IB MATHEMATICS AI HL
AHL QUESTION BOOKLET

Eigenvalues, Eigenvectors & Markov Chains

Instructions to Candidates

- This extended practice paper contains **15 questions**.
- The paper targets **Advanced Higher Level (AHL)** syllabus components 1.14 and 1.15.
- Note: In Mathematics AI, all papers require a Graphic Display Calculator (GDC). Use it efficiently to evaluate steady-state limits and 3×3 systems.
- Answer all questions, showing all step-by-step working clearly in the spaces provided.

Difficulty Progression

- **Questions 1 - 5 (Easy):** Characteristic polynomials, finding 2×2 eigenvalues, proving eigenvectors, and basic transition matrices.
- **Questions 6 - 10 (Medium):** Leslie population matrices, diagonalisation ($M = PDP^{-1}$), evaluating high matrix powers, and 3×3 Markov chains.
- **Questions 11 - 15 (Hard):** Complex eigenvalues, Google PageRank algorithms, absorbing Markov chains (fundamental matrices), and algebraic proofs.

SECTION A: EASY (Fundamentals)

Question 1 (4 Marks)

Consider the matrix $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$.

(a) Find the characteristic polynomial of A by evaluating $\det(A - \lambda I) = 0$. [**2 marks**]
(b) Hence, find the two exact eigenvalues of matrix A. [**2 marks**]

Question 2 (4 Marks)

A matrix M is given by $M = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$.

Show clearly that the column vector $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of matrix M, and state its corresponding eigenvalue λ.

Question 3 (4 Marks)

The weather in a specific town can only be Sunny (S) or Rainy (R). If it is Sunny today, there is an 80% chance it will be Sunny tomorrow. If it is Rainy today, there is a 60% chance it will be Rainy tomorrow.
(a) Construct a 2×2 transition matrix T to represent this information, ordering the columns and rows as S, then R. [**2 marks**]
(b) If it is Rainy on Monday, calculate the exact probability that it will be Sunny on Wednesday. [**2 marks**]

Question 4 (4 Marks)

A matrix A has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$, with corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.
The matrix can be diagonalised in the form $A = PDP^{-1}$.
Write down the exact matrices P and D.

Question 5 (5 Marks)

Consider the transition matrix $T = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}$.

By setting up and solving the system of equations $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ subject to $x + y = 1$, find the exact steady-state probability vector.

SECTION B: MEDIUM (Application & Modelling)

Question 6 (5 Marks)

A biologist is modelling a population of insects using a Leslie matrix. The population is divided into two age groups: Juveniles (J) and Adults (A). Each month, 40% of Juveniles survive to become Adults. Each month, every Adult produces an average of 3 new Juveniles, and 50% of the Adults survive to the next month.

(a) Write down the 2×2 Leslie matrix L for this population. [**2 marks**]

(b) The initial population consists of 100 Juveniles and 50 Adults. Calculate the exact population of each age group after 2 months. [**3 marks**]

Question 7 (6 Marks)

Matrix $M = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$.

(a) Find the eigenvalues and their corresponding eigenvectors for matrix M. [**4 marks**]

(b) Use the diagonalisation $M^n = PD^nP^{-1}$ to find an exact expression for M^4. [**2 marks**]

Question 8 (6 Marks)

A 3×3 transition matrix for a Markov chain is given by:

$$T = \begin{pmatrix} 0.5 & 0.2 & 0.1 \\ 0.3 & 0.4 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix}$$

An initial state vector is $V_0 = \begin{pmatrix} 0.4 \\ 0.4 \\ 0.2 \end{pmatrix}$.

Use your Graphic Display Calculator (GDC) to find the long-term steady-state vector V_∞. Give your values correct to 3 significant figures.

Question 9 (6 Marks)

A transition matrix is given by $T = \begin{pmatrix} 0.6 & k \\ 0.4 & 1-k \end{pmatrix}$, where k is a constant and $0 < k < 1$.

The long-term steady-state probability for the first state is known to be exactly $\frac{5}{7}$.

Calculate the exact value of k.

Question 10 (6 Marks)

Consider the sequence of vectors defined by $u_{n+1} = Au_n$, where $A = \begin{pmatrix} 0 & 1 \\ 0.24 & 1 \end{pmatrix}$ and $u_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(a) Find the eigenvalues of matrix A. [**2 marks**]

(b) Determine the values in the vector u_2. [**2 marks**]

(c) Describe the long-term behaviour of the sequence u_n as $n \to \infty$. [**2 marks**]

SECTION C: HARD (Synthesis & Proof)

Question 11 (7 Marks)

A matrix is given by $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$.

(a) Form the characteristic equation and show that the eigenvalues of A are complex numbers. [**3 marks**]
(b) Find the exact complex eigenvalues in the form $a \pm bi$. [**2 marks**]
(c) Calculate the exact modulus (magnitude) of these complex eigenvalues. [**2 marks**]

Question 12 (8 Marks)

An absorbing Markov chain has three states: 1, 2, and 3. State 1 is an absorbing state. The transition matrix is given by:

$$T = \begin{pmatrix} 1 & 0.2 & 0.1 \\ 0 & 0.3 & 0.4 \\ 0 & 0.5 & 0.5 \end{pmatrix}$$

(a) Write down the transient sub-matrix Q. [**1 mark**]
(b) Calculate the fundamental matrix $N = (I - Q)^{-1}$, giving your answers as exact fractions. [**4 marks**]
(c) Calculate the expected number of steps before absorption if the system starts in State 3. [**3 marks**]

Question 13 (8 Marks)

A simplified Google PageRank algorithm is applied to three webpages: A, B, and C. - Page A links evenly to Page B and Page C. - Page B links only to Page A. - Page C links evenly to Page A and Page B.
(a) Construct the 3×3 pure PageRank transition matrix T for this network. [**2 marks**]
(b) The PageRank of the network is the steady-state vector v such that $Tv = v$. By solving the system of equations algebraically, find the exact PageRank of all three pages. [**4 marks**]
(c) State which webpage is deemed the "most important" by the algorithm. [**2 marks**]

Question 14 (7 Marks)

The Fibonacci sequence $0, 1, 1, 2, 3, 5, \ldots$ can be generated using matrices.
Let $F_n = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$. The sequence is defined by $F_{n+1} = AF_n$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.
Find the exact eigenvalues of matrix A. Show your working clearly and express your answers in simplest surd form.

Question 15 (9 Marks)

Let T be a general 2×2 column-stochastic transition matrix for a Markov chain, given by:

$$T = \begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}$$

where $0 < a < 1$ and $0 < b < 1$.

(a) Prove algebraically that $\lambda = 1$ is always an eigenvalue for T. [**5 marks**]

(b) Find the eigenvector corresponding to $\lambda = 1$ in terms of a and b. [**4 marks**]