Topic: Worked Solutions & Mark Scheme
Optimisation, Tangents & Normals
Featuring Casio fx-CG50 Calculator Instructions

Marks are awarded for Method (M), Accuracy (A), and Reasoning (R). (M1) or (A1) indicates an implied mark.
Note on GDC usage: Solutions below explicitly use Casio fx-CG50 syntax.

1. **[Paper 1 Style, Short Answer, Easy, 4 marks]**

   (a) $f'(x) = 2x - 4$. **(M1)**
   Gradient at $x = 3$: $f'(3) = 2(3) - 4 = 2$. **A1**

   (b) Find the y-coordinate at $x = 3$: $f(3) = 3^2 - 4(3) + 5 = 9 - 12 + 5 = 2$. Point is $(3, 2)$. **(A1)**
   Use point-slope form: $y - y_1 = m(x - x_1) \implies y - 2 = 2(x - 3)$. **(M1)**
   $y = 2x - 6 + 2 \implies y = 2x - 4$. **A1**

2. **[Paper 1 Style, Short Answer, Easy, 4 marks]**

   (a) In `MENU 5 (Graph)`, enter $Y1 = x^3 - 3x^2 - 9x + 2$. Draw the graph. **(M1)**
   Press `SHIFT -> F5 (G-Solv) -> F2 (MAX)`.
   The calculator displays $X = -1, Y = 7$. Coordinates: $(-1, 7)$. **A1A1**

   (b) Press `SHIFT -> F5 (G-Solv) -> F3 (MIN)`.
   The calculator displays $X = 3, Y = -25$. Coordinates: $(3, -25)$. **A1A1**

3. **[Paper 1 Style, Short Answer, Easy, 4 marks]**

   (a) In `MENU 5 (Graph)`, enter $Y1 = 0.5x^3 - 2x + 1$ and `DRAW`. **(M1)**
   Press `SHIFT -> F4 (Sketch) -> F3 (Norm)`.
   Type `2` to select $x = 2$, then press `EXE`. **A1**

   (b) The calculator automatically draws the normal and displays the equation on screen:
   $y = -0.25x + 1.5$. **A1A1**

4. **[Paper 1 Style, Short Answer, Medium, 5 marks]**

   (a) $P'(x) = 2(-2)x + 120 = -4x + 120$. **A1A1**

   (b) Set $P'(x) = 0$ to find the maximum. **(M1)**
   $-4x + 120 = 0 \implies 4x = 120 \implies x = 30$ items. **A1**
   (Alternatively, use `G-Solv -> MAX` on the CG50).

   (c) Substitute $x = 30$ into $P(x)$:
   $P(30) = -2(30)^2 + 120(30) - 500 = -1800 + 3600 - 500 = \$1300$. **A1**

5. **[Paper 2 Style, Longer Question, Medium, 6 marks]**

   (a) The three sides of fencing are the two widths (x) and the length (L).
   $2x + L = 100 \implies L = 100 - 2x$. **(M1)**
   Area = width × length = $x(100 - 2x) = 100x - 2x^2$. **A1 (AG)**

   (b) In `MENU 5 (Graph)`, enter $Y1 = 100x - 2x^2$. **(M1)**
   Adjust `V-Window` (e.g., X from 0 to 50, Y from 0 to 1500) and `DRAW`.
   Press `SHIFT -> F5 (G-Solv) -> F2 (MAX)`. **(M1)**
   $x = 25$ m. **A1**

   (c) Read the y-value from the maximum point: $Y = 1250$.
   Maximum Area = 1250 m^2. **A1**

6. **[Paper 1 Style, Short Answer, Medium, 5 marks]**

   (a) $\frac{dy}{dx} = -3x^2 + 12x - 5$. **A1A1**

   (b) Tangent is parallel to the x-axis implies $\frac{dy}{dx} = 0$. **(M1)**
   $-3x^2 + 12x - 5 = 0$.
   In `MENU A (Equation) -> F2 (Polynomial) -> F1 (Degree 2)`, enter `a = -3, b = 12, c = -5`. **(M1)**
   $x = 3.5275\cdots \approx 3.53$ and $x = 0.4724\cdots \approx 0.472$. **A1A1**

7. **[Paper 1 Style, Short Answer, Medium, 5 marks]**

   (a) $y' = 4x - 5$. **(M1)**
   Gradient at $x = 1$: $y'(1) = 4(1) - 5 = -1$. **A1**

   (b) Normal gradient $m_n = -\frac{1}{m_t} = -\frac{1}{-1} = 1$. **A1**

   (c) y-coordinate at $x = 1$: $y = 2(1)^2 - 5(1) = -3$. Point $(1, -3)$. **(M1)**
   Equation: $y - (-3) = 1(x - 1) \implies y + 3 = x - 1$.
   Rearranging to $ax + by + d = 0 \implies x - y - 4 = 0$. **A1**

8. **[Paper 2 Style, Longer Question, Medium, 6 marks]**

(a) Perimeter $= 2x + 2L = 40 \implies 2L = 40 - 2x \implies L = 20 - x$. **A1A1**

(b) Area $=$ width $\times$ length $= x(20 - x) = 20x - x^2$. **A1 (AG)**

(c) $A'(x) = 20 - 2x$. **A1**
Set $A'(x) = 0 \implies 20 - 2x = 0 \implies 2x = 20 \implies x = 10$ cm. **(M1)**
If width is 10, length is $20 - 10 = 10$. The rectangle is a square. **R1**

9. **[Paper 2 Style, Longer Question, Hard, 7 marks]**

(a) $V = \pi r^2 h \implies 1000\pi = \pi r^2 h$. **(M1)**
$h = \frac{1000\pi}{\pi r^2} = \frac{1000}{r^2}$. **A1**

(b) $SA = 2\pi r^2 + 2\pi r h$. **(M1)**
Substitute h: $SA = 2\pi r^2 + 2\pi r \left(\frac{1000}{r^2}\right)$. **(M1)**
$SA = 2\pi r^2 + \frac{2000\pi}{r}$. **A1 (AG)**

(c) In `MENU 5 (Graph)`, enter $Y1 = 2\pi x^2 + (2000\pi)/x$. **(M1)**
Adjust `V-Window` (e.g. X: 0 to 20, Y: 0 to 2000) and `DRAW`.
Press `G-Solv -> MIN`.
$x = 7.9370\ldots \implies Radius r \approx 7.94$ cm. **A1**

10. **[Paper 2 Style, Longer Question, Hard, 7 marks]**

(a) Substitute $(1, 2)$ into $f(x)$:
$2 = p(1)^3 + q(1)^2 + 5(1) \implies p + q + 5 = 2 \implies p + q = -3$. **A1A1**

(b) $f'(x) = 3px^2 + 2qx + 5$. **A1**

(c) Minimum at $x = 1$ implies $f'(1) = 0$. **(M1)**
$0 = 3p(1)^2 + 2q(1) + 5 \implies 3p + 2q = -5$. **A1**

(d) System: $p + q = -3 \implies q = -3 - p$.
Substitute into 2nd eq: $3p+2(-3-p) = -5 \implies 3p-6-2p = -5 \implies p = 1$.
(M1)
$1 + q = -3 \implies q = -4$. **A1**
(Alternatively, use `Simultaneous Equation` solver on CG50).

11. **[Paper 1 Style, Short Answer, Hard, 5 marks]**

(a) From $y = -3x + 10$, the gradient $m = -3$. **A1**

(b) $\frac{dy}{dx} = 3x^2 - 8x + k$. **A1A1**

(c) Gradient is -3 when $x = 2$:
$3(2)^2 - 8(2) + k = -3$. **(M1)**
$12 - 16 + k = -3 \implies -4 + k = -3 \implies k = 1$. **A1**

12. **[Paper 2 Style, Longer Question, Hard, 7 marks]**

(a) Base dimensions are $(20-2x)$ by $(20-2x)$. Height is x. **(M1)**
$V(x) = x(20-2x)(20-2x) = x(20-2x)^2$. **A1A1**

(b) Lengths must be strictly positive: $x > 0$ and $20 - 2x > 0 \implies 2x < 20 \implies x < 10$.
Domain: $0 < x < 10$. **A1A1**

(c) In `MENU 5 (Graph)`, enter $Y1 = x(20-2x)^2$. `DRAW`. **(M1)**
Press `G-Solv -> MAX`.
$x = 3.333\cdots \approx 3.33$ cm.
$Y = 592.592\cdots \approx 593$ cm^3. **A1**

13. **[Paper 2 Style, Longer Question, Very Hard, 6 marks]**

(a) Distance formula: $D = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$. **(M1)**
$D(x) = \sqrt{(x-0)^2 + (x^2-2)^2} = \sqrt{x^2 + (x^2-2)^2}$. **A1 (AG)**

(b) In `MENU 5 (Graph)`, enter $Y1 = \sqrt{x^2 + (x^2-2)^2}$. `DRAW`. **(M1)**
Press `G-Solv -> MIN`.
The calculator displays $X = 1.2247\ldots$ and $Y = 1.3228\ldots$
Minimum distance ≈ 1.32. **A1A1**

(c) The points occur at $x = 1.22$ and $x = -1.22$ (due to symmetry). **A1**

14. **[Paper 1 Style, Short Answer, Very Hard, 5 marks]**

(a) Go to `MENU 5 (Graph)`, enter $Y1 = 200 + 50/x + x^2/100$. **(M1)**
Adjust `V-Window` (e.g. X: 0 to 30, Y: 150 to 300). `DRAW` to display the curve.
A1

(b) Press `G-Solv -> MIN`. **(M1)**
$X = 13.572\cdots \implies x \approx 13.6$ (hundred laptops). **A1**

(c) Read the Y value: $Y = 205.526\cdots \approx 206$ (thousands of dollars, or \$205,526).
A1

15. **[Paper 2 Style, Longer Question, Very Hard, 7 marks]**

(a) At $x = 2, y = 2^2 - 2 = 2$. Point $(2, 2)$. **(M1)**
$\frac{dy}{dx} = 2x - 1$. Gradient at $x = 2$ is $m_t = 2(2) - 1 = 3$. **(A1)**
Normal gradient $m_n = -\frac{1}{3}$. **A1**
Equation: $y - 2 = -\frac{1}{3}(x - 2) \implies y = -\frac{1}{3}x + \frac{2}{3} + 2 \implies y = -\frac{1}{3}x + \frac{8}{3}$. **A1**

(b) In `MENU 5 (Graph)`, enter $Y1 = x^2 - x$ and $Y2 = -1/3x + 8/3$. `DRAW`. **(M1)**
Press `G-Solv -> ISCT`.
First intersection is $X = 2, Y = 2$ (the original point).
Press the right arrow key to find the second intersection. **(M1)**
$X = -1.333\ldots, Y = 3.111\ldots$
Q is approximately $(-1.33, 3.11)$. **A1**