IB MATHEMATICS AI HL
UNIT 5: CALCULUS

Maclaurin Series

Instructions to Candidates

- This question booklet contains **15 questions**.
- The paper targets **AHL** syllabus component 5.19.
- Answer all questions, showing all step-by-step working clearly.

Difficulty Progression

- **Questions 1 - 5 (Easy):** Standard expansions ($e^x, \sin x, \cos x$), direct coefficient matching, and basic numerical approximations.
- **Questions 6 - 10 (Medium):** Limits evaluating to $0/0$, integration of series, and Taylor polynomials for differential equations.
- **Questions 11 - 15 (Hard):** Products of series, successive differentiation of complex functions, and extracting known functions from DE sequences.

SECTION A: EASY (Fundamentals)

Question 1 (4 Marks)

The Maclaurin series for e^x is given by $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
By substituting $-3x$ into the standard series, find the first four non-zero terms of the Maclaurin series for $f(x) = e^{-3x}$.

Question 2 (4 Marks)

The Maclaurin series for $\sin x$ is given by $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
Find the Maclaurin series for $g(x) = x^2 \sin x$ up to and including the term in x^7.

Question 3 (4 Marks)

Use the first three terms of the Maclaurin series for e^x to calculate an approximate value for $e^{0.1}$.

CG50 Tip: The Formula Booklet

You do not need to memorize the general formula for a Maclaurin series! It is given in your formula booklet as $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots$. You can generate any series simply by evaluating the function and its derivatives at $x = 0$.

Question 4 (5 Marks)

Let $f(x) = \ln(1 + x)$.
Find $f'(x)$, $f''(x)$, and $f'''(x)$. Hence, write down the first three non-zero terms of the Maclaurin series for $\ln(1 + x)$.

Question 5 (5 Marks)

The Maclaurin series for $\cos x$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
Find the exact value of the coefficient of x^4 in the Maclaurin series for $\cos(2x)$.

SECTION B: MEDIUM (Application & Modelling)

Question 6 (5 Marks)

Use the Maclaurin series for $\sin x$ to find the exact value of the following limit:

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}$$

Question 7 (6 Marks)

A function $y(x)$ satisfies the differential equation $\frac{dy}{dx} = x + y^2$, with the initial condition $y(0) = 1$. Find the values of $y'(0)$ and $y''(0)$, and hence write down the Maclaurin series for $y(x)$ up to and including the term in x^2.

Question 8 (6 Marks)

By using the binomial expansion formula for $(1+u)^n$, find the Maclaurin series for $f(x) = \frac{1}{1-x}$ up to and including the term in x^3.

Question 9 (6 Marks)

Hence, by making an appropriate substitution into your answer from Question 8, find the first four non-zero terms of the Maclaurin series for $g(x) = \frac{1}{1+x^2}$.

Question 10 (6 Marks)

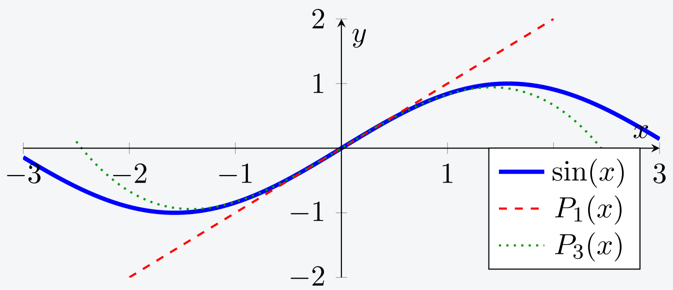
It is given that $\int \frac{1}{1+x^2}\, dx = \arctan x + C$.
By integrating the Maclaurin series for $\frac{1}{1+x^2}$ from Question 9, find the Maclaurin series for $\arctan x$ up to and including the term in x^7.

SECTION C: HARD (Synthesis & Proof)

CG50 Tip: Visualising Approximations

Maclaurin polynomials are just polynomials that "hug" the original curve near $x = 0$. You can see this visually by graphing the original function (e.g. $\sin x$) in `Y1`, and its Maclaurin expansion (e.g. $x - x^3/6$) in `Y2`. The higher the power you include, the tighter the hug!

Question 11 (7 Marks)



By multiplying the Maclaurin series for e^x and $\cos x$ together, find the first three non-zero terms of the Maclaurin series for the composite function $f(x) = e^x \cos x$.

Question 12 (8 Marks)

Use the Maclaurin series for $\cos x$ to find an approximation for the definite integral:

$$\int_0^{0.5} \cos(x^2)\, dx$$

Use only the first two non-zero terms of the expanded series in your calculation, and give your final answer as an exact decimal.

Question 13 (8 Marks)

Consider the function $f(x) = \ln(1 + \sin x)$.
By evaluating the function and its first three derivatives at $x = 0$, find the Maclaurin series for $f(x)$ up to and including the term in x^3.

Question 14 (7 Marks)

Use Maclaurin series expansions to evaluate the following exact limit:

$$\lim_{x \to 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{\sin(x^3)}$$

Question 15 (9 Marks)

A particle's displacement $s(t)$ satisfies the second-order differential equation $s''(t) = -s(t)$, with the initial conditions $s(0) = 0$ and $s'(0) = 1$.

(a) Evaluate $s''(0)$, $s'''(0)$, $s^{(4)}(0)$, and $s^{(5)}(0)$. [**4 marks**]

(b) Hence, construct the Maclaurin series for $s(t)$ up to the term in t^5. [**3 marks**]

(c) State the name of the standard mathematical function that perfectly models this particle's displacement. [**2 marks**]