PROBLEM SET 3

1. *(2021 AMC 10 A, ♯20)* In how many ways can the sequence 1, 2, 3, 4, 5 be rearranged so that no three consecutive terms are increasing and no three consecutive terms are decreasing?

   (A) 10 (B) 18 (C) 24 (D) 32 (E) 44

2. *(2017 AMC 10 A, ♯19)* Alice refuses to sit next to either Bob or Carla. Derek refuses to sit next to Eric. How many ways are there for the five of them to sit in a row of 5 chairs under these conditions?

   (A) 12 (B) 16 (C) 28 (D) 32 (E) 40

3. *(2015 AMC 10 B, ♯16)* Al, Bill, and Cal will each randomly be assigned a whole number from 1 to 10, inclusive, with no two of them getting the same number. What is the probability that Al's number will be a whole number multiple of Bill's and Bill's number will be a whole number multiple of Cal's?

   (A) $\frac{9}{1000}$ (B) $\frac{1}{90}$ (C) $\frac{1}{80}$ (D) $\frac{1}{72}$ (E) $\frac{2}{121}$

4. *(1995 AMC 12, ♯29)* For how many three-element sets of positive integers $\{a, b, c\}$ is it true that $a \cdot b \cdot c = 2310$?

   (A) 32 (B) 36 (C) 40 (D) 43 (E) 45

5. *(1996 AMC 12, ♯26)* An urn contains marbles of four colors: red, white, blue, and green. When four marbles are drawn without replacement, the following events are equally likely:

   (a) the selection of four red marbles;
   (b) the selection of one white and three red marbles;
   (c) the selection of one white, one blue, and two red marbles; and
   (d) the selection of one marble of each color.

   What is the smallest number of marbles satisfying the given conditions?

   (A) 19 (B) 21 (C) 46 (D) 69 (E) more than 69

6. *(2010 AMC 12 A, ♯15)* A coin is altered so that the probability that it lands on heads is less than $1/2$ and when the coin is flipped four times, the probability of an equal number of heads and tails is $1/6$. What is the probability that the coin lands on heads?

   (A) $\frac{\sqrt{15}-3}{6}$ (B) $\frac{6-\sqrt{6\sqrt{6}+2}}{12}$ (C) $\frac{\sqrt{2}-1}{2}$ (D) $\frac{3-\sqrt{3}}{6}$ (E) $\frac{\sqrt{3}-1}{2}$

7. *(2017 AMC 12 B, ♯17)* A coin is biased in such a way that on each toss the probability of heads is $\frac{2}{3}$ and the probability of tails is $\frac{1}{3}$. The outcomes of the tosses are independent. A player has the choice of playing Game A or Game B. In Game A she tosses the coin three times and wins if all three outcomes are the same. In Game B she tosses the coin four times and wins if both the outcomes of the first and second tosses are the same and the outcomes of the third and fourth tosses are the same. How do the chances of winning Game A compare to the chances of winning game B?

   (A) the probability of winning Game A is $\frac{4}{81}$ less than the probability of winning Game B.
   (B) the probability of winning Game A is $\frac{2}{81}$ less than the probability of winning Game B.
   (C) the probabilities are the same
   (D) the probability of winning Game A is $\frac{2}{81}$ greater than the probability of winning Game B.
   (E) the probability of winning Game A is $\frac{4}{81}$ greater than the probability of winning Game B.

8. *(2008 AMC 10 A, ♯22)* Jacob uses the following procedure to write down a sequence of numbers. First he chooses the first term to be 6. To generate each succeeding term, he flips a fair coin. If it comes up heads, he doubles the previous term and subtracts 1. If it comes up tails, he takes half the previous term and subtracts 1. What is the probability that the fourth term in Jacob's sequence is an integer?

   (A) $\dfrac{1}{6}$ (B) $\dfrac{1}{3}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{5}{8}$ (E) $\dfrac{3}{4}$

9. *(2018 AMC 12 B, ♯22)* Consider polynomials $P(x)$ of degree at most 3, each of whose coefficients is an element of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many such polynomials satisfy $P(-1) = -9$?

   (A) 110 (B) 143 (C) 165 (D) 220 (E) 286

10. *(2021 AMC 10 A, ♯23)* Frieda the frog begins a sequence of hops on a 3×3 grid of squares, moving one square on each hop and choosing at random the direction of each hop - up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she "wraps around" and jumps to the opposite edge. For example, if Frieda begins in the center square and makes two hops "up", the first hop will place her in the top row middle square, and the second hop would cause Frieda to jump to the opposite edge, landing in the bottom row middle square. Suppose Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops?

    (A) $\dfrac{9}{16}$ (B) $\dfrac{5}{8}$ (C) $\dfrac{3}{4}$ (D) $\dfrac{25}{32}$ (E) $\dfrac{13}{16}$

11. *(2021 AMC 10 A, ♯21)* Let $ABCDEF$ be an equiangular hexagon. The lines AB, CD, and EF determine a triangle with area $192\sqrt{3}$, and the lines BC, DE, and FA determine a triangle with area $324\sqrt{3}$. The perimeter of the hexagon $ABCDEF$ can be expressed as $m + n\sqrt{p}$, where m, n, and p are positive integers and p is not divisible by the square of any prime. What is $m + n + p$?

    (A) 47 (B) 52 (C) 55 (D) 58 (E) 63

12. *(2013 AMC 10 B, ♯16)* In triangle ABC, medians AD and CE intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$?

(A) 13 (B) 13.5 (C) 14 (D) 14.5 (E) 15

13. *(2005 AMC 10 A, ♯25)* In $\triangle ABC$ we have $AB = 25$, $BC = 39$, and $AC = 42$. Points D and E are on $\overline{AB}$ and $\overline{AC}$, respectively, with $AD = 19$ and $AE = 14$. What is the ratio of the area of triangle ADE to the area of quadrilateral $BCED$?

(A) $\frac{266}{1521}$ (B) $\frac{19}{75}$ (C) $\frac{1}{3}$ (D) $\frac{19}{56}$ (E) 1

14. *(2019 AMC 10 B, ♯16)* In $\triangle ABC$ with a right angle at C, point D lies in the interior of $\overline{AB}$ and point E lies in the interior of $\overline{BC}$ so that $AC = CD$, $DE = EB$, and the ratio $AC : DE = 4 : 3$. What is the ratio $AD : DB$?

(A) $2 : 3$ (B) $2 : \sqrt{5}$ (C) $1 : 1$ (D) $3 : \sqrt{5}$ (E) $3 : 2$

15. *(2009 AMC 12 B, ♯16)* Trapezoid $ABCD$ has $\overline{AD} \parallel \overline{BC}$, $BD = 1$, $\angle DBA = 23^\circ$, and $\angle BDC = 46^\circ$. The ratio $BC : AD$ is $9 : 5$. What is CD?

(A) $\frac{7}{9}$ (B) $\frac{4}{5}$ (C) $\frac{13}{15}$ (D) $\frac{8}{9}$ (E) $\frac{14}{15}$

16. *(2013 AMC 12 A, ♯19)* In $\triangle ABC$, $AB = 86$ and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover, $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?

(A) 11 (B) 28 (C) 33 (D) 61 (E) 72

17. *(2017 AMC 10 B, ♯22)* The diameter AB of a circle of radius 2 is extended to a point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and line ED is perpendicular to line AD. Segment AE intersects the circle at a point C between A and E. What is the area of $\triangle ABC$?

(A) $\frac{120}{37}$ (B) $\frac{140}{39}$ (C) $\frac{145}{39}$ (D) $\frac{140}{37}$ (E) $\frac{120}{31}$

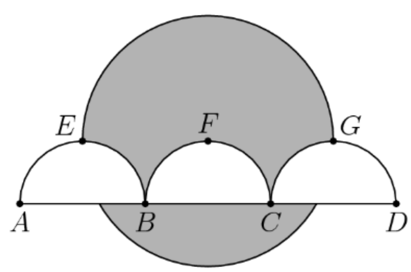
18. *(2007 AMC 10 B, ♯23)* A pyramid with a square base is cut by a plane that is parallel to its base and 2 units from the base. The surface area of the smaller pyramid that is cut from the top is half the surface area of the original pyramid. What is the altitude of the original pyramid?

(A) 2 (B) $2 + \sqrt{2}$ (C) $1 + 2\sqrt{2}$ (D) 4 (E) $4 + 2\sqrt{2}$

19. *(2021 AMC 10 A, ♯24)* The interior of a quadrilateral is bounded by the graphs of $(x+ay)^2 = 4a^2$ and $(ax-y)^2 = a^2$, where a is a positive real number. What is the area of this region in terms of a, valid for all $a > 0$?

(A) $\dfrac{8a^2}{(a+1)^2}$ (B) $\dfrac{4a}{a+1}$ (C) $\dfrac{8a}{a+1}$ (D) $\dfrac{8a^2}{a^2+1}$ (E) $\dfrac{8a}{a^2+1}$

20. *(2019 AMC 10 B, ♯20)* As shown in the figure, line segment $\overline{AD}$ is trisected by points B and C so that $AB = BC = CD = 2$. Three semicircles of radius 1, AEB, BFC, and CGD, have their diameters on $\overline{AD}$, and are tangent to line EG at E, F, and G, respectively. A circle of radius 2 has its center at F. The area of the region inside the circle but outside the three semicircles, shaded in the figure, can be expressed in the form $\frac{a}{b} \cdot \pi - \sqrt{c} + d$, where a, b, c, and d are positive integers and a and b are relatively prime. What is $a+b+c+d$?



(A) 13 (B) 14 (C) 15 (D) 16 (E) 17