IB MATHEMATICS AI HL
AHL QUESTION BOOKLET

Laws of Logarithms & Infinite Geometric Series

Instructions to Candidates

- This extended practice paper contains **15 questions**.
- The paper targets **Advanced Higher Level (AHL)** syllabus components 1.9, 1.11, and 2.10.
- Note: In Mathematics AI, all papers require a Graphic Display Calculator (GDC). Use it efficiently to solve equations.
- Base a for logarithms in AI HL will strictly be limited to 10 or e.
- Answer all questions, showing all step-by-step working clearly in the spaces provided.

Difficulty Progression

- **Questions 1 - 5 (Easy):** Fundamental laws of ln and $\log_{10}$, basic exponential solving, and introductory infinite series.
- **Questions 6 - 10 (Medium):** Bouncing ball models, steady-state drug concentrations, Richter scale applications, and semi-log graph modelling.
- **Questions 11 - 15 (Hard):** Log-log graph transformations, hidden quadratics, solving simultaneous logarithmic equations, and limits of fractal areas.

SECTION A: EASY (Fundamentals)

Question 1 (4 Marks)

By using the laws of logarithms, write the following expression as a single natural logarithm. Show your working clearly.

$$\ln(100) - 2\ln(5)$$

Question 2 (4 Marks)

An infinite geometric series has a first term of $u_1 = 50$ and a common ratio of $r = e^{-0.2}$.
(a) Write down the value of the common ratio correct to three decimal places. **[1 mark]**
(b) Calculate the exact sum to infinity, S_∞, of this geometric series. Give your answer in the form $\frac{A}{1-e^B}$. **[3 marks]**

Question 3 (4 Marks)

Solve the following exponential equation for t. Give your answer correct to three significant figures.

$$1500e^{0.04t} = 4500$$

Question 4 (4 Marks)

Solve the following equation for x.

$$\log_{10}(x) + \log_{10}(2) = 3$$

Question 5 (5 Marks)

The sum to infinity of a geometric sequence is exactly 125. The common ratio is $r = 0.6$. Calculate the exact value of the first term, u_1, and the third term, u_3.

SECTION B: MEDIUM (Application & Modelling)

Question 6 (5 Marks)

The Richter scale measures the magnitude, M, of an earthquake based on its seismic intensity, I, using the base-10 logarithmic formula:

$$M = \log_{10}\left(\frac{I}{I_0}\right)$$

where I_0 is a constant baseline intensity.
An earthquake in Japan had a magnitude of 6.5. An earthquake in California had a magnitude of 4.0.
Determine how many times more intense the earthquake in Japan was compared to the earthquake in California.

Question 7 (6 Marks)

A highly elastic rubber ball is dropped from a vertical height of 15 m onto a hard concrete floor. After each bounce, the ball rebounds to exactly 80% of its previous maximum height.
(a) Calculate the maximum height the ball reaches immediately after its 4th bounce. **[2 marks]**
(b) Determine the total vertical distance travelled by the ball from the moment it is initially dropped until it eventually comes to rest. **[4 marks]**

Question 8 (6 Marks)

A patient is prescribed a daily dose of 20 mg of a specific medication. Every 24 hours, the patient's body naturally processes and removes 15% of the medication currently in their bloodstream.
If the patient takes this 20 mg dose at the same time every day indefinitely, the total amount of medication in the bloodstream immediately after taking a dose approaches a steady-state limit.
By treating this as an infinite geometric series, calculate this maximum steady-state amount of medication in the patient's bloodstream.

Question 9 (6 Marks)

A biologist is studying the exponential growth of a bacteria colony. To create a linear model, she plots the natural logarithm of the population, $\ln(P)$, against the time in days, t, on a semi-log graph.
The resulting straight line passes through the points $(0, \ln 50)$ and $(4, \ln 200)$.
(a) Find the equation of the straight line in the form $\ln P = mt + c$. **[3 marks]**
(b) Hence, express the population P as an exponential model in the form $P = Ae^{kt}$. **[3 marks]**

Question 10 (6 Marks)

The sum of an infinite geometric series is 100. The second term of the series is $u_2 = 24$.
By setting up an algebraic equation, find the two possible values for the common ratio, r.

SECTION C: HARD (Synthesis & Proof)

Question 11 (7 Marks)

Solve the following equation for x.

$$\ln(x^2 - 9) - \ln(x - 3) = \ln(8)$$

State clearly why one step of algebraic simplification requires the assumption that $x > 3$.

Question 12 (8 Marks)

A system of simultaneous equations is given below:

$$\ln x + \ln y = 5$$
$$\ln x - \ln y = 1$$

(a) Solve the system to find the exact values of $\ln x$ and $\ln y$. **[4 marks]**
(b) Hence, write down the exact values of x and y. **[2 marks]**
(c) Calculate the exact value of $x \times y$, giving your answer as a power of e. **[2 marks]**

Question 13 (8 Marks)

Astronomers are measuring the distance, D, of newly discovered galaxies and their recession velocity, v. Because the numbers are extremely large, they plot the data on a log-log graph. They plot $\log_{10}(v)$ on the vertical axis against $\log_{10}(D)$ on the horizontal axis. The resulting straight line has a gradient of 1.5 and passes through the coordinate point $(2, 5)$.
Find the algebraic model linking v and D, expressing your final answer in the form $v = aD^n$, where a and n are constants to be found.

Question 14 (7 Marks)

Consider the exponential equation:

$$e^{2x} - 7e^x + 10 = 0$$

(a) Write down a substitution that transforms this into a quadratic equation. **[2 marks]**
(b) Use your graphic display calculator (GDC) to find the exact solutions for x, giving your answers in terms of natural logarithms. **[5 marks]**

Question 15 (9 Marks)

A mathematician is generating a fractal. At Stage 1, the area of the central shape is $A_1 = 50$ cm^2.
At each subsequent stage n, new shapes are added to the perimeter of the fractal. The total area added at stage n is given by:

$$A_n = 50 \times \left(\frac{4}{9}\right)^{n-1}$$

(a) Find the total area added during Stage 4. **[2 marks]**
(b) Write down an expression using sigma (Σ) notation for the total combined area of the fractal after k stages. **[2 marks]**
(c) As $k \to \infty$, the fractal is completely formed. Show that the total area of the fully formed fractal converges to a finite value, and calculate this exact area. **[5 marks]**