THE AREA OF A QUADRILATERAL

1. Let $ABCD$ be a quadrilateral, and let ϕ be the angle between the two diagonals $\overline{AC}$ and $\overline{BD}$. Prove that the area S of the quadrilateral is given by:

$$S = \frac{1}{2}(AC)(BD)\sin\phi.$$

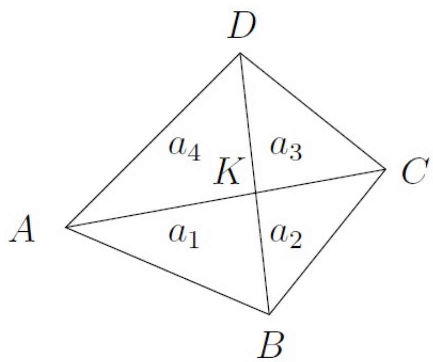
2. *(1993 ARML, Team Questions, ♯2)* The sides of a rectangle are 1 and 3, and its diagonals intersect forming an angle of $\theta°$. Compute $\sin\theta°$.

3. Let $ABCD$ be a quadrilateral with diagonals $\overline{AC}$ and $\overline{BD}$ and area S. Show that

$$S \leq \frac{1}{2}(AC)(BD),$$

and conclude that equality holds if and only if $ABCD$ has perpendicular diagonals (i.e. $ABCD$ is orthodiagonal).

4. *(2015 UNC Charlotte, Super Competition, Comprehensive Test, ♯14)* The diagonals of the convex four-gon $ABCD$ divide the four-gon into four triangles, whose areas are a_1, a_2, a_3, and a_4, respectively, as shown in the figure.



Which of the following identities must hold for these areas?

(A) $a_1 + a_2 = a_3 + a_4$ (B) $a_1 + a_3 = a_2 + a_4$ (C) $a_1 \cdot a_2 = a_3 \cdot a_4$ (D) $a_1 \cdot a_3 = a_2 \cdot a_4$
(E) $a_1 - a_3 = a_2 - a_4$

5. *(2002 AMC 12 B, ♯24)* A convex quadrilateral $ABCD$ with area 2002 contains a point P in its interior such that $PA = 24$, $PB = 32$, $PC = 28$, and $PD = 45$. Find the perimeter of $ABCD$.

   (A) $4\sqrt{2002}$ (B) $2\sqrt{8465}$ (C) $2(48+\sqrt{2002})$ (D) $2\sqrt{8633}$ (E) $4(36+\sqrt{113})$

6. *(1977 AMC 12, ♯26)* Let a, b, c, and d be the lengths of sides MN, NP, PQ, and QM, respectively, of quadrilateral $MNPQ$. If A is the area of $MNPQ$, then:

   (A) $A = \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is convex
   (B) $A = \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is a rectangle
   (C) $A \leq \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is a rectangle
   (D) $A \leq \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is a parallelogram
   (E) $A \geq \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is a parallelogram

7. *(BRETSCHNEIDER'S FORMULA, 1842)* Let $ABCD$ be a quadrilateral with sides of length $AB = a$, $BC = b$, $CD = c$, and $DA = d$. Let $p = (a+b+c+d)/2$ be its semiperimeter, let S be its area, and let ϕ be the average of a pair of opposite angles of $ABCD$. Then:

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d) - abcd\cos^2\phi}.$$

8. Find the area of a quadrilateral $ABCD$ all of whose sides have length 6 and such that the sum of a pair of opposite angles of $ABCD$ is $120°$.

9. *(HOBSON'S FORMULA, 1918)* Let $ABCD$ be a quadrilateral with sides of length $AB = a$, $BC = b$, $CD = c$, $DA = d$, and diagonals of length $AC = e$ and $BD = f$. Then the area S of $ABCD$ is given by:

$$S = \frac{1}{4}\sqrt{4e^2f^2 - (a^2+c^2-b^2-d^2)^2}.$$

10. Prove that if $ABCD$ is a quadrilateral with sides of length $AB = a$, $BC = b$, $CD = c$, $DA = d$, area S, diagonals intersecting at P, and $\theta = \angle BPC$, then:

    - $S = \frac{1}{4}(a^2+c^2-b^2-d^2)\tan\theta$
    - If $ABCD$ is circumscribed, then $S = \frac{1}{2}(bd-ac)\tan\theta$.

11. Prove that if $ABCD$ is a circumscribed quadrilateral with sides of length $AB = a$, $BC = b$, $CD = c$, $DA = d$, and diagonals of length $AC = e$ and $BD = f$. Then the area S of $ABCD$ is given by:

$$S = \frac{1}{2}\sqrt{e^2 f^2 - (ac - bd)^2}.$$

12. *(COOLIDGE'S FORMULA, 1939)* Let $ABCD$ be a quadrilateral with sides of length $AB = a$, $BC = b$, $CD = c$, $DA = d$, and diagonals of length $AC = e$ and $BD = f$. Let $p = \dfrac{a+b+c+d}{2}$ be its semiperimeter and let S be its area. Then

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d) - \frac{1}{4}(ac+bd+ef)(ac+bd-ef)}.$$

13. *(BRAHMAGUPTA'S THEOREM, circa 728 A.D.)* Let $ABCD$ be a quadrilateral with sides of length $AB = a$, $BC = b$, $CD = c$, $DA = d$, area S and semiperimeter p. Then $ABCD$ is cyclic if and only if

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d)}.$$

14. *(HERON-ARCHIMEDES FORMULA, circa 60 A.D.)* Let ABC be a triangle with sides of length $AB = a$, $BC = b$, and $CA = c$, area S, and semiperimeter $p = \dfrac{a+b+c}{2}$. Use Brahmagupta's Formula to prove the Heron-Archimedes Formula:

$$S = \sqrt{p(p-a)(p-b)(p-c)}.$$

15. Prove that if $ABCD$ is a cyclic and circumscribed quadrilateral with sides of length $AB = a$, $BC = b$, $CD = c$, $DA = d$ and area S, then

$$S = \sqrt{abcd}.$$

16. Prove that if a quadrilateral $ABCD$ has side lengths $AB = a$, $BC = b$, $CD = c$, $DA = d$, then the area S of $ABCD$ is maximum if and only if $ABCD$ is cyclic.

17. *(2000 Turkish National Math Olympiad, Round 1, ♯17)* What is the largest possible area of a quadrilateral with sides 1, 4, 7, and 8?

    (A) $7\sqrt{2}$ (B) $10\sqrt{3}$ (C) 18 (D) $12\sqrt{3}$ (E) $9\sqrt{5}$

18. Prove that if a quadrilateral $ABCD$ has side lengths $AB = a$, $BC = b$, $CD = c$, $DA = d$, and area S, then

$$S \leq \frac{1}{4}(a+b)(c+d).$$

19. Let $ABCD$ be a cyclic quadrilateral with area S and semiperimeter p. If $S = \left(\frac{p}{2}\right)^2$, prove that $ABCD$ is a square.

20. A quadrilateral $ABCD$ has sides of length $AB = 6$, $BC = 7$, $CD = 8$, $DA = 11$, and area 60. Prove that $ABCD$ is cyclic.

21. *(2007 Princeton University Math Competition, Geometry A, ♯2)* Let $ABCD$ be a convex cyclic quadrilateral. If $AB = 13$, $BC = 13$, $CD = 37$, and $AD = 47$, what is the area of $ABCD$?

22. *(2011 AMC 12 A, ♯24)* Consider all quadrilaterals $ABCD$ such that $AB = 14$, $BC = 9$, $CD = 7$, and $DA = 12$. What is the radius of the largest possible circle that fits inside or on the boundary of such a quadrilateral?

    (A) $\sqrt{15}$ (B) $\sqrt{21}$ (C) $2\sqrt{6}$ (D) 5 (E) $2\sqrt{7}$

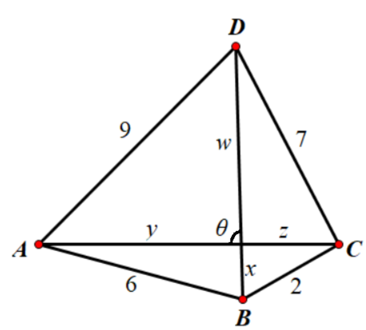
23. *(2012 Princeton University Math Competition, Geometry A, ♯8)* Cyclic quadrilateral $ABCD$ has side lengths $AB = 2$, $BC = 3$, $CD = 5$, and $AD = 4$. Then $\sin A \sin B \left(\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} + \cot \frac{D}{2}\right)^2$ can be written as $\frac{a}{b}$, where a and b relatively prime positive integers. Find $a + b$.

24. *(1973 Romanian Math Olympiad)* In convex quadrilateral $ABCD$, $[ABC] \le [BCD] \le [CDA] \le [ABD]$. Prove that $ABCD$ is a trapezoid.

25. *(2007 Mandelbrot Competition, Individual, Round 2, ♯7)* Let convex quadrilateral $ABCD$ have side lengths $AB = 12$, $BC = 6$, and $CD = 20$. Suppose that $ABCD$ has an inscribed circle which is tangent to side $\overline{BC}$ at its midpoint. What is the area of quadrilateral $ABCD$?

26. *(2008 Mandelbrot Competition, Team Play, Round 3, ♯1)* Let $ABCD$ be a quadrilateral with sides of length $AB = 6$, $BC = 2$, $CD = 7$, and $DA = 9$. Suppose that the diagonals intersect in an angle θ, and that the diagonals split one another into segments of length w, x, y, and z, as shown in the figure below. Prove that:

$$w^2 + x^2 + y^2 + z^2 = 85 + (2wy + 2xz)\cos\theta.$$



27. *(2008 Mandelbrot Competition, Team Play, Round 3, ♯2)* Let $ABCD$ be a quadrilateral with sides of length $AB = 6$, $BC = 2$, $CD = 7$, and $DA = 9$. Prove that $ABCD$ is orthodiagonal, i.e. the diagonals of $ABCD$ are perpendicular.

28. *(2008 Mandelbrot Competition, Team Play, Round 3, ♯6)* Let P be a point in the plane. Find the largest possible area of a rectangle $ABCD$ such that $PA = 9$, $PB = 7$, $PC = 2$, and $PD = 6$.

29. *(1976 IMO, ♯1)* In a plane convex quadrilateral of area 32, the sum of the lengths of two opposite sides and one diagonal is 16. Determine all possible lengths of the other diagonal.

30. *(1985 Austrian-Polish Math Olympiad, ♯3)* Let $ABCD$ be a quadrilateral with area 1. Prove that the sum of the lengths of its sides and diagonals is not less than $4 + \sqrt{8}$.