IB MATHEMATICS AI HL
UNIT 3: GEOMETRY

Graph Theory

Instructions to Candidates

- This question booklet contains **15 questions**.
- The paper targets **AHL** syllabus component 3.15.
- Answer all questions, showing all step-by-step working clearly.

Difficulty Progression

- **Questions 1 - 5 (Easy):** Drawing graphs, adjacency matrices, vertex degrees, and simple complete graphs.
- **Questions 6 - 10 (Medium):** Minimum spanning trees (Kruskal/Prim), A^k walks, and Hamiltonian/Eulerian paths.
- **Questions 11 - 15 (Hard):** Chinese Postman Problem, Google PageRank algorithm, and bipartite graph proofs.

SECTION A: EASY (Fundamentals)

CG50 Tip: Graph Theory Matrix Powers

In Graph Theory, the number of walks of length k between vertices is found by raising the Adjacency Matrix to the power of k. In `Run-Matrix`, press `F3 (MAT/VCT)`, define your matrix as `Mat A`, and type `Mat A ^5` to instantly find all 5-step walks!

Question 1 (4 Marks)

An undirected graph G has 4 vertices: A, B, C, and D.

Its adjacency matrix is given by: $M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

Sketch the graph G clearly showing all vertices and edges.

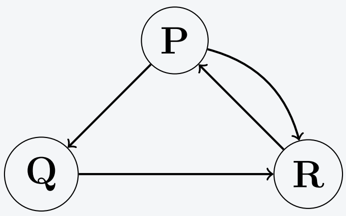
Question 2 (4 Marks)

Determine the degree of every vertex in the graph drawn in Question 1. Hence, state whether the graph contains an Eulerian trail, and justify your answer.

Question 3 (4 Marks)

A complete graph K_n has n vertices where every vertex is connected to every other vertex by exactly one edge. Calculate the exact number of edges in a complete graph K_6.

Question 4 (4 Marks)



Construct the probability transition matrix T for a random walk on the directed graph above. (Assume a walker at a node is equally likely to pick any outgoing directed edge).

Question 5 (5 Marks)

A connected graph is known as a "Tree" if it contains no cycles. State the exact number of edges a Tree with 15 vertices must contain.

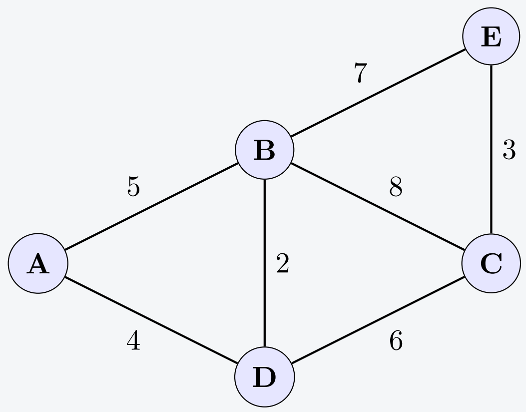
SECTION B: MEDIUM (Application & Modelling)

Question 6 (5 Marks)

An unweighted adjacency matrix for graph G is $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Calculate A^3. State the geometric meaning of the number in Row 1, Column 3 of the matrix A^3.

Question 7 (6 Marks)



Use Kruskal's or Prim's algorithm to find the Minimum Spanning Tree (MST) for the weighted graph above. State the order in which you select the edges, and calculate the total weight.

Question 8 (6 Marks)

Using the same weighted graph from Question 7, a delivery driver must start at A, visit every node exactly once, and return to A (a Travelling Salesman Problem).
Use the Nearest Neighbour Algorithm starting from A to find a Hamiltonian cycle, and calculate its total distance. Is this distance optimal?

Question 9 (6 Marks)

A graph G has an adjacency matrix where the sum of all elements in the matrix is exactly 24. How many edges does the graph G have? Justify your answer.

Question 10 (6 Marks)

Identify whether it is possible to draw a simple graph with 5 vertices having degrees $1, 2, 3, 4$, and 5. Give a mathematical reason for your answer.

SECTION C: HARD (Synthesis & Proof)

Question 11 (7 Marks)

A simplified internet network has 3 webpages. - Page 1 links to Page 2. - Page 2 links evenly to Page 1 and Page 3. - Page 3 links to Page 1.
(a) Write down the Google PageRank transition matrix T for this network. [**2 marks**]
(b) By solving the matrix equation $Tv = v$, algebraically find the exact steady-state PageRank vector v. [**5 marks**]

Question 12 (8 Marks)

The town of Königsberg has a snow-plowing network defined by a weighted graph with the following odd-degree vertices: V_1, V_2. The shortest path between V_1 and V_2 has a weight of 12. The total weight of all edges in the network is 150.
A snowplow must traverse every edge in the graph at least once and return to its starting position (The Chinese Postman Problem).
Calculate the minimum total distance the snowplow must travel. Explain your reasoning regarding Eulerian circuits.

Question 13 (8 Marks)

Let G be a bipartite graph with two distinct disjoint sets of vertices U and V. Set U contains 3 vertices and set V contains 4 vertices.
Determine the maximum possible number of edges G can have. Draw this maximal bipartite graph, and write out its 7×7 adjacency matrix.

Question 14 (7 Marks)

Consider a complete bipartite graph $K_{m,n}$. Prove algebraically that if a complete bipartite graph has an Eulerian circuit, then m and n must both be even numbers.

Question 15 (9 Marks)

A graph G has adjacency matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.
The number of walks of length k from Vertex 1 to itself is given by the top-left entry of A^k.
Using your GDC, find the exact number of walks of length 10 from Vertex 1 to itself.