CHALLENGING PROBLEMS IN TRIGONOMETRY

1. *(AOPS Precalculus, Problem 4.62, page 156)* Find $\sin 18°$. (See if you can find both a geometric and an algebraic solution.)

2. *(1978 NYSML, Team Contest, ♯8)* The positive root of the equation $4x^2 + ax + b = 0$ is $\sin 18°$. Find the ordered pair of numbers (a, b).

3. Find the exact value of $\sin 18° \sin 54°$.

4. *(1975 AMC 12, ♯30)* Let $x = \cos 36° - \cos 72°$. Then x equals:

   (A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $3 - \sqrt{6}$ (D) $2\sqrt{3} - 3$ (E) none of these

5. Find the exact value of the expression: $\tan 9° - \tan 27° - \tan 63° + \tan 81°$.

6. *(1974 Timisoara Mathematics Magazine, Romania, Problem 2020)* Find the value of $\tan^2 27° + 2 \tan 27° \tan 36°$.

7. *(2009 Romanian Math Olympiad, Local Round, Vrancea)* Prove that $\cos \frac{\pi}{5} + \cos \frac{3\pi}{5} = \frac{1}{2}$.

8. Show that $\sin 3° = \frac{1}{16}\left[(\sqrt{5}-1)(\sqrt{6}+\sqrt{2}) - 2(\sqrt{3}-1)\sqrt{5+\sqrt{5}}\right]$.

9. Show that $\sin 6° = \dfrac{\sqrt{30-6\sqrt{5}} - \sqrt{6+2\sqrt{5}}}{8}$ and $\cos 6° = \dfrac{\sqrt{18+6\sqrt{5}} + \sqrt{10-2\sqrt{5}}}{8}$.

10. Prove the following identity: $\sin\theta + \sin(108° - \theta) - \sin(\theta - 36°) - \sin(72° - \theta) = \cos(54° - \theta)$.

11. *(2009 Mandelbrot Competition, Round 1 Individual, ♯8)* Find an acute angle θ for which

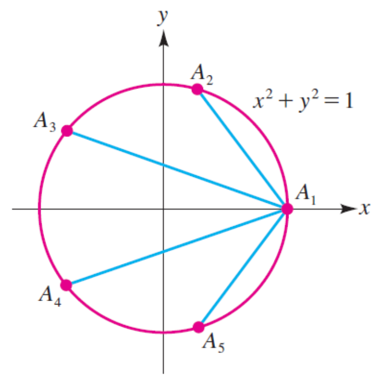
$$\cos\theta = \sin 60° + \cos 42° - \sin 12° - \cos 6°.$$

12. (a) Prove that $\cos 18° = \frac{1}{4}\sqrt{10+2\sqrt{5}}$ and $\sin 36° = \frac{1}{4}\sqrt{10-2\sqrt{5}}$.

    (b) Find the exact value of $\cos 72° \cos 144°$.

(c) Find the exact value of $\cos 72° + \cos 144°$.

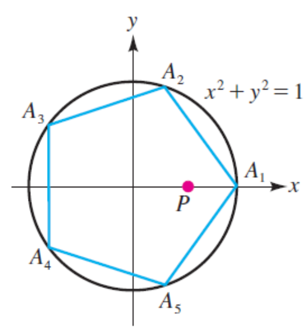
13. In the figure below, the points A_1, A_2, A_3, A_4, and A_5 are the vertices of a regular pentagon.



Prove that $(A_1A_2)(A_1A_3)(A_1A_4)(A_1A_5) = 5$.

This is a particular case of a general result due to Roger Cotes (1682-1716): Suppose that a regular n-gon is inscribed in the unit circle. Let the vertices of the n-gon be A_1, A_2, ..., A_n. Then the product $(A_1A_2)(A_1A_3)\ldots(A_1A_n)$ is equal to n, then number of sides of the polygon.

14. In the figure below, $A_1A_2A_3A_4A_5$ is a regular pentagon inscribed in the unit circle and $P(x, 0)$.



Prove that $(PA_1)(PA_2)(PA_3)(PA_4)(PA_5) = 1 - x^5$.

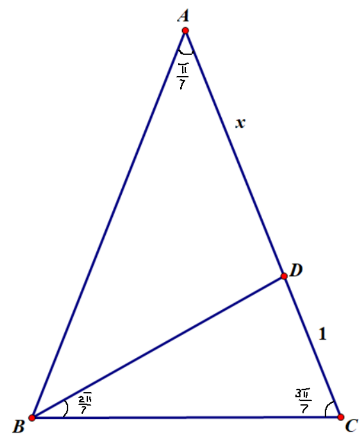
This is a particular case of the following theorem due to Roger Cotes (1682-1716): Suppose that a regular n-gon $A_1A_2\ldots A_n$ is inscribed in the unit circle. Suppose further that A_1 is the point $(1, 0)$ and that P is a point with coordinates $(x, 0)$, where $0 \le x \le 1$. Then $(PA_1)(PA_2)\ldots(PA_n) = 1 - x^n$.

15. In triangle ABC with sides a, b, and c and circumradius R, $B = 18°$ and $C = 36°$. Prove that $a - b = R$.

16. Let A, B, and C be three acute angles such that $\cos A = \tan B$, $\cos B = \tan C$, $\cos C = \tan A$. Prove that

$$\sin A = \sin B = \sin C = 2\sin 18^\circ.$$

17. In $\triangle ABC$, point D is chosen on side AC such that $CD = 1$ and $DA = x$, with angles (in radians) as shown.



(a) Prove that x is a solution of the cubic equation $x^3 + x^2 - 2x - 1 = 0$.

(b) Prove that $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{1}{8}$.

18. *(1963 IMO, ♯5)* Prove that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$.

19. *(2009 Matematica - Modus Vivendi Interstate Competition, Romania, grade 10)* Show that $\cos\frac{\pi}{7}$ is a solution of the equation $8x^3 - 4x^2 - 4x + 1 = 0$.

20. *(2009 Romanian Math Olympiad, Grade 10, Local Round, Dambovita)* Let $z = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$.

(a) Determine the real constants a, b, c, d, and e such that $az + bz^2 + cz^3 + dz^4 + ez^5 = \frac{1}{1-z}$.

(b) Deduce that $\left|z^4 + z^2 + 1\right| = \frac{1}{2\sin\frac{\pi}{14}}$.

(c) Deduce that $6\sin\frac{\pi}{14} - 32\sin^3\frac{\pi}{14} + 32\sin^5\frac{\pi}{14} = 1$.

21. *(2005 Mandelbrot Competition, Individual Round 4, ♯8)* Determine the exact value of

$$\sqrt{\left(2-\sin^2\frac{\pi}{7}\right)\left(2-\sin^2\frac{2\pi}{7}\right)\left(2-\sin^2\frac{3\pi}{7}\right)}.$$

22. *(AOPS Precalculus, Problem 7.72, page 254)* Find a cubic polynomial whose roots are $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, and $\cos\frac{6\pi}{7}$.

23. Prove that $\csc\frac{\pi}{7} = \csc\frac{2\pi}{7} + \csc\frac{3\pi}{7}$.

24. Find the exact value of $\sin 10^\circ \sin 50^\circ \sin 70^\circ$.

25. *(1980 NYSML, Short Answer Questions, ♯6)* Find the degree measure of the least positive angle θ satisfying

$$\tan 20^\circ \tan 40^\circ \tan 80^\circ = \tan\theta.$$

26. Find the degree measure of the least positive angle θ such that $\tan 20^\circ + \tan 40^\circ + \tan 80^\circ = 8\sin\theta + \sqrt{3}$.

27. Find the exact value of $\tan^2 20^\circ + \tan^2 40^\circ + \tan^2 80^\circ$.

28. *(V. Lidsky at al, Problems in Elementary Mathematics, Moscow, 1973)* Simplify the following expression:

$$\frac{1}{2\sin 10^\circ} - 2\sin 70^\circ.$$

29. Simplify the following expression: $\dfrac{\tan 50^\circ - \tan 40^\circ}{\tan 10^\circ}$.

30. Prove that $\sin 3\theta = \sin\theta + 2\sin\theta\cos 2\theta$ and deduce that $\sin\frac{\pi}{9} + 2\sin\frac{\pi}{9}\cos\frac{2\pi}{9} = \frac{\sqrt{3}}{2}$.