IB MATHEMATICS AA HL
AHL TOPIC 3 PRACTICE

Interactions of Lines in 3D Space & The Vector Product

Instructions to Candidates

- This practice paper contains **20** questions progressing from Easy to Very Hard.
- Each question indicates whether it is styled for **Paper 1 (No Calculator)** or **Paper 2 (Calculator Allowed)**.
- The paper tests syllabus topics AHL 3.15 and 3.16: Coincident, parallel, intersecting, and skew lines in 3D space, finding points of intersection, calculating the vector (cross) product, and using the cross product to find areas of triangles and parallelograms.
- Answer all questions, showing all your working clearly.
- Total marks available: **95**.

Difficulty Progression

- **SECTION A (Easy):** Basic cross product calculations, extracting direction vectors, recognizing parallel vectors, and basic formulas for cross product magnitude.
- **SECTION B (Medium):** Proving two lines intersect and finding the coordinates, showing lines are skew, algebraic expansion of cross products, and calculating triangle/-parallelogram areas in 3D.
- **SECTION C (Hard):** Complex intersections with unknown parameters, deriving trigonometric/vector identities (e.g., Lagrange's Identity), kinematics of colliding paths, and finding common perpendicular vectors.

SECTION A: EASY (Fundamentals)

Question 1 (3 Marks) — Paper 1 (No Calculator Allowed)

Find the vector product $\mathbf{a} \times \mathbf{b}$ where $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

Question 2 (2 Marks) — Paper 1 (No Calculator Allowed)

Given that $\mathbf{v} \times \mathbf{w} = \begin{pmatrix} -4 \\ 7 \\ 2 \end{pmatrix}$, write down the exact vector for $\mathbf{w} \times \mathbf{v}$.

Question 3 (3 Marks) — Paper 1 (No Calculator Allowed)

Two lines in 3D space are given by $L_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 6 \\ 9 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$.

Show that L_1 and L_2 are parallel.

Question 4 (3 Marks) — Paper 2 (Calculator Allowed)

Vectors $\mathbf{p}$ and $\mathbf{q}$ have magnitudes $|\mathbf{p}| = 4$ and $|\mathbf{q}| = 5$. The angle between them is $30°$. Calculate the exact area of the parallelogram formed by $\mathbf{p}$ and $\mathbf{q}$.

Question 5 (3 Marks) — Paper 1 (No Calculator Allowed)

Given the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, evaluate the following exact vector products:

(a) $\mathbf{i} \times \mathbf{j}$ **[1 mark]**

(b) $\mathbf{j} \times \mathbf{j}$ **[1 mark]**

(c) $\mathbf{k} \times \mathbf{i}$ **[1 mark]**

Question 6 (2 Marks) — Paper 1 (No Calculator Allowed)

State the geometric significance of the condition $|\mathbf{a} \times \mathbf{b}| = 0$, assuming neither $\mathbf{a}$ nor $\mathbf{b}$ is the zero vector.

Question 7 (3 Marks) — Paper 1 (No Calculator Allowed)

A line L passes through the point $(4, -2, 7)$ and is parallel to the vector $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$. Write down the vector equation of the line L in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.

SECTION B: MEDIUM (Application & Algebraic Methods)

Question 8 (6 Marks) — Paper 1 (No Calculator Allowed)

Consider the two lines defined by:

$$L_1 : \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad L_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Show algebraically that L_1 and L_2 intersect, and find the coordinates of their point of intersection.

Question 9 (5 Marks) — Paper 2 (Calculator Allowed)

Find the area of the triangle with vertices $A(1, 2, 3)$, $B(4, -1, 2)$, and $C(0, 5, 1)$. Give your answer correct to 3 significant figures.

Question 10 (6 Marks) — Paper 1 (No Calculator Allowed)

Lines L_3 and L_4 have the following equations:

$$L_3 : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad L_4 : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Prove that L_3 and L_4 are skew lines.

Question 11 (4 Marks) — Paper 1 (No Calculator Allowed)

Using the distributive properties of the vector product, expand and fully simplify the expression:

$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})$$

Question 12 (5 Marks) — Paper 2 (Calculator Allowed)

Find a unit vector that is perpendicular to both $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$.

Question 13 (6 Marks) — Paper 1 (No Calculator Allowed)

A parallelogram is defined by the adjacent vectors $\mathbf{a} = 3\mathbf{i} + m\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$. Given that the area of the parallelogram is $\sqrt{195}$, find the possible values of the real constant m.

Question 14 (5 Marks) — Paper 1 (No Calculator Allowed)

Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be three non-zero vectors. Given that $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ and $\mathbf{a}$ is not perpendicular to $(\mathbf{b} - \mathbf{c})$, state the geometric relationship between the vector $\mathbf{a}$ and the vector $(\mathbf{b} - \mathbf{c})$. Justify your answer.

SECTION C: HARD / VERY HARD (Synthesis & Proof)

Question 15 (7 Marks) — Paper 2 (Calculator Allowed)

Two planes in a 3D airspace are being tracked. The path of Airplane 1 follows the line L_1 and Airplane 2 follows the line L_2, where parameters t and s represent time in minutes:

$$L_1 : \mathbf{r}_1 = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \quad \text{and} \quad L_2 : \mathbf{r}_2 = \begin{pmatrix} 0 \\ k \\ 5 \end{pmatrix} + s \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

(a) Find the exact value of the constant k such that the flight paths of the two airplanes intersect. **[4 marks]**

(b) Determine whether the airplanes will collide in mid-air. Justify your answer. **[3 marks]**

Question 16 (6 Marks) — Paper 1 (No Calculator Allowed)

Prove Lagrange's Identity for vectors in 3D space:

$$|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = |\mathbf{a}|^2 |\mathbf{b}|^2$$

Question 17 (7 Marks) — Paper 1 (No Calculator Allowed)

Let L_1 be the line passing through points $A(1, 0, -1)$ and $B(2, 2, 1)$. Let L_2 be the line passing through $C(3, -1, 2)$ with direction vector $\mathbf{d} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Find the Cartesian equation of a third line, L_3, which passes through the origin $(0, 0, 0)$ and is mutually perpendicular to both L_1 and L_2.

Question 18 (7 Marks) — Paper 2 (Calculator Allowed)

The shortest distance D from a point P to a line defined by $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ is given by the formula:

$$D = \frac{|(\mathbf{p} - \mathbf{a}) \times \mathbf{b}|}{|\mathbf{b}|}$$

Where $\mathbf{p}$ is the position vector of point P. Find the exact shortest distance from the point $P(2, -1, 4)$ to the line $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.

Question 19 (5 Marks) — Paper 1 (No Calculator Allowed)

Prove that the area of a triangle with vertices defined by position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ from the origin is given by:

$$\text{Area} = \frac{1}{2}|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|$$

Question 20 (6 Marks) — Paper 2 (Calculator Allowed)

Vectors $\mathbf{x}$ and $\mathbf{y}$ are such that $|\mathbf{x}| = 3$, $|\mathbf{y}| = 2$, and the angle between them is θ. Given that $|(\mathbf{x} + \mathbf{y}) \times (\mathbf{x} - \mathbf{y})| = 6\sqrt{2}$, find the exact value of θ in radians, where $0 \leq \theta \leq \pi$.