Unit 5: Applications of Differentiation
IB Math AA SL

Answer all 15 questions. Show all working. For Paper 1 questions, use analytical algebraic methods. For Paper 2 questions, use your graphic display calculator (GDC) efficiently.

1. **[Paper 1 Style, Non-Calculator, Easy, 4 marks]**
   The equation of a curve is given by $f(x) = x^2 - 4x + 5$.

   (a) Find the exact gradient of the curve at $x = 3$.

   (b) Hence, find the equation of the tangent to the curve at $x = 3$. Give your answer in the form $y = mx + c$.

2. **[Paper 2 Style, Calculator Required, Easy, 4 marks]**
   Let $y = 0.5x^3 - 2x + 1$.

   (a) Use the numerical derivative feature on your graphic display calculator to evaluate the gradient of the curve at $x = 2$.

   (b) Hence, find the equation of the normal to the curve at the point where $x = 2$, giving your answer in the form $y = mx + c$.

3. **[Paper 1 Style, Non-Calculator, Easy, 5 marks]**
   A company's daily profit, P in dollars, from manufacturing x items is modelled by the quadratic function $P(x) = -2x^2 + 120x - 500$.

   (a) Find the marginal profit function, $P'(x)$.

   (b) Determine the exact number of items that must be manufactured to maximise the daily profit.

   (c) Calculate the maximum daily profit.

4. **[Paper 2 Style, Calculator Required, Easy, 4 marks]**
A farmer wants to build a rectangular sheep pen against an existing long stone wall. No fencing is needed along the wall. He has exactly 100 metres of fencing available for the other three sides. Let x be the width of the pen in metres.

   (a) Show that the area of the pen, A, can be written as $A(x) = 100x - 2x^2$.

   (b) Use your graphic display calculator to find the width x that maximises the area of the pen, and state this maximum area.

5. **[Paper 1 Style, Non-Calculator, Medium, 5 marks]**
Consider the curve $y = 2x^2 - 5x$.

   (a) Find the gradient of the tangent to the curve at $x = 1$.

   (b) Find the gradient of the normal to the curve at $x = 1$.

   (c) Find the equation of the normal line at $x = 1$, giving your answer in the form $ax + by + d = 0$, where a, b, and d are integers.

6. **[Paper 2 Style, Calculator Required, Medium, 6 marks]**
An open-topped box is made from a 20 cm × 20 cm square piece of cardboard by cutting out identical squares of side length x cm from each of the four corners and folding up the sides.

   (a) Write down an expression for the volume of the box, V, in terms of x.

   (b) State the practical domain for x (the possible values x can take).

   (c) Use your graphic display calculator to find the value of x that gives the maximum volume, and state this maximum volume correct to 3 significant figures.

7. **[Paper 1 Style, Non-Calculator, Medium, 5 marks]**
The curve C is given by the equation $y = -x^3 + 6x^2 - 5x + 4$.

   (a) Find $\frac{dy}{dx}$.

   (b) The tangent to the curve is parallel to the x-axis at two specific points. Find the exact x-coordinates of these two points.

8. **[Paper 2 Style, Calculator Required, Medium, 6 marks]**
A manufacturer needs to design a cylindrical tin can to hold 1000π cm^3 of liquid. The radius of the base is r cm, and the height is h cm.

   (a) Write down an equation linking r, h, and the volume. Hence, express h in terms of r.

   (b) Show that the total surface area of the can, SA, is given by $SA = 2\pi r^2 + \frac{2000\pi}{r}$.

   (c) Graph this function on your calculator and find the radius r that minimises the surface area. Give your answer to 3 significant figures.

9. **[Paper 1 Style, Non-Calculator, Hard, 5 marks]**
The equation of a curve is $y = x^3 - 4x^2 + kx$, where k is a constant. The equation of the tangent to the curve at the point where $x = 2$ is $y = -3x + 10$.

   (a) State the gradient of the tangent line.

   (b) Find $\frac{dy}{dx}$ in terms of x and k.

   (c) By substituting $x = 2$ into your derivative, find the exact value of k.

10. **[Paper 2 Style, Calculator Required, Hard, 6 marks]**
A function is defined by $f(x) = px^3 + qx^2 + 5x$. The curve has a local minimum point at $x = 1$ and passes through the point $(1, 2)$.

   (a) Using the fact that the curve passes through $(1, 2)$, write down an equation in terms of p and q.

   (b) Find $f'(x)$.

   (c) Using the fact that the local minimum occurs at $x = 1$, write down a second equation in terms of p and q.

   (d) Solve the system of equations to find the exact values of p and q.

11. **[Paper 1 Style, Non-Calculator, Hard, 6 marks]**
The curve C has equation $y = 3x^2 - 6 + \frac{4}{x}$. The point $P(1, 1)$ lies on C.

   (a) Find an expression for $\frac{dy}{dx}$.

   (b) Show that an equation of the normal to C at point P is $x + 2y = 3$.

   (c) This normal cuts the x-axis at the point Q. Find the exact length of the line segment PQ.

12. **[Paper 2 Style, Calculator Required, Hard, 6 marks]**
A company produces and sells cricket bats. The daily cost C, in hundreds of dollars, to produce x hundred cricket bats is $C(x) = 6x^3 - 10x^2 + 10x + 4$. The daily revenue R from selling x hundred cricket bats is $R(x) = 42x$.

   (a) Determine a function for the profit, $P(x)$, in hundreds of dollars.

   (b) Find $P'(x)$.

   (c) The production of bats will reach its profit-maximising level when the marginal profit $P'(x)$ equals zero. Use your GDC to find the profit-maximising production level (number of bats) and the expected maximum profit in dollars.

13. **[Paper 1 Style, Non-Calculator, Very Hard, 6 marks]**
Consider the function f defined by $f(x) = 90e^{-0.5x}$ for $x \geq 0$. The line L has a gradient of -1 and is a tangent to the graph of f at the point Q.

   (a) Find $f'(x)$.

   (b) Find the exact coordinates of Q.

   (c) Show that the equation of L is $y = -x + 2\ln 45 + 2$.

14. **[Paper 2 Style, Calculator Required, Very Hard, 7 marks]**
We want to find the shortest distance from the point $(0, 2)$ to the curve $y = x^2$. Let $D(x)$ be the distance from $(0, 2)$ to a general point (x, x^2) on the curve.

   (a) Using the distance formula, show that $D(x) = \sqrt{x^4 - 3x^2 + 4}$.

   (b) Graph $Y = \sqrt{x^4 - 3x^2 + 4}$ on your calculator. Using the minimum feature, find the minimum distance from the point to the curve.

   (c) State the x-coordinates of the points on the curve $y = x^2$ that are closest to $(0, 2)$.

15. **[Paper 1 Style, Non-Calculator, Very Hard, 7 marks]**
A small cylindrical drum, closed at the top but open at the bottom, has a radius r cm and a volume of 1000 cm^3. The material to make the top skin of the drum costs 25 cents per cm^2, and the curved surface costs 20 cents per cm^2. It can be shown that the total cost C, in cents, of the material is given by $C(r) = 25\pi r^2 + \frac{40000}{r}$.

   (a) Find $\frac{dC}{dr}$.

   (b) The function $C(r)$ has a local minimum at the point where $r = p$. Find the exact value of p.

   (c) Find $\frac{d^2C}{dr^2}$ and hence mathematically justify that the cost is minimised at the radius $r = p$.