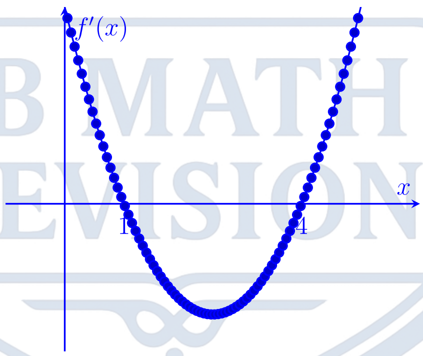
Unit 5: Graphical Behaviour & The Second Derivative
IB Math AA SL

Answer all 15 questions. Show all working. For Paper 1 questions, use analytical algebraic methods. For Paper 2 questions, use your graphic display calculator (GDC) efficiently.

1. **[Paper 1 Style, Non-Calculator, Easy, 4 marks]**
The diagram below shows the graph of the **derivative** function $y = f'(x)$ for a continuous function $f(x)$. The graph of $f'(x)$ is a parabola that crosses the x-axis at $x = 1$ and $x = 4$.



   (a) Write down the intervals of x for which the original function $f(x)$ is increasing.

   (b) State the x-coordinate of the local maximum of $f(x)$, giving a clear reason for your answer.

2. **[Paper 2 Style, Calculator Required, Easy, 4 marks]**
Consider the function $f(x) = x^3 - 3x^2 - 9x + 2$.

   (a) Graph the function on your graphic display calculator and find the exact coordinates of the local maximum point.

   (b) Find the exact coordinates of the local minimum point.

3. **[Paper 1 Style, Non-Calculator, Easy, 4 marks]**
Consider the function $g(x) = e^{2x} - 4x$.

   (a) Find $g'(x)$ and $g''(x)$.

   (b) Hence, find the exact x-coordinate of the stationary point and use the second derivative test to determine its nature (maximum or minimum).

4. **[Paper 2 Style, Calculator Required, Easy, 5 marks]**
Let $h(x) = \sin x + \cos x$ for $0 \leq x \leq 2\pi$.

   (a) Find $h''(x)$.

   (b) A point of inflexion occurs where the second derivative is zero and changes sign (a change in concavity). Use your GDC to solve $h''(x) = 0$ and find the exact x-coordinates of the points of inflexion.

5. **[Paper 1 Style, Non-Calculator, Medium, 5 marks]**
The curve C has the equation $f(x) = x^3 - 6x^2 + 9x - 1$.

   (a) Find the x-coordinates of the two stationary points of C.

   (b) Determine the x-coordinate of the point of inflexion on the graph.

   (c) Explain mathematically why the point of inflexion found in part (b) is **not** a stationary point.

6. **[Paper 2 Style, Calculator Required, Medium, 5 marks]**
A function f is defined for $x > 0$. The derivative of f is given by:

$$f'(x) = 3x^2 + \frac{2}{x^3} - 25$$

   (a) Find $f''(x)$.

   (b) The graph of f is concave up when $x > n$, where n is the least possible real number that makes this inequality true. Find the exact value of n.

7. **[Paper 1 Style, Non-Calculator, Medium, 6 marks]**
Let $f(x) = \frac{1}{3}x^3 - 2x^2 - 21x - 24$.

   (a) Find $f'(x)$.

   (b) The graph of f has horizontal tangents at the points where $x = a$ and $x = b$, with $a < b$. Find the values of a and b.

   (c) Find $f''(x)$. Hence, show that the graph of f has a local maximum point at $x = a$.

8. **[Paper 2 Style, Calculator Required, Medium, 5 marks]**
Consider the function $f(x) = 4x^2 + 4x - 4x^3$.

   (a) Use your GDC to find the coordinates of the local maximum point of $f(x)$.

   (b) Find $f''(x)$ algebraically.

   (c) Hence, determine the exact interval of x for which the graph of f is concave down.

9. **[Paper 1 Style, Non-Calculator, Hard, 5 marks]**
A function is given by $f(x) = (x-2)^3 + 4$.

   (a) Find $f'(x)$ and $f''(x)$.

   (b) Show that there is a stationary point of inflexion at $x = 2$, clearly justifying your answer by analyzing both the first and second derivatives.

10. **[Paper 2 Style, Calculator Required, Hard, 6 marks]**
A function f is defined for $x \neq 0$. The derivative of f is given by:

$$f'(x) = 48x^2 + \frac{1}{x^3} - 22$$

   (a) Find $f''(x)$.

   (b) Show that the curve $y = f(x)$ has exactly one point of inflexion, and determine its x-coordinate.

   (c) Find the exact gradient of the **normal** line to the curve at this point of inflexion.

11. **[Paper 1 Style, Non-Calculator, Hard, 6 marks]**
Consider the function $g(x) = x^2 \ln x$ for $x > 0$.

   (a) Use the product rule to find $g'(x)$ and $g''(x)$.

   (b) Find the exact x-coordinate of the point of inflexion of $g(x)$.

   (c) State the exact interval for which the graph of g is concave up.

12. **[Paper 2 Style, Calculator Required, Hard, 6 marks]**
The second derivative of a function f is given by $f''(x) = \frac{2\ln(px)-3}{qx^3}$ for $x > 0$, where p and q are positive real constants. The graph of f has exactly one point of inflexion at point B.

   (a) Show algebraically that the x-coordinate of B is given by $\frac{e^{1.5}}{p}$.

   (b) Given that the x-coordinate of B is exactly 4.481689..., find the integer value of p.

13. **[Paper 1 Style, Non-Calculator, Very Hard, 6 marks]**
A general cubic curve has the equation $y = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$ and $a \neq 0$.

(a) Show algebraically that the curve will always have exactly one point of inflexion, and determine its x-coordinate in terms of a and b.

(b) In the case where the curve has two distinct stationary points, let their x-coordinates be x_1 and x_2. By finding the roots of the first derivative, show that the x-coordinate of the point of inflexion lies exactly halfway between x_1 and x_2.

14. **[Paper 2 Style, Calculator Required, Very Hard, 7 marks]**
A function is defined by $f(x) = px^3 + qx^2 + 5x$, where p and q are constants. The curve has a local minimum point at $x = 1$ and passes through the point $(1, 2)$.

(a) Using the given information, set up a system of two linear equations in terms of p and q.

(b) Solve the system to find the exact values of p and q.

(c) Determine the x-coordinate of the point of inflexion.

(d) Determine, with a reason, whether the graph is concave up or concave down at $x = -1$.

15. **[Paper 1 Style, Non-Calculator, Very Hard, 7 marks]**
Let $f(x) = xe^{kx}$, where k is a non-zero real constant.

(a) Find $f'(x)$ and $f''(x)$, giving both answers in fully factorised form.

(b) The graph of f has a single local extremum (turning point). Find its x-coordinate in terms of k.

(c) By evaluating the second derivative at this point, deduce whether the turning point is a local maximum or a local minimum in the case where $k < 0$.

(d) Find the coordinates of the point of inflexion in terms of k.