IB MATHEMATICS AA HL
AHL TOPIC 4 PRACTICE

Discrete Variance, Continuous RVs, and Linear Transformations

Instructions to Candidates

- This practice paper contains **20** questions progressing from Easy to Very Hard.
- Each question indicates whether it is styled for **Paper 1 (No Calculator)** or **Paper 2 (Calculator Allowed)**.
- The paper tests syllabus topic AHL 4.14: Variance of discrete random variables, continuous random variables and their probability density functions (PDFs), mode, median, expected values, and linear transformations ($\mathrm{E}(aX+b)$ and $\mathrm{Var}(aX+b)$).
- Answer all questions, showing all your working clearly.
- Total marks available: **83**.

Difficulty Progression

- **SECTION A (Easy):** Evaluating constants k for simple PDFs, basic linear transformations of mean and variance, and setting up equations for the median.
- **SECTION B (Medium):** Calculating $\mathrm{E}(X^2)$ to find discrete variance, finding conditional probabilities using continuous PDFs, solving for mode using calculus, and practical payout transformation problems.
- **SECTION C (Hard):** Improper integral expectations, formal proofs of variance rules ($\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$), using integration by parts for continuous expected values, and exponential PDF median derivations.

SECTION A: EASY (Fundamentals)

Question 1 (3 Marks) — Paper 1 (No Calculator Allowed)

A discrete random variable X has an expected value $\mathrm{E}(X) = 3$ and a variance $\mathrm{Var}(X) = 4$. Find the exact values of $\mathrm{E}(5X - 2)$ and $\mathrm{Var}(5X - 2)$.

Question 2 (3 Marks) — Paper 2 (Calculator Allowed)

A continuous random variable X has a probability density function given by $f(x) = k(3 - x)$ for $0 \leq x \leq 3$, and $f(x) = 0$ otherwise. Find the exact value of the constant k.

Question 3 (2 Marks) — Paper 1 (No Calculator Allowed)

The probability distribution of a discrete random variable Y is given in the table below:

y	1	2	3
$P(Y = y)$	0.2	0.5	0.3

Calculate the expected value $\mathrm{E}(Y)$.

Question 4 (4 Marks) — Paper 1 (No Calculator Allowed)

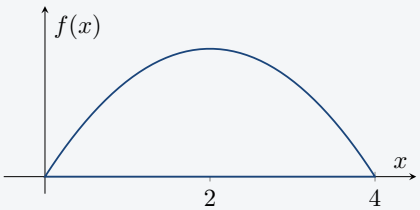
Using the probability distribution from Question 3, calculate $\mathrm{E}(Y^2)$ and hence find the exact value of the variance, $\mathrm{Var}(Y)$.

Question 5 (2 Marks) — Paper 2 (Calculator Allowed)

A continuous random variable T follows a uniform distribution with probability density function $f(t) = 0.5$ for $2 \leq t \leq 4$. Find $P(T \geq 3.5)$.

Question 6 (3 Marks) — Paper 1 (No Calculator Allowed)

A continuous random variable X has the probability density function $f(x) = \frac{3}{32}(4x - x^2)$ for $0 \leq x \leq 4$, as shown in the graph below.



By using calculus to find where $f(x)$ is a maximum, state the mode of X.

Question 7 (3 Marks) — Paper 1 (No Calculator Allowed)

A continuous random variable X has the probability density function $f(x) = \frac{1}{2}x$ for $0 \leq x \leq 2$. Find the exact value of the median of X.

SECTION B: MEDIUM (Application & Algebraic Methods)

Question 8 (5 Marks) — Paper 1 (No Calculator Allowed)

A discrete random variable X has the following probability distribution:

x	0	1	2
$P(X = x)$	p	q	0.4

Given that the expected value $\mathrm{E}(X) = 1.3$, find the exact values of p and q.

Question 9 (6 Marks) — Paper 2 (Calculator Allowed)

A continuous random variable X has the probability density function $f(x) = \frac{3}{8}x^2$ for $0 \leq x \leq 2$. Calculate the expected value $\mathrm{E}(X)$ and the variance $\mathrm{Var}(X)$.

Question 10 (4 Marks) — Paper 1 (No Calculator Allowed)

A continuous random variable Θ has the probability density function $f(\theta) = \cos\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$. Find the exact probability that $\Theta < \frac{\pi}{6}$.

Question 11 (4 Marks) — Paper 2 (Calculator Allowed)

Let S be the score on a biased die where $\mathrm{E}(S) = 4.2$ and $\mathrm{Var}(S) = 1.5$. A game pays out a prize, Y dollars, based on the formula $Y = 3S - 5$. Find the expected payout and the standard deviation of the payout. Give your answers correct to 3 significant figures.

Question 12 (6 Marks) — Paper 2 (Calculator Allowed)

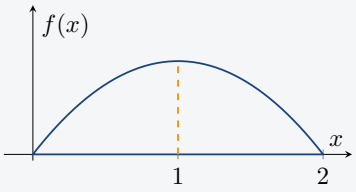
A continuous random variable X has the probability density function $f(x) = \frac{k}{x^2}$ for $1 \leq x \leq 2$.

(a) Find the exact value of the constant k. **[2 marks]**

(b) Find the exact median of X. **[4 marks]**

Question 13 (5 Marks) — Paper 1 (No Calculator Allowed)

A continuous random variable X has the probability density function $f(x) = a(2x - x^2)$ for $0 \leq x \leq 2$.



(a) Find the exact value of a. **[3 marks]**

(b) Using the geometrical symmetry of the probability density function, state the expected value $\mathrm{E}(X)$. **[2 marks]**

Question 14 (5 Marks) — Paper 1 (No Calculator Allowed)

Using the probability density function $f(x) = \frac{3}{4}(2x - x^2)$ for $0 \leq x \leq 2$ from Question 13, calculate the exact conditional probability $P(X > 1.5 \mid X > 1)$.

SECTION C: HARD / VERY HARD (Synthesis & Proof)

Question 15 (6 Marks) — Paper 1 (No Calculator Allowed)

A discrete random variable X has the probability distribution $P(X = x) = \frac{c}{x}$ for $x \in \{1, 2, 3\}$. Find the exact value of the constant c, and hence calculate $\text{Var}(X)$.

Question 16 (6 Marks) — Paper 1 (No Calculator Allowed)

A continuous random variable X has the probability density function $f(x) = \frac{1}{2}\sin x$ for $0 \leq x \leq \pi$. Using integration by parts, prove that the expected value $\text{E}(X) = \frac{\pi}{2}$.

Question 17 (6 Marks) — Paper 2 (Calculator Allowed)

A continuous random variable X has the probability density function $f(x) = \frac{2x}{9}$ for $0 \leq x \leq 3$. Find the exact value of $\text{Var}(2X + 1)$.

Question 18 (5 Marks) — Paper 1 (No Calculator Allowed)

A continuous random variable T models the lifespan of a component, with probability density function $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$. Given that the median lifespan of the component is $m = \frac{\ln 2}{3}$, find the exact value of the parameter λ.

Question 19 (6 Marks) — Paper 2 (Calculator Allowed)

A continuous random variable X has the probability density function $f(x) = \frac{3}{(x+1)^4}$ for $x \geq 0$. Calculate the expected value $\text{E}(X)$.

Question 20 (5 Marks) — Paper 1 (No Calculator Allowed)

For any continuous random variable X defined on the domain $[a, b]$ with probability density function $f(x)$ and mean $\mu = \text{E}(X)$, the variance is strictly defined as $\text{Var}(X) = \int_a^b (x - \mu)^2 f(x)\, dx$. Starting from this definition, mathematically prove the expectation algebra identity:

$$\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$$