IB MATHEMATICS AI HL
UNIT 5: CALCULUS

Differential Equations & Euler's Method

Instructions to Candidates

- This question booklet contains **15 questions**.
- The paper targets **AHL** syllabus components 5.14 to 5.18.
- Answer all questions, showing all step-by-step working clearly.

Difficulty Progression

- **Questions 1 - 5 (Easy):** Basic separation of variables, verifying solutions, 1-step Euler's method, and matrix forms of coupled DEs.
- **Questions 6 - 10 (Medium):** Multi-step Euler's method, Integrating Factors (IF), eigenvalues of 2×2 matrices, and phase portrait classifications.
- **Questions 11 - 15 (Hard):** Exact solutions to coupled systems, homogeneous substitutions ($y = vx$), predator-prey equilibrium, and logistic growth.

SECTION A: EASY (Fundamentals)

Question 1 (4 Marks)

Use the method of separation of variables to find the general solution to the differential equation:

$$\frac{dy}{dx} = \frac{x}{y}$$

Express your answer in the form $y^2 = f(x)$.

Question 2 (4 Marks)

A differential equation is given in the standard linear form $\frac{dy}{dx} + P(x)y = Q(x)$:

$$\frac{dy}{dx} + \frac{2}{x}y = 5x$$

Find the exact expression for the Integrating Factor, $I(x) = e^{\int P(x)dx}$.

Question 3 (4 Marks)

Verify algebraically that the function $y = Ce^{-4x}$, where C is an arbitrary constant, is a valid solution to the differential equation $\frac{dy}{dx} + 4y = 0$.

CG50 Tip: Euler's Method

For Euler's Method, $y_{n+1} = y_n + h \times f(x_n, y_n)$. You can set this up quickly in the `Spreadsheet` app! Put your x-values in Column A, your y-values in Column B, and enter the formula '=B1 + 0.1*(A1+B1)' into cell B2. Drag it down to instantly solve for as many steps as you need!

Question 4 (4 Marks)

Consider the differential equation $\frac{dy}{dx} = x + y$, with the initial condition $y(0) = 2$.
Using Euler's method with a step size of $h = 0.1$, calculate the approximate value of y when $x = 0.1$.

Question 5 (5 Marks)

A system of coupled linear differential equations is given by:

$$\frac{dx}{dt} = 3x - y$$
$$\frac{dy}{dt} = 2x + 4y$$

Write this system in the matrix form $\vec{x}' = A\vec{x}$, clearly stating the coefficient matrix A.

SECTION B: MEDIUM (Application & Modelling)

Question 6 (5 Marks)

Consider the differential equation $\frac{dy}{dx} = x - y^2$, with initial condition $y(1) = 1$.
Use Euler's method with a step size of $h = 0.2$ to find the approximate value of $y(1.4)$. Show your intermediate steps in a table.

Question 7 (6 Marks)

Solve the following differential equation using an Integrating Factor, given the initial condition $y(0) = 5$:

$$\frac{dy}{dx} - 2y = e^{3x}$$

Question 8 (6 Marks)

Find the exact particular solution to the differential equation:

$$\frac{dy}{dx} = y\cos x$$

given the boundary condition that $y(0) = e^2$.

CG50 Tip: Eigenvalues of a Matrix

Need to find eigenvalues quickly to classify a phase portrait? Go to `MENU 1 (Run-Matrix)`. Enter your 2×2 matrix as `Mat A`. Then press `OPTN` → `MAT/VCT (F2)` → `Eigenval (F6, F6, F1)` of `Mat A`. It will instantly output the eigenvalues λ_1 and λ_2!

Question 9 (6 Marks)

The phase portrait of a system of coupled differential equations $\vec{x}' = A\vec{x}$ has a coefficient matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$.

(a) Find the exact eigenvalues of matrix A. [**4 marks**]

(b) Using the signs of the eigenvalues, classify the origin $(0, 0)$ as a stable node, an unstable node, or a saddle point. [**2 marks**]

Question 10 (6 Marks)

A homogeneous differential equation is given by $\frac{dy}{dx} = \frac{x+y}{x}$.
By using the substitution $y = vx$, where v is a function of x, show that the equation can be

transformed into the separable form:

$$x\frac{dv}{dx} = 1$$

Hence, find the general solution for y in terms of x.

SECTION C: HARD (Synthesis & Proof)

Question 11 (7 Marks)

A system of coupled differential equations is given by $\vec{x}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \vec{x}$.

The exact solution takes the form $\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$.
Find the exact solution for $\vec{x}(t)$ given the initial conditions $x(0) = 2$ and $y(0) = 1$.

Question 12 (8 Marks)

A predator-prey model for a population of rabbits (x) and foxes (y) is modelled by the coupled non-linear differential equations:

$$\frac{dx}{dt} = 0.2x - 0.01xy$$
$$\frac{dy}{dt} = -0.1y + 0.002xy$$

Determine the exact coordinates (x, y) of both equilibrium points for this system, and state what the non-zero equilibrium point represents in the context of the populations.

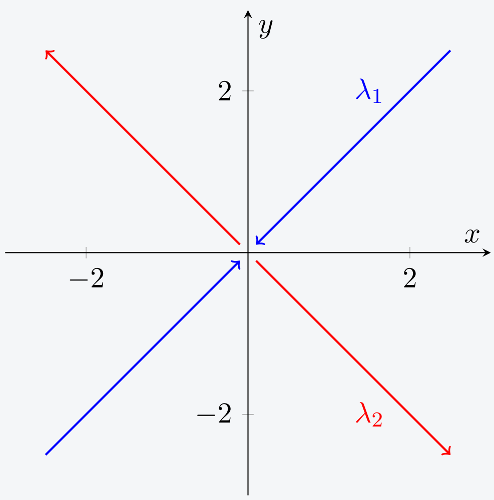
Question 13 (8 Marks)

Consider the differential equation:

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$$

Use the homogeneous substitution $y = vx$ to find the general solution of the differential equation. Express your final answer in the form $y^2 = f(x)$.

Question 14 (7 Marks)



The diagram above shows the phase portrait of a system $\vec{x}' = A\vec{x}$, possessing a saddle point at

the origin. The eigenvectors are $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
By analyzing the directional arrows on the invariant manifolds, state whether the corresponding eigenvalues λ_1 and λ_2 are positive or negative, and justify your reasoning.

Question 15 (9 Marks)

The population P of a fish farm is modelled by the exact logistic differential equation:

$$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{1000}\right)$$

By separating the variables and using the method of partial fractions on $\frac{1}{P(1-P/1000)}$, solve the differential equation to find an exact expression for t in terms of P.