HOMEWORK 22

Please do the following textbook exercises:

- *Geometry for Challenge and Enjoyment*
  §16.3: page 723: ♯ 1, 2, 3, 4, 5
  Review Problems: page 433: ♯ 32
  Review Problems: page 738: ♯ 1, 4, 7

- *AOPS Introduction to Geometry*
  §7.3: page 182: ♯ 7.3.5, 7.3.6
  §7.4: page 187: ♯ 7.4.2, 7.4.4
  §7.5: page 192: ♯ 7.5.3, 7.5.4, 7.5.5
  Review Problems: page 168: ♯ 6.55
  Review Problems: page 202: ♯ 7.38, 7.39

1. Let ABC be a triangle with $BC = a$, $AC = b$, and $AB = c$. Let m_a, m_b, and m_c be the lengths of the medians from A, B, and C of triangle ABC. Prove that:
   (a) $\dfrac{b+c-a}{2} < m_a < \dfrac{b+c}{2}$.
   (b) Prove that in any triangle the sum of the medians is greater than $\dfrac{3}{4}$ of the perimeter but less than the perimeter.
   (c) Prove that $m_a^2 + m_b^2 + m_c^2 = \dfrac{3(a^2+b^2+c^2)}{4}$.
   (d) Prove that $m_a = \dfrac{a}{2}$ if and only if triangle ABC is right with $\angle A = 90°$.
   (e) Prove that $a = b$ if and only if $m_a = m_b$.
   (f) Prove that if $a > b$, then $m_a < m_b$.
   (g) Prove that $a^2 + b^2 > \dfrac{1}{2}c^2$, and deduce that $m_a^2 + m_b^2 > \dfrac{9}{8}c^2$.
   (h) Let $x = ab + bc + ca$, and let $x_1 = m_a m_b + m_b m_c + m_c m_a$. Prove that: $\dfrac{9}{20} < \dfrac{x_1}{x} < \dfrac{5}{4}$.

2. *(2007 Archimedes Contest, Romania, Grade 7, Final Round, ♯3(b))* In triangle ABC, D and E are the midpoints of the sides $\overline{BC}$ and $\overline{AC}$, respectively. Prove that if $AD^2 + 2BE^2 = 3AB^2$, then $BC = AB\sqrt{2}$.

3. In $\triangle ABC$, $a = 1$ and $m_a = 3$. Find the perimeter of the triangle and determine if the triangle is equilateral, isosceles, or scalene.

4. *(1990 Chinese National Middle School Math League, Preliminary Form, ♯II.4)* $\triangle ABC$ is isosceles with $AB = AC = 2$. Let $P_1, P_2, \ldots, P_{100}$ be 100 points on the side BC. Let $m_i = AP_i^2 + BP_i \cdot P_iC$, for $i = 1, 2, \ldots, 100$. Find the value of $m_1 + m_2 + \cdots + m_{100}$.

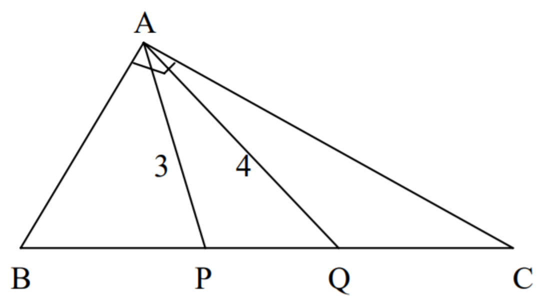
5. *(2015 USC High School Math Contest, ♯23)* In triangle ABC, $AB = 7$, $BC = 5$, and $AC = 6$. Locate points P_1, P_2, P_3, and P_4 on $\overline{BC}$ so that the side is divided into 5 equal segments, each of length 1. Let $q_k = AP_k$ for $k \in \{1, 2, 3, 4\}$. What is $q_1^2 + q_2^2 + q_3^2 + q_4^2$?

   (A) 142 (B) 150 (C) 155 (D) 160 (E) 168

6. *(1980 AMC 12, ♯23)* Line segments drawn from the vertex opposite the hypotenuse of a right triangle to the points trisecting the hypotenuse have lengths $\sin x$ and $\cos x$, where x is a real number such that $0 < x < \dfrac{\pi}{2}$. The length of the hypotenuse is:

   (A) $\dfrac{4}{3}$ (B) $\dfrac{3}{2}$ (C) $\dfrac{3\sqrt{5}}{5}$ (D) $\dfrac{2\sqrt{5}}{3}$ (E) not uniquely determined by the given information

7. *(1999 Canadian Open Math Challenge, ♯B3)* Triangle ABC is right angled with its right angle at A. The points P and Q are on the hypotenuse BC such that $BP = PQ = QC$, $AP = 3$, and $AQ = 4$. Determine the length of each side of $\triangle ABC$.



8. *(2005 AMC 10 B, ♯10)* In $\triangle ABC$, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is BD?

   (A) 3 (B) $2\sqrt{3}$ (C) 4 (D) 5 (E) $4\sqrt{2}$

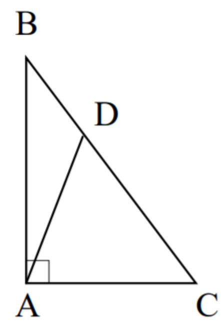
9. *(1977 Alberta High School Math Competition, Canada, ♯17)* The lengths of the medians of a right triangle which are drawn from the vertices of the acute angles are $\sqrt{73}$ and $2\sqrt{13}$. The length of the third median is:

   (A) $5\sqrt{5}$ (B) $\sqrt{73} + 2\sqrt{13}$ (C) 5 (D) 10 (E) none of these

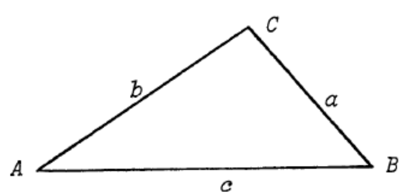
10. *(1983 AMC 12, ♯19)* Point D is on side CB of triangle ABC. If $\angle CAD = \angle BAD = 60°$, $AC = 3$, and $AB = 6$, then the length of AD is:

    (A) 2 (B) 2.5 (C) 3 (D) 3.5 (E) 4

11. *(1998 Canadian Open Math Challenge, ♯A6)* The lengths of the sides of $\triangle ABC$ are 60, 80, and 100, with $\angle A = 90°$. The line AD divides triangle ABC into two triangles of equal perimeter. Calculate the length of AD.



12. *(1985 AMC 12, ♯28)* In $\triangle ABC$, we have $\angle C = 3\angle A$, $a = 27$, and $c = 48$. What is b?



    (A) 33 (B) 35 (C) 37 (D) 39 (E) not uniquely determined

13. *(Posamentier and Salkind, Problem 10-8, page 47)* In a right triangle, the bisector of the right angle divides the hypotenuse into segments that measure 3 and 4. Find the measure of the angle bisector of the larger acute angle of the right triangle.

14. *(2008 Canadian Open Math Challenge, ♯B4)* A triangle is called automedian if its three medians can be used to form a triangle that is similar to the original triangle.

    (a) Show that the triangle with sides of length 7, 13, and 17 is automedian.

    (b) $\triangle ABC$ has side lengths $AB = c$, $AC = b$, and $BC = a$, with $a < b < c$. If $\triangle ABC$ is automedian, prove that $a^2 + c^2 = 2b^2$.

    (c) Determine, with proof, an infinite family of automedian triangles with integer side lengths, such that no two of the triangles in the family are similar.

15. *(1995 China Math Competitions for Secondary Schools)* In $\triangle ABC$, $\angle A = 90^\circ$, $\overline{AB} \cong \overline{AC}$, and D is a point on $\overline{BC}$. Prove that
$$BD^2 + CD^2 = 2AD^2.$$