A1

4. [Paper 1 Style, Non-Calculator, Medium, 5 marks]
 Since $\tan \theta = -\frac{4}{3}$ is negative, θ must be in Quadrant 2 or Quadrant 4. (R1)
 The hypotenuse of the reference triangle is $\sqrt{4^2 + 3^2} = 5$. (M1)
Case 1 (Quadrant 2): Sine is positive, cosine is negative.
 $\sin \theta = \frac{4}{5}$ and $\cos \theta = -\frac{3}{5}$. A1A1
Case 2 (Quadrant 4): Sine is negative, cosine is positive.
 $\sin \theta = -\frac{4}{5}$ and $\cos \theta = \frac{3}{5}$. A1
5. [Paper 1 Style, Non-Calculator, Medium, 4 marks]
 Since θ is acute ($0 < \theta < \frac{\pi}{2}$) and $\sin \theta = p$, we have $p > 0$.
 From the Pythagorean identity, $\cos \theta = \sqrt{1 - p^2}$. (M1)
- (a) $\sin(\pi - \theta) = \sin \theta = p$. A1
 (b) $\cos(\pi + \theta) = -\cos \theta = -\sqrt{1 - p^2}$. A1
 (c) $\tan(2\pi - \theta) = -\tan \theta = -\frac{\sin \theta}{\cos \theta} = -\frac{p}{\sqrt{1 - p^2}}$. A1
6. [Paper 1 Style, Non-Calculator, Medium, 5 marks]
 $3 \tan^2 x - 1 = 0 \implies 3 \tan^2 x = 1 \implies \tan^2 x = \frac{1}{3}$. (M1)
 Take the square root of both sides: $\tan x = \pm \frac{1}{\sqrt{3}}$. (A1)
 The reference angle is 30° . Since the ratio is \pm , there is a solution in all four quadrants. (M1)
 Q1: $x = 30^\circ$. A1
 Q2: $x = 180^\circ - 30^\circ = 150^\circ$. A1
 Q3: $x = 180^\circ + 30^\circ = 210^\circ$. A1
 Q4: $x = 360^\circ - 30^\circ = 330^\circ$. A1
 (Maximum 5 marks awarded for 4 correct solutions and valid method).
7. [Paper 1 Style, Non-Calculator, Medium, 5 marks]
 Substitute the Pythagorean identity $\sin^2 x = 1 - \cos^2 x$ into the equation. (M1)
 $2(1 - \cos^2 x) + 3 \cos x = 0$. (M1)
 Expand the brackets: $2 - 2 \cos^2 x + 3 \cos x = 0$. (A1)
 Multiply the entire equation by -1 to make the leading term positive: (M1)
 $2 \cos^2 x - 3 \cos x - 2 = 0$. A1 (AG)
8. [Paper 1 Style, Non-Calculator, Medium, 4 marks]
 Factorise the quadratic in terms of $\cos x$:
 $(2 \cos x + 1)(\cos x - 2) = 0$. (M1)A1
 This gives two cases: $\cos x = 2$ or $\cos x = -\frac{1}{2}$.
 Since $-1 \leq \cos x \leq 1$, the equation $\cos x = 2$ has no real solutions. (R1)
 For $\cos x = -\frac{1}{2}$, the reference angle is 60° and cosine is negative in Q2 and Q3.
 $x = 180^\circ - 60^\circ = 120^\circ$. A1
 $x = 180^\circ + 60^\circ = 240^\circ$. A1

9. [Paper 1 Style, Non-Calculator, Hard, 4 marks]
 Substitute the double angle identities: $\sin 2\theta = 2 \sin \theta \cos \theta$ and $\cos 2\theta = 1 - 2 \sin^2 \theta$.
 (M1)
 $\text{LHS} = 3(2 \sin \theta \cos \theta) + (1 - 2 \sin^2 \theta) - 1.$ (A1)
 $\text{LHS} = 6 \sin \theta \cos \theta + 1 - 2 \sin^2 \theta - 1.$ (A1)
 $\text{LHS} = 6 \sin \theta \cos \theta - 2 \sin^2 \theta.$ (A1)
 Factorise out $2 \sin \theta$:
 $\text{LHS} = 2 \sin \theta(3 \cos \theta - \sin \theta) = \text{RHS}.$ A1 (AG)
10. [Paper 1 Style, Non-Calculator, Hard, 4 marks]
 Use double angle identities: $\sin 2x = 2 \sin x \cos x$ and $\cos 2x = 1 - 2 \sin^2 x$. (M1)
 $\text{LHS} = 2 \sin x \cos x + 1 - 2 \sin^2 x - 1.$ (A1)
 $\text{LHS} = 2 \sin x \cos x - 2 \sin^2 x.$ (A1)
 Factor out $2 \sin x$:
 $\text{LHS} = 2 \sin x(\cos x - \sin x) = \text{RHS}.$ A1 (AG)
11. [Paper 1 Style, Non-Calculator, Hard, 6 marks]
 Substitute the result from Question 10 into the equation:
 $2 \sin x(\cos x - \sin x) + \cos x - \sin x = 0.$ (M1)
 Factorise by grouping the common factor $(\cos x - \sin x)$:
 $(2 \sin x + 1)(\cos x - \sin x) = 0.$ (M1)A1
Case 1: $2 \sin x + 1 = 0 \implies \sin x = -\frac{1}{2}.$
 Reference angle is $\frac{\pi}{6}$. Q3 and Q4 $\implies x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}.$ A1
Case 2: $\cos x - \sin x = 0 \implies \sin x = \cos x.$
 Divide by $\cos x \implies \tan x = 1.$ (M1)
 Reference angle is $\frac{\pi}{4}$. Q1 and Q3 $\implies x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}.$ A1
12. [Paper 1 Style, Non-Calculator, Very Hard, 6 marks]
 Substitute the double angle identity $\cos 2x = 1 - 2 \sin^2 x$. (M1)
 $1 - 2 \sin^2 x = \sin x \implies 2 \sin^2 x + \sin x - 1 = 0.$ (A1)
 Factorise the quadratic: $(2 \sin x - 1)(\sin x + 1) = 0.$ (M1)A1
Case 1: $2 \sin x - 1 = 0 \implies \sin x = \frac{1}{2}.$
 Within the domain $-\pi \leq x \leq \pi$, solutions are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}.$ A1
Case 2: $\sin x + 1 = 0 \implies \sin x = -1.$
 Within the domain $-\pi \leq x \leq \pi$, the solution is $x = -\frac{\pi}{2}.$ A1
13. [Paper 1 Style, Non-Calculator, Very Hard, 5 marks]
 Recognize that $\sin x \cos x = \frac{1}{2} \sin 2x.$ (M1)
 The equation becomes $\frac{\sqrt{3}}{2} \cos 2x = \frac{1}{2} \sin 2x.$ (A1)
 Multiply both sides by 2 and divide by $\cos 2x$:
 $\sqrt{3} = \frac{\sin 2x}{\cos 2x} \implies \tan 2x = \sqrt{3}.$ (M1)
 Since $0 \leq x \leq \pi$, the domain for $2x$ is $0 \leq 2x \leq 2\pi.$ (R1)
 The reference angle for $\tan(2x) = \sqrt{3}$ is $\frac{\pi}{3}.$
 In $0 \leq 2x \leq 2\pi$, $2x = \frac{\pi}{3}$ and $2x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}.$ (A1)
 Divide by 2: $x = \frac{\pi}{6}$ and $x = \frac{2\pi}{3}.$ A1

14. [Paper 2 Style, Calculator Required, Hard, 4 marks]

- (a) **CG50:** MENU 5 (Graph). $Y_1 = \sin(e^x)$. Set V-Window X: $0 \rightarrow 1.5$, Y: $-1 \rightarrow 1$. DRAW. (M1)
Sketch should show a wave that begins to compress (increase in frequency) as x increases, crossing the x -axis once and dipping down. A1
- (b) The x -intercept occurs where $\sin(e^x) = 0$. (M1)
This means $e^x = k\pi$ for $k \in \mathbb{Z}$.
Since $0 \leq x \leq 1.5$, the range of e^x is $e^0 \leq e^x \leq e^{1.5} \implies 1 \leq e^x \leq 4.48$.
The only multiple of π in this range is $1\pi \approx 3.14159$. (R1)
Therefore, $e^x = \pi \implies x = \ln \pi$. A1
(Using GDC G-Solv ROOT gives $x \approx 1.1447$, which confirms the exact value $\ln \pi$).

15. [Paper 2 Style, Calculator Required, Very Hard, 5 marks]

- (a) Amplitude = $|2| = 2$. A1
Period = $\frac{2\pi}{b} = \frac{2\pi}{3}$. A1
- (b) **CG50:** Ensure calculator is in **Radian** mode.
MENU 5 (Graph). $Y_1 = 2\cos(3x) + 4$ and $Y_2 = 3$. DRAW. (M1)
Set V-Window from $-\pi$ to π .
Press G-Solv (F5) \rightarrow ISCT (F5). (M1)
Record the x -coordinates for all intersections in the domain:
 $x = -2.79, -1.40, -0.698, 0.698, 1.40, 2.79$. A2
(Full marks for all 6 correct values to 3sf. Deduct 1 mark if some are missing or incorrectly rounded).