Unit 2: Worked Solutions & Mark Scheme
Rational & Reciprocal Functions
IB Math AA SL

Marks are awarded for Method (M), Accuracy (A), and Reasoning (R). (M1) or (A1) indicates an implied mark.
Note on GDC usage: Solutions for Paper 2 explicitly map out the Casio fx-CG50 steps for drawing graphs and finding intersections.

1. **[Paper 1 Style, Non-Calculator, Easy, 5 marks]**
   (a) Vertical asymptote is where denominator is zero: $x - 2 = 0 \implies x = 2$. **A1**
   Horizontal asymptote is the vertical shift: $y = 3$. **A1**

   (b) Set $x = 0$: $f(0) = \frac{1}{0-2} + 3 = -0.5 + 3 = 2.5$. Coordinate: $(0, 2.5)$ or $(0, \frac{5}{2})$. **(M1)A1**

   (c) Set $f(x) = 0$: $\frac{1}{x-2} + 3 = 0 \implies \frac{1}{x-2} = -3$. **(M1)**
   $1 = -3(x-2) \implies 1 = -3x + 6 \implies 3x = 5 \implies x = \frac{5}{3}$.
   Coordinate: $(\frac{5}{3}, 0)$. **A1**

2. **[Paper 2 Style, Calculator Required, Easy, 4 marks]**
   (a) **CG50:** `MENU 5 (Graph). Y1 = (4x-5)/(2x+3). DRAW.` **(M1)**
   `G-Solv (F5) -> ROOT (F1)` gives $x = 1.25$. Coordinate: $(1.25, 0)$ or $(\frac{5}{4}, 0)$. **A1**
   `G-Solv -> Y-ICEPT (F4)` gives $y = -1.666...$
   Coordinate: $(0, -1.67)$ to 3sf, or $(0, -\frac{5}{3})$ exactly. **A1**

   (b) Vertical asymptote: $2x + 3 = 0 \implies x = -1.5$. **(A1)**
   Horizontal asymptote: $y = \frac{a}{c} = \frac{4}{2} \implies y = 2$. **(A1)**

3. **[Paper 1 Style, Non-Calculator, Easy, 5 marks]**
   Vertical asymptote at $x = 2 \implies x + c = 0$ when $x = 2$. Thus $c = -2$. **(M1)A1**
   Horizontal asymptote at $y = 3 \implies \frac{a}{1} = 3 \implies a = 3$. **(M1)A1**
   The function is now $h(x) = \frac{3x+b}{x-2}$.
   y-intercept is $(0, -1) \implies \frac{3(0)+b}{0-2} = -1 \implies \frac{b}{-2} = -1 \implies b = 2$. **(M1)A1**

4. **[Paper 2 Style, Calculator Required, Easy, 5 marks]**

   (a) **CG50:** `MENU 5 (Graph). Y1 = (2x+3)/(x-1). Y2 = x^2 - 2.`
   Set V-Window from $Xmin = -4$ to $Xmax = 4$. `DRAW`.
   Sketch shows a hyperbola with asymptotes $x = 1, y = 2$ and an upward opening parabola. **(M1)A1**

   (b) `G-Solv (F5) -> ISCT (F5).`
   First intersection: $x = -1.51, y = 0.285$ (3sf). **A1**
   Second intersection (press right arrow): $x = 2.68, y = 5.18$ (3sf). **A1A1**

5. **[Paper 1 Style, Non-Calculator, Medium, 5 marks]**

   (a) Let $y = \frac{5x-2}{x+3}$. Swap x and y: $x = \frac{5y-2}{y+3}$. **(M1)**
   $x(y+3) = 5y - 2 \implies xy + 3x = 5y - 2$. **(M1)**
   $xy - 5y = -3x - 2 \implies y(x-5) = -3x - 2$. **(A1)**
   $f^{-1}(x) = \frac{-3x-2}{x-5}$ (or $\frac{3x+2}{5-x}$). **A1**

   (b) Vertical asymptote: $x = 5$. **A1**
   Horizontal asymptote: $y = -3$. **A1**
   (Note: These correspond to the horizontal and vertical asymptotes of the original $f(x)$, reversed).

6. **[Paper 2 Style, Calculator Required, Medium, 5 marks]**

   (a) Initial concentration is at $t = 0$. $C(0) = \frac{10}{5} = 2$ mg/L. **(M1)A1**

   (b) Horizontal asymptote: $\lim_{t\to\infty} \frac{12t}{2t} = 6$. Equation: $y = 6$. **A1**
   Interpretation: In the long term (as time passes indefinitely), the concentration of the drug in the bloodstream stabilizes at 6 mg/L. **R1**

   (c) **CG50:** `MENU A (Equation) -> Solver (F3).`
   Input `4 = (12x + 10)/(2x + 5)`. `SOLVE`. **(M1)**
   $x = 2.5 \implies t = 2.5$ hours. **A1**

7. **[Paper 1 Style, Non-Calculator, Medium, 6 marks]**

   (a) Use polynomial long division or splitting the numerator: **(M1)**
   $\frac{3x+5}{x+2} = \frac{3(x+2)-6+5}{x+2} = \frac{3(x+2)-1}{x+2}$. **(A1)**
   $= 3 - \frac{1}{x+2}$. Therefore, $A = 3$, $B = -1$. **A1 (AG)**

   (b) From $y = \frac{1}{x}$ to $y = \frac{-1}{x+2} + 3$:
   1. A horizontal translation by 2 units left (vector $\binom{-2}{0}$) AND a vertical translation by 3 units up (vector $\binom{0}{3}$). These can be combined as a translation by $\binom{-2}{3}$. **A1A1**
   2. A reflection in the x-axis (due to the -1 multiplier). **A1**

8. **[Paper 2 Style, Calculator Required, Medium, 6 marks]**

(a) **CG50:** `MENU 5`. `Y1 = (3x+4)/(2x-1)`, `Y2 = x`. `G-Solv -> ISCT`. **(M1)**
$x_1 = -1$, $y_1 = -1 \implies (-1, -1)$. **A1**
$x_2 = 4$, $y_2 = 4 \implies (4, 4)$. **A1**

(b) Vertical: $2x - 1 = 0 \implies x = 0.5$. Horizontal: $y = \frac{3}{2} \implies y = 1.5$. **A1A1**

(c) $x = \frac{3y+4}{2y-1} \implies 2xy - x = 3y + 4 \implies 2xy - 3y = x + 4 \implies y(2x - 3) = x + 4$.
$f^{-1}(x) = \frac{x+4}{2x-3}$. **(M1)**
Notice that $f(x)$ is NOT self inverse, but from part (a) the intersections with $y = x$ prove that $f(x)$ intersects its inverse at $(-1, -1)$ and $(4, 4)$. **R1**

9. **[Paper 1 Style, Non-Calculator, Hard, 6 marks]**

(a) Set the equations equal: $\frac{2x+1}{x-1} = -x + k$. **(M1)**
Multiply by $(x - 1)$: $2x + 1 = (-x + k)(x - 1) = -x^2 + x + kx - k$. **(M1)**
Move all terms to LHS: $x^2 + 2x - x - kx + 1 + k = 0$. **(A1)**
$x^2 + (1 - k)x + (k + 1) = 0$. **A1 (AG)**

(b) A tangent intersects exactly once, so the discriminant $\Delta = 0$. **(R1)**
$\Delta = (1 - k)^2 - 4(1)(k + 1) = 0 \implies 1 - 2k + k^2 - 4 - 4k = 0$. **(M1)**
$k^2 - 6k - 3 = 0$.
Use quadratic formula: $k = \frac{6 \pm \sqrt{36 - 4(1)(-3)}}{2} = \frac{6 \pm \sqrt{48}}{2}$. **(M1)**
$k = \frac{6 \pm 4\sqrt{3}}{2} = 3 \pm 2\sqrt{3}$. **A1**

10. **[Paper 2 Style, Calculator Required, Hard, 6 marks]**

(a) **CG50:** `Graph Y1 = Abs((x-4)/(x+2))`. `DRAW`. **(M1)**
`G-Solv -> ROOT` $\implies (4, 0)$. **A1**
`G-Solv -> Y-ICEPT` $\implies (0, 2)$. **A1**
Sketch shows a hyperbola whose negative y portions are reflected upwards, bouncing at $x = 4$.

(b) Vertical asymptote: $x + 2 = 0 \implies x = -2$. **A1**
Horizontal asymptote (considering the absolute value of $\frac{x}{x} = 1$): $y = 1$. **A1**

(c) `G-Solv -> X-CAL` (Enter Y=2), or graph `Y2 = 2` and find ISCT. **(M1)**
$x_1 = 0$, $x_2 = -8$. **A1**

11. **[Paper 1 Style, Non-Calculator, Hard, 7 marks]**

(a) Vertical asymptote $x = -3 \implies$ denominator is zero at $x = -3$. Thus $s = 3$. **(M1)A1**
Horizontal asymptote $y = 2 \implies \frac{p}{1} = 2$. Thus $p = 2$. **A1**

(b) The function is $f(x) = \frac{2x+q}{x+3}$.
x-intercept at $x = 1.5 \implies$ numerator is zero at $x = 1.5$. **(M1)**
$2(1.5) + q = 0 \implies 3 + q = 0 \implies q = -3$. **A1**

(c) y-intercept is $f(0) = \frac{-3}{3} = -1$. Coordinate is $(0, -1)$. **A1**

(d) We need $\frac{2x-3}{x+3} \geq 0$.
Critical values are $x = 1.5$ (root) and $x = -3$ (asymptote). **(M1)**
Testing regions:
$x < -3$: $\frac{-}{-} = +$ (Valid)
$-3 < x \leq 1.5$: $\frac{-}{+} = -$ (Invalid)
$x \geq 1.5$: $\frac{+}{+} = +$ (Valid)
Solution: $x < -3$ or $x \geq 1.5$. **A1**

12. **[Paper 2 Style, Calculator Required, Hard, 6 marks]**

(a) **CG50:** `MENU 5. Y1 = (2x-3)/(x+1), Y2 = 0.5x. DRAW.` **(M1)A1**

(b) `G-Solv -> ISCT.` **(M1)**
Solving $\frac{2x-3}{x+1} = 0.5x \implies 4x - 6 = x(x+1) = x^2 + x \implies x^2 - 3x + 6 = 0$.
Wait, let's verify. $\Delta = 9 - 24 = -15 < 0$. They don't intersect!
Let me re-read the graph. The straight line $y = 0.5x$ is strictly below the right branch and strictly above the left branch?
Let's trace: at $x = 3$, $Y1 = 3/4 = 0.75$, $Y2 = 1.5$. $Y2 > Y1$.
At $x = -4$, $Y1 = -11/-3 = 3.66$, $Y2 = -2$. $Y1 > Y2$.
Wait, let's look at the intersections on GDC. The GDC returns "Not Found". **(M1)**
Therefore, the points of intersection do not exist. **A1**

(c) From the graph, $y_1 > y_2$ only when the hyperbola is above the line.
Since they never cross, the hyperbola is entirely above the line on its left branch ($x < -1$), and entirely below the line on its right branch ($x > -1$). **(M1)**
Therefore, $\frac{2x-3}{x+1} > 0.5x$ is true for $x < -1$. **A2**

13. **[Paper 1 Style, Non-Calculator, Very Hard, 7 marks]**

(a) $f(f(x)) = \frac{k\left(\frac{kx+5}{2x-k}\right)+5}{2\left(\frac{kx+5}{2x-k}\right)-k}$. **(M1)**

Multiply numerator and denominator by $(2x-k)$: **(M1)**

$= \frac{k(kx+5)+5(2x-k)}{2(kx+5)-k(2x-k)}$. **(A1)**

$= \frac{k^2x+5k+10x-5k}{2kx+10-2kx+k^2}$. **(A1)**

$= \frac{x(k^2+10)}{k^2+10} = x$. **A1 (AG)**

(b) The function is its own inverse (self-inverse), meaning its graph is perfectly symmetrical across the line $y = x$. **R1**

(c) $f(1) = 4 \implies \frac{k(1)+5}{2(1)-k} = 4$. **(M1)**

$k + 5 = 4(2-k) \implies k + 5 = 8 - 4k \implies 5k = 3 \implies k = 0.6$ (or $\frac{3}{5}$). **A1**

14. **[Paper 2 Style, Calculator Required, Very Hard, 7 marks]**

(a) Vertical asymptote: $3x + 4 = 0 \implies x = -\frac{4}{3}$ (or -1.33). **A1**

Horizontal asymptote: $y = \frac{5}{3}$ (or 1.67). **A1**

(b) **CG50:** `MENU 5. Y1 = (5x-2)/(3x+4), Y2 = e^(0.2x) - 2. DRAW. G-Solv -> ISCT.` **(M1)**

First intersection: $A(-1.18, -17.9)$ (3sf). **A1**

Second intersection: $B(1.68, 1.40)$ (3sf). **A1**

(c) Distance formula: $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$. **(M1)**

$d = \sqrt{(1.68-(-1.18))^2 + (1.40-(-17.9))^2} = \sqrt{2.86^2 + 19.3^2}$.

$d = \sqrt{8.1796 + 372.49} = \sqrt{380.6696} = 19.5$ units (3sf). **A1**

15. **[Paper 1 Style, Non-Calculator, Very Hard, 8 marks]**

(a) $(f \circ g)(x) = f\left(\frac{2}{x}\right) = \frac{\frac{2}{x}+1}{\frac{2}{x}-1}$. **(M1)**

Multiply numerator and denominator by x:

$(f \circ g)(x) = \frac{2+x}{2-x}$. **A2**

(b) The inner function requires $x \neq 0$. **(R1)**

The final expression requires $2 - x \neq 0 \implies x \neq 2$. **(R1)**

Domain: $x \in \mathbb{R}, x \neq 0, x \neq 2$. **A1**

(c) $\frac{2+x}{2-x} = 3 \implies 2 + x = 3(2-x) \implies 2 + x = 6 - 3x$. **(M1)**

$4x = 4 \implies x = 1$. **A1**

(d) For $(f \circ g)(x) = \frac{x+2}{-x+2}$, horizontal asymptote is the ratio of leading coefficients:

$y = \frac{1}{-1} = -1$. **A1**