RADICALS AND RADICAL EQUATIONS

1. *(1989 AIME, ♯1)* Compute $\sqrt{(31)(30)(29)(28)+1}$.

2. *(2003 Harvard-MIT Math Tournament, Guts Round, ♯7)* a and b are integers such that $a+\sqrt{b}=\sqrt{15+\sqrt{216}}$. Compute a/b.

3. *(2011 AMC 10 A, ♯16)* Which of the following is equal to $\sqrt{9-6\sqrt{2}}+\sqrt{9+6\sqrt{2}}$?

   (A) $3\sqrt{2}$ (B) $2\sqrt{6}$ (C) $\frac{7\sqrt{2}}{2}$ (D) $3\sqrt{3}$ (E) 6

4. *(1999 Middle School Math Contest, Shandong Province, China)* The closest integer to $\frac{1}{\sqrt{17-12\sqrt{2}}}$ is:

   (A) 5 (B) 6 (C) 7 (D) 8

5. *(2003 Middle School Math Contest, Tianjun City, China)* Simplify: $2\sqrt{4+2\sqrt{3}}-\sqrt{21-12\sqrt{3}}$.

   (A) $5-4\sqrt{3}$ (B) $4\sqrt{3}-1$ (C) 5 (D) 1

6. *(2010 Romanian Math Olympiad, Grade 7, Local Round, Giurgiu)* Show that the number

   $$a=\sqrt{124+11\sqrt{12}}+\sqrt{28-5\sqrt{12}}$$

   is a perfect square.

7. *(1993 New Mexico Mathematics Contest, ♯5)*

   (a) Find positive integers u and v satisfying $\sqrt{18-2\sqrt{65}}=\sqrt{u}-\sqrt{v}$.

   (b) Find positive integers x and y satisfying: $\sqrt{14+3\sqrt{3+2\sqrt{5-12\sqrt{3-2\sqrt{2}}}}}=x+\sqrt{y}$.

8. *(1990 AIME, ♯2)* Find the value of $(52+6\sqrt{43})^{3/2}-(52-6\sqrt{43})^{3/2}$.

9. *(2006 AIME I, ♯5)* The number $\sqrt{104\sqrt{6}+468\sqrt{10}+144\sqrt{15}+2006}$ can be written as $a\sqrt{2}+b\sqrt{3}+c\sqrt{5}$, where a, b, and c are positive integers. Find abc.

10. *(2005 USAMTS, Round 4, Problem 2/4/16)* Find positive integers a, b, and c such that:

$$\sqrt{a}+\sqrt{b}+\sqrt{c}=\sqrt{219+\sqrt{10080}+\sqrt{12600}+\sqrt{35280}}.$$

This problem is based on Problem 80 on page 42 of *Problems from the History of Mathematics*, by Levardi and Sain, a book published in Hungarian in Budapest in 1982. Problem 80 was attributed to the Indian mathematician Bhaskara (1114 - ca. 1185).

11. *(2012 Exeter Math Club Competition, Individual Accuracy Test, ♯9)* Let $f(x)=\sqrt{2x+1+2\sqrt{x^2+x}}$. Determine the value of

$$\frac{1}{f(1)}+\frac{1}{f(2)}+\frac{1}{f(3)}+\cdots+\frac{1}{f(24)}.$$

12. For any positive integer n, let $f(n)=\dfrac{4n+\sqrt{4n^2-1}}{\sqrt{2n+1}+\sqrt{2n-1}}$. Evaluate the sum $f(1)+f(2)+\cdots+f(40)$.

13. *(2005 AIME II, ♯7)* Let $x=\dfrac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}$. Find $(x+1)^{48}$.

14. *(1975 AMC 12, ♯29)* What is the smallest integer larger than $(\sqrt{3}+\sqrt{2})^6$?

(A) 972 (B) 971 (C) 970 (D) 969 (E) 968

15. *(2005 UK Senior Mathematical Challenge, ♯25)* Which of the following is equal to $\dfrac{1}{\sqrt{2005+\sqrt{2005^2-1}}}$?

(A) $\sqrt{1003}-\sqrt{1002}$ (B) $\sqrt{1005}-\sqrt{1004}$ (C) $\sqrt{1007}-\sqrt{1005}$ (D) $\sqrt{2005}-\sqrt{2003}$ (E) $\sqrt{2007}-\sqrt{2005}$

16. *(1992 UK National Mathematics Contest, ♯25)* If $x=\sqrt{1+1992^2+\dfrac{1992^2}{1993^2}}+\dfrac{1992}{1993}$, then which statement is true?

(A) $1992<x<1993$ (B) $x=1993$ (C) $1993<x<1994$ (D) $x=1994$ (E) $x>1994$

17. *(1974 AMC 12, ♯20)* Let $T=\dfrac{1}{3-\sqrt{8}}-\dfrac{1}{\sqrt{8}-\sqrt{7}}+\dfrac{1}{\sqrt{7}-\sqrt{6}}-\dfrac{1}{\sqrt{6}-\sqrt{5}}+\dfrac{1}{\sqrt{5}-2}$. Then:

(A) $T<1$ (B) $T=1$ (C) $1<T<2$ (D) $T>2$

(E) $T=\dfrac{1}{(3-\sqrt{8})(\sqrt{8}-\sqrt{7})(\sqrt{7}-\sqrt{6})(\sqrt{6}-\sqrt{5})(\sqrt{5}-2)}$

18. *(1976 AMC 12, ♯27)* If $N = \dfrac{\sqrt{\sqrt{5}+2}+\sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}} - \sqrt{3-2\sqrt{2}}$, then N equals:

    (A) 1 (B) $2\sqrt{2}-1$ (C) $\dfrac{\sqrt{5}}{2}$ (D) $\sqrt{\dfrac{5}{2}}$ (E) none of these

19. *(2009 Georgia Tech High School Math Contest, Junior Varsity Multiple Choice, ♯1)* Simplify:

$$\frac{\sqrt{\sqrt{10}-3}+\sqrt{\sqrt{10}+3}}{\sqrt{\sqrt{10}+1}}.$$

    (A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2 (E) $2\sqrt{2}$

20. *(2012 Louisiana State University Math Contest, Open Session, ♯17)* $f(x) = x^2 + \sqrt{x^4+1} + \dfrac{1}{x^2 - \sqrt{x^4+1}}$. Find $f(2011^{2012})$.

21. *(1991 AMC 12, ♯27)* If $x + \sqrt{x^2-1} + \dfrac{1}{x - \sqrt{x^2-1}} = 20$, then $x^2 + \sqrt{x^4-1} + \dfrac{1}{x^2+\sqrt{x^4-1}} =$

    (A) 5.05 (B) 20 (C) 51.005 (D) 61.25 (E) 400

22. *(2018 Awesome Math Summer Program, Admission Test C, ♯5)* Let a, b, and c be positive real numbers such that

$$\sqrt{a} + 9\sqrt{b} + 44\sqrt{c} = \sqrt{2018(a+b+c)}.$$

    Evaluate $\dfrac{b+c}{a}$.

23. *(1980 AMC 12, ♯27)* The sum $\sqrt[3]{5+2\sqrt{13}} + \sqrt[3]{5-2\sqrt{13}}$ equals:

    (A) $\dfrac{3}{2}$ (B) $\dfrac{\sqrt[3]{65}}{4}$ (C) $\dfrac{1+\sqrt[6]{13}}{2}$ (D) $\sqrt[3]{2}$ (E) none of these

24. Simplify: $\sqrt[3]{\sqrt{\dfrac{980}{27}}+6} - \sqrt[3]{\sqrt{\dfrac{980}{27}}-6}$.

25. *(1943 Gazeta Matematica, Romania)* Find $\sqrt[3]{20+14\sqrt{2}} + \sqrt[3]{20-14\sqrt{2}}$.

26. *(1971 Revista Matematica a Elevilor din Timisoara, Romania, proposed by Titu Andreescu)*

    (a) $\sqrt[5]{41+29\sqrt{2}} + \sqrt[5]{41-29\sqrt{2}} = 2$.

    (b) $\sqrt[6]{26+15\sqrt{3}} + \sqrt[6]{26-15\sqrt{3}} = 6$.

27. *(1963 AMC 12, ♯40)* If x is a number satisfying the equation $\sqrt[3]{x+9} - \sqrt[3]{x-9} = 3$, then x^2 is between:

    (A) 55 and 65 (B) 65 and 75 (C) 75 and 85 (D) 85 and 95 (E) 95 and 105

28. *(1997 ARML, Individual Contest, ♯8)* If $\sqrt[3]{\sqrt[3]{2}-1}$ is written as $\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}$, where a, b, and c are rational numbers, compute the sum $a+b+c$.

29. *(2004 UK Senior Math Challenge, ♯25)* Positive integers x and y satisfy the equation

$$\sqrt{x+\frac{1}{2}\sqrt{y}} - \sqrt{x-\frac{1}{2}\sqrt{y}} = 1.$$

    Which of the following is a possible value of y?

    (A) 5 (B) 6 (C) 7 (D) 8 (E) 9

30. *(2018 Awesome Math Summer Program, Admission Test C, ♯3(a))* Solve in positive real numbers the equation:

$$x - \sqrt{2018+\sqrt{x}} = 4.$$

31. *(1983 AIME, ♯3)* What is the product of the real roots of the equation $x^2+18x+30 = 2\sqrt{x^2+18x+45}$?

32. *(1991 AIME, ♯7)* Find A^2, where A is the sum of the absolute values of all roots of the following equation:

$$x = \sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{x}}}}}.$$

33. *(1987 British Mathematical Olympiad, ♯1)* Find all real solutions x of the equation:

$$\sqrt{x+1972098-1986\sqrt{x+986049}} + \sqrt{x+1974085-1988\sqrt{x+986049}} = 1.$$

34. Solve the equation: $\sqrt[3]{x+1}+\sqrt[3]{4x+7} = \sqrt[3]{3x+5}+\sqrt[3]{2x+3}$.

35. Solve the equation: $\sqrt[5]{x+\sqrt{x^2+32}} + \sqrt[5]{x-\sqrt{x^2+32}} = 2$.