NUMBER BASES

For base ten as well as for any base b, where b is a positive integer greater than 1, the following theorem holds:

The Fundamental Theorem of Systems of Numeration: Every natural number n can be uniquely written as

$$n = a_m b^m + a_{m-1} b^{m-1} + a_{m-2} b^{m-2} + \cdots + a_2 b^2 + a_1 b + a_0$$

where a_0, a_1, $\ldots$, a_m and m are integers satisfying the following conditions: $0 \leqslant a_i < b$ for all $i = 0, 1, 2, \ldots, m$, $a_m \neq 0$ and $m \geqslant 0$.

Notation and terminology: We abbreviate

$$n = a_m b^m + a_{m-1} b^{m-1} + a_{m-2} b^{m-2} + \cdots + a_2 b^2 + a_1 b + a_0$$

as $n = a_m a_{m-1} \ldots a_1 a_0$ and we say that n is written in base b. The numbers 0, 1, 2, $\ldots$, $b-2$, $b-1$ are called digits in base b. The numbers 1, 2, 3, $\ldots$, $b-1$ are called significant digits.

If in a certain problem one deals with multiple bases, then the base is added in subscript to the right of the number. For example, 17 in base 3 is written as 17_3 or $17_{(3)}$. Unless specified by the context, numbers without subscript are considered to be decimal (in base 10).

History: The base-10 system is the most commonly used today. The binary system (base 2) is used to perform integer arithmetic in almost all digital computers. Modern computers today also use the octal (base 8) and hexadecimal (base 16) systems. A base-5 system has been used in many cultures for counting; the system was based on the number of fingers of the human hand. The Yuki tribe in Northern California still uses a quaternary (base-4) system, by counting the spaces between the fingers rather than the fingers themselves. Base-12 systems (duodecimal) have been popular because multiplication and division are easier than in base 10. They have been used extensively by the British, where 12 was a common unit of measurement (12 inches in a foot, 12 pennies in a shilling, etc). Twelve is also used as unit for analog and digital printing; printer resolutions like 360, 600, 720, 1200, 1440 dpi are everywhere. The Mayan civilization used a base-20 system, possibly originating from the number of a person's fingers and toes. The sexagesimal system (base 60) was used by Sumerians and their successors in Mesopotamia and survives today in our systems of time (hence the division of an hour into 60 minutes and a minute into 60 seconds) and angular measure (a degree is divided into 60 minutes and a minute into 60 seconds). The Chinese calendar still uses today a base-60 system to denote years.

Converting from base b to base 10:

(i) The conversion of a whole number from base b to base 10 uses the fundamental theorem:

$$(a_m a_{m-1} \ldots a_1 a_0)_b = a_m b^m + a_{m-1} b^{m-1} + \cdots + a_1 b + a_0$$

For example:

$$621_8 = 6 \cdot 8^2 + 2 \cdot 8 + 1 = 401$$
$$23010_5 = 2 \cdot 5^4 + 3 \cdot 5^3 + 0 \cdot 5^2 + 1 \cdot 5 + 0 = 1630$$

(ii) If the number in base b is negative, use the exact same formula:

$$-(a_m a_{m-1} \dots a_1 a_0)_b = -(a_m b^m + a_{m-1} b^{m-1} + \dots + a_1 b + a_0)$$

For example,

$$-5312_6 = -(5 \cdot 6^3 + 3 \cdot 6^2 + 1 \cdot 6 + 2) = -1196$$

(iii) Finally, decimal numbers in base b are converting to base 10 using:

$$(a_m a_{m-1} \dots a_1 a_0 . c_1 c_2 c_3 \dots)_b = a_m b^m + a_{m-1} b^{m-1} + \dots + a_1 b + a_0 + c_1 b^{-1} + c_2 b^{-2} + c_3 b^{-3} + \dots$$

For example,

$$11.011_2 = 1 \cdot 2 + 1 + 0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 1 \cdot 2^{-3} = \frac{27}{8} = 3.375$$

Converting from base 10 to an arbitrary base b:

For converting a positive integer n to an arbitrary base b, use the following algorithm:

$$\begin{aligned}
n &= b \cdot q_0 + a_0 & 0 \leqslant a_0 < b \\
q_0 &= b \cdot q_1 + a_1 & 0 \leqslant a_1 < b \\
q_1 &= b \cdot q_2 + a_2 & 0 \leqslant a_2 < b \\
\vdots \quad & \quad \vdots & \vdots \\
q_{m-2} &= b \cdot q_{m-1} + a_{m-1} & 0 \leqslant a_{m-1} < b \\
q_{m-1} &= b \cdot 0 + a_m & 0 \leqslant a_m < b
\end{aligned}$$

Then $n = (a_m a_{m-1} \dots a_1 a_0)_b$.

For example, suppose we want to convert 641 from base 10 to base 7. Using the above algoritm we get:

$$\begin{aligned}
641 &= 7 \cdot 91 + 4 \\
91 &= 7 \cdot 13 + 0 \\
13 &= 7 \cdot 1 + 6 \\
1 &= 7 \cdot 0 + 1
\end{aligned}$$

Hence $641 = 1604_7$.

1. *(2004 Harker Math Invitational, Individual Contest, ♯17)* In a strange land they have a slightly different place value system than our base ten system. For example, $4 \times 6 = 30$ and $4 \times 7 = 34$. Based on this system give the value of $5 \times 4 \times 7$.

2. *(2005 Harker Math Invitational, Individual Contest, ♯10)* A base-two numeral consists of 15 digits all of which are ones. This number when tripled and written in base two, contains how many digits?

3. *(2006 Alabama Statewide Mathematics Contest, Algebra II with Trigonometry Exam, ♯24)* The symbol $(37)_b$ represents a two-digit number written in base b. For instance, $(37)_8 = 31$, where the last number is written in base 10. Suppose that $(37)_b$ is exactly one-half of $(73)_b$. What is b?

   (A) 11 (B) 10 (C) 9 (D) 8 (E) 12

4. *(2006 Alabama Statewide Mathematics Contest, Algebra II with Trigonometry Exam, ♯31)* The symbol $(x)_7$ means that the number is written in base 7; so, for instance, $(23)_7$ equals $2 \times 7 + 3 = 17$ in base 10. Suppose that a certain integer x can be written as a two-digit number in both bases 5 and 6, so that $x = (4z)_5$ and $x = (4y)_6$. What is x (in base 10)?

   (A) 30 (B) 20 (C) 16 (D) 24 (E) 19

5. *(2011 Purple Comet Math Meet, Middle School Contest, ♯6)* The following addition problem is not correct if the numbers are interpreted as base 10 numbers. In what base is the problem correct?

```
  66
  87
  85
 +48
 ---
 132
```

6. *(2006 Alabama Statewide Mathematics Contest, Algebra II with Trigonometry Exam, ♯50)* James did the following addition problem: $440 + 340 = 1000$. This problem is done correctly! In what base was James writing his numbers?

   (A) 9 (B) 5 (C) 6 (D) 7 (E) 8

7. Let x and y be digits in base 7 such that

$$1111_2 + 1111_3 + 1111_4 + 1111_5 + 1111_6 + 1111_7 + 1111_8 + 1111_9 = \overline{xyxyx}_7 - 91.$$

   Find $x + y$.

8. A positive integer n can be written in base 7 as the three-digit number $\overline{xyy}$. The same number n can be written in base 6 as the three-digit number $\overline{yxx}$. Find n.

9. A three digit number $\overline{xyz}$ in base 7 when written in base 9 becomes the three-digit number $\overline{zyx}$. What is the decimal representation of the number?

10. In what base is 165 a divisor of 561?

11. Prove that, in any base, 10101 is divisible by 111.

12. In what base does the following multiplication hold: $25 \times 314 = 10274$?

13. *(2005 "Math is Cool" Masters, 6th Grade, Individual Contest, ♯30)* 1580_{10} is 2145 in what base?

14. Show that the number 1320 is divisible by 6 in any base larger than 3.

15. Show that in any base larger than 7 the number 1,367,631 is a perfect cube.

16. Find $\sqrt{1031^2 + 10(420^2 + 452)}$ given that all numbers are written in base 7.

17. Determine a number of the form $\overline{abba}$ knowing that in any base of a numerical system it is a perfect cube.

18. Show that any integer of the form $111\ldots11_9$ is triangular.

19. Determine all positive integers n such that 11111_n is a perfect square.

20. *(2003 AMC 10 A, ♯20)* A base-10 three digit number n is selected at random. Which of the following is closest to the probability that the base-9 representation and the base-11 representation of n are both three-digit numerals?

    (A) 0.3 (B) 0.4 (C) 0.5 (D) 0.6 (E) 0.7

21. *(2007 AMC 12 B, ♯21)* The first 2007 positive integers are each written in base 3. How many of these base-3 representations are palindromes? (A palindrome is a number that reads the same forward and backward.)

    (A) 100 (B) 101 (C) 102 (D) 103 (E) 104

22. *(2017 AIME II, ♯4)* Find the number of positive integers less than or equal to 2017 whose base-three representation contains no digit equal to 0.

23. *(2012 AIME I, ♯5)* Let B be the set of all binary integers that can be written using exactly 5 zeros and 8 ones where the leading zeros are allowed. If all possible subtractions are performed in which one element of B is subtracted from another, find the number of times the answer 1 is obtained.

24. *(2012 AIME II, ♯7)* Let S be the increasing sequence of positive integers whose binary representation has exactly 8 ones. Let N be the 1000^{th} number in S. Find the remainder when N is divided by 1000.

25. *(1987 AMC 12, ♯16)* A cryptographer devises the following method for encoding positive integers. First, the integer is expressed in base 5. Second, a 1-to-1 correspondence is established between the digits that appear in the expressions in base 5 and the elements of the set $\{V, W, X, Y, Z\}$. Using this correspondence, the cryptographer finds that three consecutive integers in increasing order are coded as VYZ, VYX, VVW, respectively. What is the base ten expression for the integer coded as XYZ?

    (A) 48 (B) 71 (C) 82 (D) 108 (E) 113

26. *(2005 AMC 12 A, ♯19)* A faulty car odometer proceeds from digit 3 to digit 5, always skipping the digit 4, regardless of position. For example, after traveling one mile the odometer changed from 000039 to 00050. If the odometer now reads 002005, how many miles has the car actually traveled?

    (A) 1404 (B) 1462 (C) 1604 (D) 1605 (E) 1804

27. *(2004 AMC 12 A, ♯25)* For each integer $n \geqslant 4$, let a_n denote the base-n number $0.\overline{133}_n$. The product $a_4a_5 \cdots a_{99}$ can be expressed as $\dfrac{m}{n!}$, where m and n are positive integers and n is as small as possible. What is the value of m?

    (A) 98 (B) 101 (C) 132 (D) 798 (E) 962

28. *(2000 Junior Balkan Math Olympiad Short-List)* Find all integers written as $\overline{abcd}$ in decimal representation and $\overline{dcba}$ in base 7.

29. *(2001 AIME I, ♯8)* Call a positive integer N a 7-10 double if the digits of the base 7-representation of N form a base-10 number that is twice N. For example, 51 is a 7-10 double because its base-7 representation is 102. What is the largest 7-10 double?

30. *(2018 AIME I, ♯2)* The number n can be written in base 14 as $\underline{a}\,\underline{b}\,\underline{c}$, can be written in base 15 as $\underline{a}\,\underline{c}\,\underline{b}$, and can be written in base 6 as $\underline{a}\,\underline{c}\,\underline{a}\,\underline{c}$, where $a > 0$. Find the base-10 representation of n.

31. *(2018 AIME II, ♯3)* Find the sum of all positive integers $b < 1000$ such that the base-b integer 36_b is a perfect square and the base-b integer 27_b is a perfect cube.

32. *(2017 AIME I, ♯5)* A rational number written in base eight is $\underline{ab}.\underline{cd}$, where all digits are nonzero. The same number in base twelve is $\underline{bb}.\underline{ba}$. Find the base-ten number $\underline{abc}$.

33. *(2008 AIME II, ♯4)* There exist r unique nonnegative integers $n_1 > n_2 > \cdots > n_r$ and r integers a_k $(1 \le k \le r)$ with each a_k either 1 or -1 such that

$$a_1 3^{n_1} + a_2 3^{n_2} + \cdots + a_r 3^{n_r} = 2008.$$

Find $n_1 + n_2 + \cdots + n_r$.

34. *(2010 AIME I, ♯10)* Let N be the number of ways to write 2010 in the form

$$2010 = a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0,$$

where the a_i's are integers, and $0 \le a_i \le 99$. An example of such a representation is $1 \cdot 10^3 + 3 \cdot 10^2 + 67 \cdot 10^1 + 40 \cdot 10^0$. Find N.

35. *(2004 AIME II, ♯10)* Let S be the set of integers between 1 and 2^{40} whose binary expansions have exactly two 1's. If a number is chosen at random from S, the probability that it is divisible by 9 is p/q, where p and q are relatively prime positive integers. Find $p + q$.

36. *(2013 AMC 12 B, ♯23)* Bernardo chooses a three-digit positive integer N and writes both its base-5 and base-6 representations on a blackboard. Later LeRoy sees the two numbers Bernardo has written. Treating the two numbers as base-10 integers, he adds them to obtain an integer S. For example, if $N = 749$, Bernardo writes the numbers 10,444 and 3,245, and LeRoy obtains the sum $S = 13,689$. For how many choices of N are the two rightmost digits of S, in order, the same as those of $2N$?

(A) 5 (B) 10 (C) 15 (D) 20 (E) 25

37. An evil king wrote three secret two-digit numbers a, b, and c. A handsome prince must name three numbers X, Y, and Z, after which the king will tell him the sum $aX + bY + cZ$. The prince must then name all three of the king's numbers. Otherwise he will be executed. Help him out of this dangerous situation.

38. Frederik Pohl, a top writer of science fiction, thought of this stunt, which appeared in a magic magazine called "Epilogue". Computer programmers are likely to solve it more quickly than others.

Ask someone to draw a horizontal row of small circles on a sheet of paper to indicate a row of coins. Your back is turned while he does this. He then places the tip of his right thumb on the first circle so that his thumb completely covers the row of circles. You turn around and bet that you can immediately put on the sheet a number that will indicate the total number of combinations of heads and tails that are possible if each coin is flipped. (For example, two coins can fall in four different ways, three coins in eight different ways, and so on.) You have no way of knowing how many coins he drew and yet you win the bet easily. How?