CSEC Add Maths

Paper 2

June 2021

Solutions

SECTION I

Answer all questions.

ALL working must be clearly shown.

1. (a) (i) Determine the remainder when $f(x) = ax^3 + 7x^2 - 7x - 3$ is divided by $x - 1$. [2]

Consider,

$x - 1 = 0$

$x = 1$

When $f(x)$ is divided by $(x - 1)$, then by the Remainder Theorem, $f(1)$ is the remainder.

$f(x) = ax^3 + 7x^2 - 7x - 3$

$f(1) = a(1)^3 + 7(1)^2 - 7(1) - 3$

$= a + 7 - 7 - 3$

$= a - 3$

$\therefore$ The remainder is $f(1) = a - 3$.

(ii) If the remainder when $f(x)$ is divided by $(x+3)$ is equal to the remainder determined in (a)(i), find the value of a. [3]

Consider,

$x + 3 = 0$

$x = -3$

When $f(x)$ is divided by $(x+3)$, then by the Remainder Theorem, $f(-3)$ is the remainder.

$f(x) = ax^3 + 7x^2 - 7x - 3$

$f(-3) = a(-3)^3 + 7(-3)^2 - 7(-3) - 3$

$= a(-27) + 7(9) - 7(-3) - 3$

$= -27a + 63 + 21 - 3$

$= -27a + 81$

Now, this remainder is equal to the remainder found in part (a)(i).

$-27a + 81 = a - 3$

$-27a - a = -81 - 3$

$-28a = -84$

$a = \frac{-84}{-28}$

$a = 3$

$\therefore$ The value of $a = 3$.

(b) Consider the function $g(x) = x^2 + (m+4)x + 4m = 0$, which has real and equal roots. Use the discriminant of the given equation to determine the value for m. You may use the grid provided to assist you. [5]

$x^2 + (m+4)x + 4m = 0$ which is in the form $ax^2 + bx + c = 0$, where $a = 1, b = m + 4$ and $c = 4m$.

Since the function has real and equal roots, then

$$b^2 - 4ac = 0$$

$$(m+4)^2 - 4(1)(4m) = 0$$

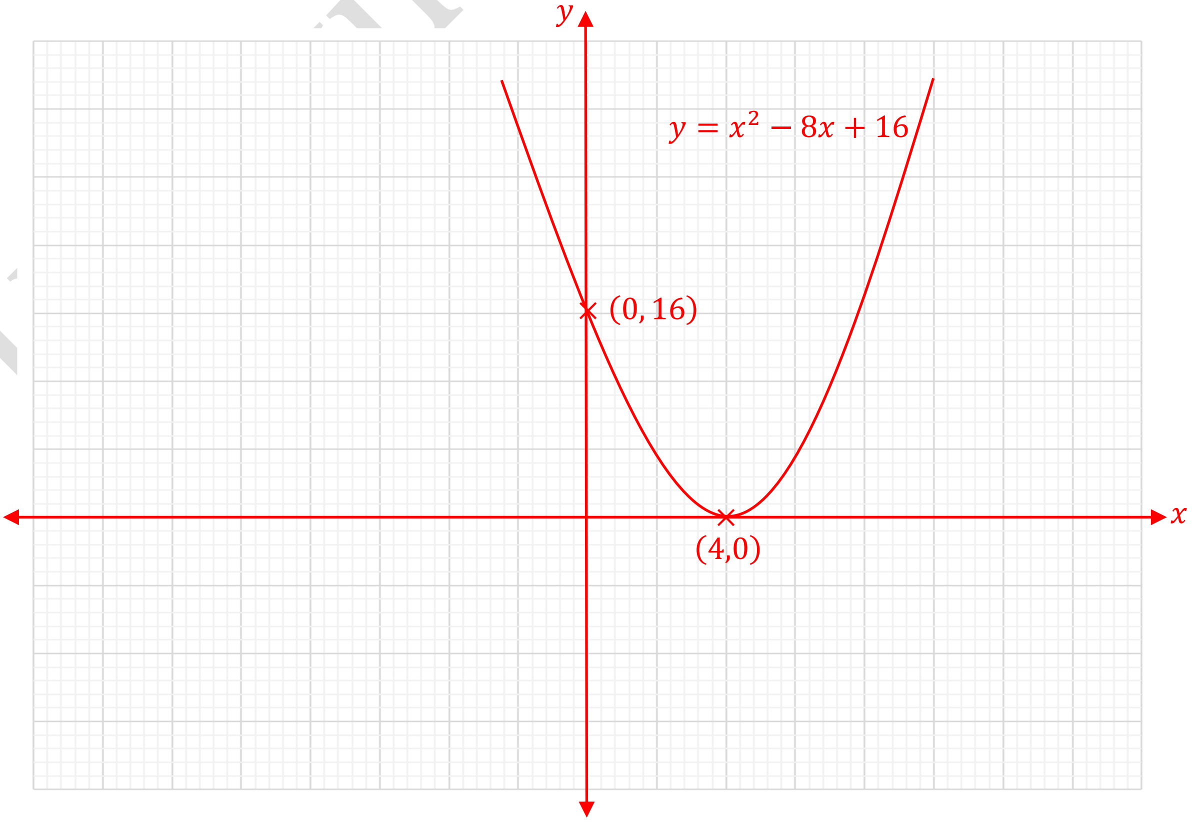
$$m^2 + 8m + 16 - 16m = 0$$

$$m^2 - 8m + 16 = 0$$

$$(m-4)^2 = 0$$

$$m - 4 = 0$$

$$m = 4$$



(c) Let $h(x) = 2x^2 + 8x - 10$.

(i) Express $h(x)$ in the form $a(x+b)^2 + c$. [3]

$2x^2 + 8x - 10$

$= 2(x^2 + 4x) - 10$

$= 2(x^2 + 4x + 4) - 10 - 2(4)$

$= 2(x+2)^2 - 10 - 8$

$= 2(x+2)^2 - 18$ which is in the form $a(x+b)^2 + c$,

where $a = 2, b = -2$ and $c = -18$.

(ii) State the minimum value of $h(x)$. [1]

$h(x) = 2(x+2)^2 - 18$

The minimum value of $h(x)$ is -18.

(iii) Determine the value of x for which $h(x)$ is a minimum. [1]

$h(x) = 2(x+2)^2 - 18$

Consider,

$x + 2 = 0$

$x = -2$

$\therefore$ The value of x for which $h(x)$ is a minimum is $x = -2$.

Total: 15 marks

2. (a) Given that $\log_2(6+\sqrt{12}) - \log_2(3+\sqrt{a}) = \log 10$, find the value of a. [5]

$$\log_2(6+\sqrt{12}) - \log_2(3+\sqrt{a}) = \log 10$$

$$\log_2 \frac{(6+\sqrt{12})}{(3+\sqrt{a})} = 1$$

Converting to exponential form gives:

$$\frac{(6+\sqrt{12})}{(3+\sqrt{a})} = 2^1$$

$$\frac{6+\sqrt{12}}{3+\sqrt{a}} = 2$$

$$6+\sqrt{12} = 2(3+\sqrt{a})$$

$$6+\sqrt{12} = 6+2\sqrt{a}$$

$$\sqrt{12} = 2\sqrt{a}$$

Squaring both sides gives:

$$(\sqrt{12})^2 = (2\sqrt{a})^2$$

$$12 = 4a$$

$$\frac{12}{4} = a$$

$$3 = a$$

$\therefore$ The value of $a = 3$.

(b) Determine the set of values of x for which $\frac{2-x}{4x-9} < 0$. [4]

$\frac{2-x}{4x-9} < 0$

$\times\ (4x-9)^2$

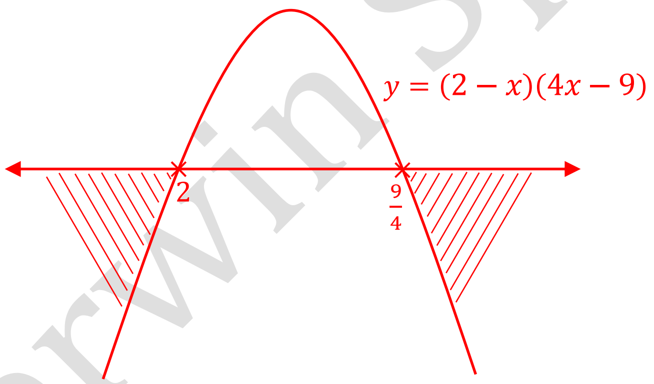
$(2-x)(4x-9) < 0$

The critical values are:

$2 - x = 0$ and $4x - 9 = 0$

$x = 2$ $x = \frac{9}{4}$

Consider the sketch below:



Since we are considering $(2-x)(4x-9) < 0$, then we look where the graph is negative which is below the x-axis.

$\therefore$ The solution set is $\left\{x: x < 2 \cup x > \frac{9}{4}\right\}$.

(c) Alice deposited \$4000 into her new savings account at Bank of Fortune, which pays interest at 8% per annum. The bank's compounded interest is represented by the geometric progression

$A = P\left(1 + \frac{R}{100}\right)^T$ where A is the amount of money accumulated after T years, R, the percentage rate of interest per annum and T, a positive integer, the time in years.

Determine the number of years it would take Alice's money to **at least triple**. [6]

$P = \$4000$

$R = 8\%$

$A = 3 \times \$4000$

$= \$12\,000$

Now,

$$A = P\left(1 + \frac{R}{100}\right)^T$$

$$12\,000 = 4000\left(1 + \frac{8}{100}\right)^T$$

$$\frac{12000}{4000} = \left(1 + \frac{8}{100}\right)^T$$

$$3 = (1.08)^T$$

$$(1.08)^T = 3$$

Taking logs on both sides gives:

$\log(1.08)^T = \log 3$

$T \log 1.08 = \log 3$

$$T = \frac{\log 3}{\log 1.08}$$

$T = 14.3$ years (to 3 significant figures)

Since T is a positive integer, then $T = 15$ years.

$\therefore$ The number of years it would take Alice's money to at least triple is 15 years.

Total: 15 marks

SECTION II

Answer ALL questions.

ALL working must be clearly shown.

3. (a) The coordinates for the centre of a circle is $(2, 1)$ and the coordinates for a point on its circumference is $(3, 3)$.

(i) Determine the equation of the circle in the form

$x^2 + y^2 + ax + by + c = 0.$ [4]

Points are $(2, 1)$ and $(3, 3)$.

Length of radius, $r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(3-2)^2 + (3-1)^2}$

$= \sqrt{1^2 + 2^2}$

$r = \sqrt{5}$ units

The standard form of the equation of a circle is:

$(x - a)^2 + (y - b)^2 = r^2$

where $C(a, b)$ and radius $= r$.

The centre is $C(2, 1)$ and the radius, $r = \sqrt{5}$.

$(x - 2)^2 + (y - 1)^2 = \left(\sqrt{5}\right)^2$

$(x - 2)^2 + (y - 1)^2 = 5$

Expanding gives:

$$x^2 - 4x + 4 + y^2 - 2y + 1 = 5$$

$$x^2 + y^2 - 4x - 2y + 4 + 1 - 5 = 0$$

$$x^2 + y^2 - 4x - 2y = 0$$

which is in the form $x^2 + y^2 + ax + by + c = 0$,

where $a = -4, b = -2$ and $c = 0$.

(ii) The circle intersects the x and y-axes at three points. Determine the coordinates of the three points of intersection. [4]

The equation of the circle is $(x - 2)^2 + (y - 1)^2 = 5$.

When $x = 0$,

$$(0 - 2)^2 + (y - 1)^2 = 5$$

$$(2)^2 + (y - 1)^2 = 5$$

$$4 + (y - 1)^2 = 5$$

$$(y - 1)^2 = 5 - 4$$

$$(y - 1)^2 = 1$$

$$y^2 - 2y + 1 = 1$$

$$y^2 - 2y = 0$$

$$y(y - 2) = 0$$

Either $y = 0$ or $y - 2 = 0$

$y = 2$

The points are $(0, 0)$ and $(0, 2)$.

When $y = 0$,

$$(x-2)^2 + (0-1)^2 = 5$$

$$(x-2)^2 + (-1)^2 = 5$$

$$(x-2)^2 + 1 = 5$$

$$(x-2)^2 = 5 - 1$$

$$(x-2)^2 = 4$$

$$x^2 - 4x + 4 = 4$$

$$x^2 - 4x = 0$$

$$x(x-4) = 0$$

Either $x = 0$ or $x - 4 = 0$

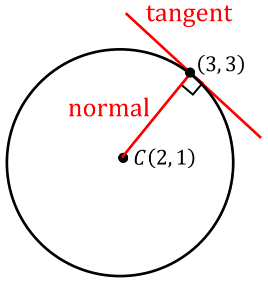
$x = 4$

The points are $(0, 0)$ and $(4, 0)$.

$\therefore$ The coordinates of the three points of intersection are $(0,0)$, $(0,2)$ and $(4,0)$.

(iii) Determine the equation of the tangent to the circle at the point (3,3). [4]

Consider the sketch below.



Points are (2, 1) and (3, 3).

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{3-1}{3-2}$

$= 2$

Gradient of normal $= 2$

Gradient of tangent $= -\frac{1}{2}$

Substituting $m = -\frac{1}{2}$ and point $(3, 3)$ into $y - y_1 = m(x - x_1)$ gives:

$y - 3 = -\frac{1}{2}(x - 3)$

$y - 3 = -\frac{1}{2}x + \frac{3}{2}$

$y = -\frac{1}{2}x + \frac{3}{2} + 3$

$y = -\frac{1}{2}x + \frac{9}{2}$ → Equation of tangent

(b) The position vectors of two points, P and Q, relative to a fixed origin, O, are given by $OP = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $OQ = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$. Determine the unit vector in the direction of $\boldsymbol{PQ}$, giving your answer in simplest surd form. [4]

$\overrightarrow{OP} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

Using the triangle law,

$\overrightarrow{PQ} = OQ - OP$

$= \begin{pmatrix} -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} -4 - 2 \\ 1 - (-3) \end{pmatrix}$

$= \begin{pmatrix} -6 \\ 4 \end{pmatrix}$

The magnitude of $\overrightarrow{PQ}$ is,

$|\overrightarrow{PQ}| = \sqrt{(-6)^2 + (4)^2}$

$= \sqrt{36 + 16}$

$= \sqrt{52}$

$= \sqrt{4 \times 13}$

$= \sqrt{4}\sqrt{13}$

$= 2\sqrt{13}$

Now,

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The unit vector is the direction of $\overrightarrow{PQ} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}$

$$= \frac{1}{2\sqrt{13}}\begin{pmatrix}-6\\4\end{pmatrix}$$

$$= \frac{1}{\sqrt{13}}\begin{pmatrix}\frac{-6}{2}\\\frac{4}{2}\end{pmatrix}$$

$$= \frac{1}{\sqrt{13}}\begin{pmatrix}-3\\2\end{pmatrix}$$

$$= \frac{1}{\sqrt{13}}(-3\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})$$

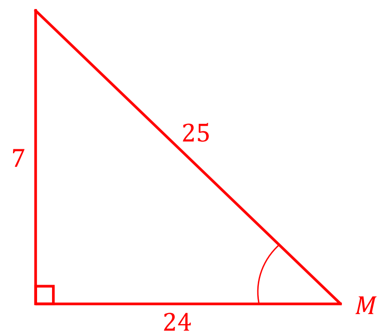
(c) Given that $\cos M = \frac{24}{25}$ and that angle M is acute, determine the value for $\tan 2M$. [4]

$\tan 2M = \frac{2\tan M}{1-\tan^2 M}$

We are given that $\cos M = \frac{24}{25}$

$\cos M = \frac{adj}{hyp}$

Consider the sketch below:



Using Pythagoras' Theorem,

$$a^2 + b^2 = c^2$$

$$a^2 + (24)^2 = (25)^2$$

$$a^2 + 576 = 625$$

$$a^2 = 625 - 576$$

$$a^2 = 49$$

$$a = \sqrt{49}$$

$$a = 7$$

So, we have, $\tan M = \frac{opp}{adj}$

$$= \frac{7}{24}$$

Now,

$$\tan 2M = \frac{2\tan M}{1 - \tan^2 M}$$

$$= \frac{2\left(\frac{7}{24}\right)}{1 - \left(\frac{7}{24}\right)^2}$$

$$= \frac{\frac{7}{12}}{1 - \frac{49}{576}}$$

$$= \frac{\frac{7}{12}}{\frac{527}{576}}$$

$$= \frac{7}{12} \div \frac{527}{576}$$

$$= \frac{7}{12} \times \frac{576}{527}$$

$$= \frac{336}{527}$$

Total: 20 marks

SECTION III

Answer ALL questions.

ALL working must be clearly shown.

4. (a) (i) Differentiate $\sin x + \cos 4x$ with respect to x. [2]

$$\frac{d}{dx}(\sin x + \cos 4x)\ dx = \cos x + (-4\sin 4x)$$

$$= \cos x - 4\sin 4x$$

(ii) Differentiate $\frac{2x^3+2}{2x+1}$ with respect to x. [3]

Let $y = \frac{2x^3+2}{2x+1}$.

Let $u = 2x^3 + 2$ and $v = 2x + 1$

$\frac{du}{dx} = 6x^2$ $\frac{dv}{dx} = 2$

Using the quotient rule,

$$\frac{d^2y}{dx^2} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(2x+1)(6x^2)-(2x^3+2)(2)}{(2x+1)^2}$$

$$= \frac{12x^3+6x^2-(4x^3+4)}{(2x+1)^2}$$

$$= \frac{12x^3+6x^2-4x^3-4}{(2x+1)^2}$$

$$= \frac{8x^3+6x^2-4}{(2x+1)^2}$$

(b) Use the principles of differentiation to compute the stationary value of the function $y = x^2 - 4x + 2$. [4]

$y = x^2 - 4x + 2$

$\frac{dy}{dx} = 2x - 4$

At stationary values, $\frac{dy}{dx} = 0$.

When $\frac{dy}{dx} = 0$,

$2x - 4 = 0$

$2x = 4$

$x = \frac{4}{2}$

$x = 2$

When $x = 2$,

$y = (2)^2 - 4(2) + 2$

$= 4 - 8 + 2$

$= -2$

$\therefore$ The stationary value of the function is -2.

(c) A motorist starts from a point X and travels 60 m due north to a point Y at a constant speed of 4 ms^{-1}. He stays at Y for 25 seconds and then travels at a constant speed of 10 ms^{-1} for 100 m due south to a point, Z. Calculate

(i) the average speed for the whole journey. [4]

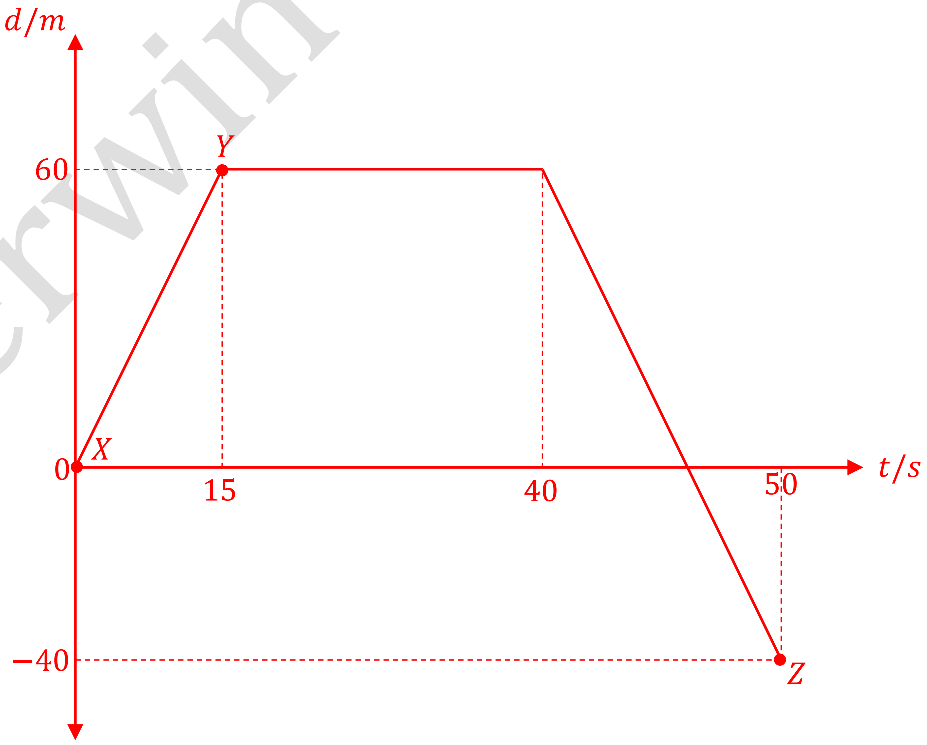
He travelled 60 m at a constant speed of 4 ms^{-1}.

$$\text{Time} = \frac{Distance}{Speed}$$

$$= \frac{60}{4}$$

$$= 15\ s$$

Consider the sketch below:



He travelled 100 m at a constant speed of 10 ms^{-1}.

$$\text{Time} = \frac{Distance}{Speed}$$

$$= \frac{100}{10}$$

$$= 10\ s$$

$$\text{Total distance travelled} = 60 + 100$$

$$= 160\ m$$

$$\text{Total time taken} = 15 + 25 + 10$$

$$= 50\ s$$

Hence,

$$\text{Average speed} = \frac{Total\ distance}{Total\ time}$$

$$= \frac{160}{50}$$

$$= 3.2\ ms^{-1}$$

∴ The average speed for the whole journey is 3.2 ms^{-1}.

(ii) the average velocity of the whole journey. [2]

$$\text{Average velocity} = \frac{Displacement}{Time}$$

$$= \frac{-40}{50}$$

$$= -0.8\ ms^{-1}$$

∴ The average velocity of the whole journey is $-0.8\ ms^{-1}$.

Total: 15 marks

5. (a) Determine the following integrals, giving each answer in its simplest form.

(i) $\int 2x^2 + 3x \ dx$ [2]

$$\int 2x^2 + 3x \ dx = \frac{2x^3}{3} + \frac{3x^2}{2} + c$$

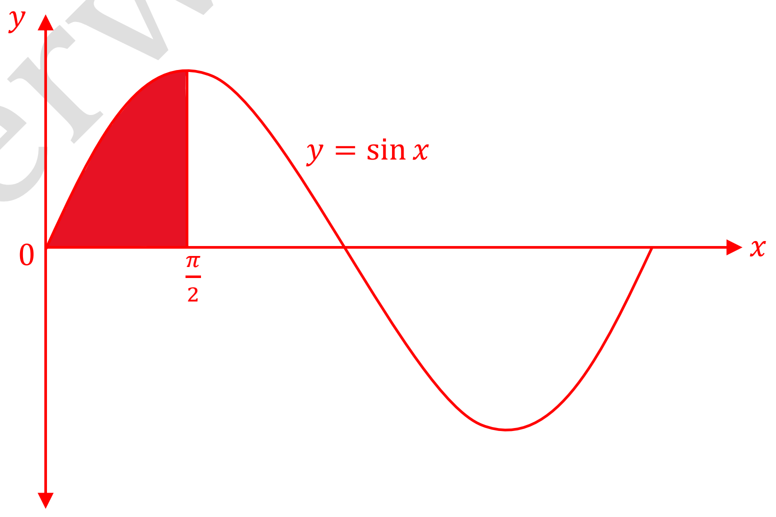
(ii) $\int 2\sin 3x \ dx$ [3]

$$\int 2\sin 3x \ dx = 2\left(-\frac{1}{3}\cos 3x\right) + c$$

$$= -\frac{2}{3}\cos 3x + c$$

(b) Using an integration method, calculate the area of the region in the first quadrant under the graph $y = 3\sin x$. [4]

Consider the sketch below:



Area of the region $= \int_a^b y\, dx$

$= \int_0^{\frac{\pi}{2}} (3 \sin x)\, dx$

$= [-3 \cos x]_0^{\frac{\pi}{2}}$

$= -3 \cos \frac{\pi}{2} - (-3 \cos 0)$

$= -3(0) + 3(1)$

$= 0 + 3$

$= 3$ square units

$\therefore$ The area of the region in the first quadrant is 3 units^2.

(c) A particle starting from rest travels in a straight line with an acceleration, a, given by $a = t^2$ where t is the time in seconds.

(i) Determine the velocity, v, of the particle in terms of time, t. [2]

$a = t^2$

$v = \int a\ dt$

$v = \int t^2\ dt$

$v = \frac{t^3}{3} + c$

When $t = 0$ and $v = 0$,

$0 = \frac{(0)^3}{3} + c$

$0 = 0 + c$

$c = 0$

$\therefore$ The velocity of the particle is: $v = \frac{t^3}{3}$.

(ii) Calculate the displacement, s, of the particle in the interval of time $t = 0$ to $t = 2$. [4]

$s = \int_a^b v \, dt$

$= \int_0^2 \frac{t^3}{3} \, dt$

$= \frac{1}{3}\left[\frac{t^4}{4}\right]_0^2$

$= \frac{1}{3}\left(\frac{2^4}{4} - \frac{0^4}{4}\right)$

$= \frac{1}{3}\left(\frac{16}{4} - 0\right)$

$s = \frac{1}{3}(4)$

$s = \frac{4}{3}$ units

$\therefore$ The displacement of the particle is $\frac{4}{3}$ units.

Total: 15 marks

SECTION IV

Answer ALL questions.

ALL working must be clearly shown.

6. (a) Two fair tetrahedral dice with faces numbered 1, 2, 3, 4 are rolled. The numbers obtained on the turned-down face of each dice are noted.

Create a sample space table listing ALL possible outcomes for the two dice. [3]

The sample space table is shown below:

Dice 1				
4	1,4	2,4	3,4	4,4
3	1,3	2,3	3,3	4,3
2	1,2	2,2	3,2	4,2
1	1,1	2,1	3,1	4,1
	1	2	3	4

Dice 2

(b) Using your sample space table created in (a), or otherwise, determine the probability of obtaining a 4

(i) on **both** dice [1]

$$\text{Probability} = \frac{\textit{Number of possible outcomes}}{\textit{Total number of outcomes}}$$

$$= \frac{1}{16}$$

∴ The probability of obtaining a 4 on both dice is $\frac{1}{16}$.

(ii) on **at least one** dice [1]

$$\text{Probability} = \frac{\textit{Number of possible outcomes}}{\textit{Total number of outcomes}}$$

$$= \frac{7}{16}$$

∴ The probability of obtaining a 4 on at least one dice is $\frac{7}{16}$.

(iii) on **exactly one** dice. [1]

$$\text{Probability} = \frac{\textit{Number of possible outcomes}}{\textit{Total number of outcomes}}$$

$$= \frac{6}{16}$$

$$= \frac{3}{8}$$

∴ The probability of obtaining a 4 on exactly one dice is $\frac{3}{8}$.

(iv) Show that obtaining a 4 on **both** dice are independent events. [2]

Let A be the event of obtaining a 4 on Dice 1.

Let B be the event of obtaining a 4 on Dice 2.

$$P(A \cap B) = \frac{Number\ of\ possible\ outcomes}{Total\ number\ of\ outcomes}$$

$$= \frac{1}{16}$$

$$P(A) = \frac{Number\ of\ possible\ outcomes}{Total\ number\ of\ outcomes}$$

$$= \frac{1}{4}$$

$$P(B) = \frac{Number\ of\ possible\ outcomes}{Total\ number\ of\ outcomes}$$

$$= \frac{1}{4}$$

So,

$$P(A) \times P(B) = \frac{1}{4} \times \frac{1}{4}$$

$$= \frac{1}{16}$$

Since $P(A \cap B) = \frac{1}{16} = P(A) \times P(B)$, then obtaining a 4 on both dice are independent events.

Q.E.D.

(v) Determine the probability of obtaining a 4 on **both** dice, given that a 4 was obtained on **at least one** dice. [2]

Let A be the event of obtaining a 4 on both dice.

Let B be the event of obtaining a 4 on at least one dice.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{\left(\frac{1}{16}\right)}{\left(\frac{7}{16}\right)}$$

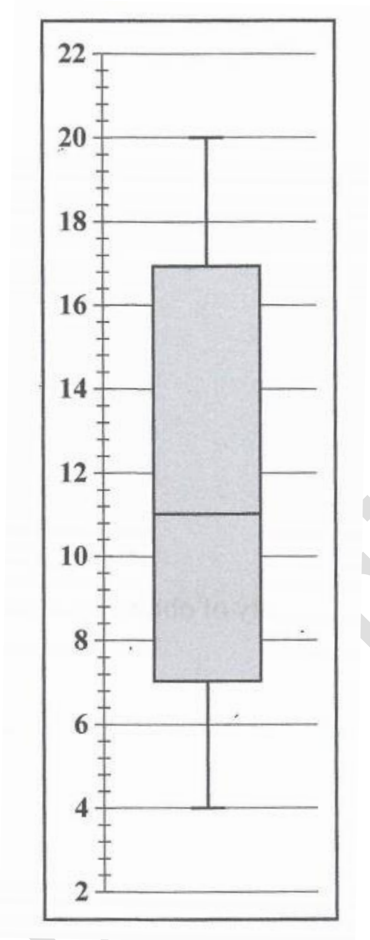
$$= \frac{1}{16} \div \frac{7}{16}$$

$$= \frac{1}{16} \times \frac{16}{7}$$

$$= \frac{1}{7}$$

$\therefore$ The probability of obtaining a 4 on both dice, given that a 4 was obtained on at least one dice is $\frac{1}{7}$.

(c) The scores of a class of 30 students on a Mathematics test were used to draw the box plot below. (The total score possible is 20 marks.)



Using the box plot, determine the following:

(i) The median score [2]

The median score is 11 marks.

(ii) The range of the scores [1]

Range = Highest Value – Lowest Value

$= 20 - 4$

$= 16$ marks

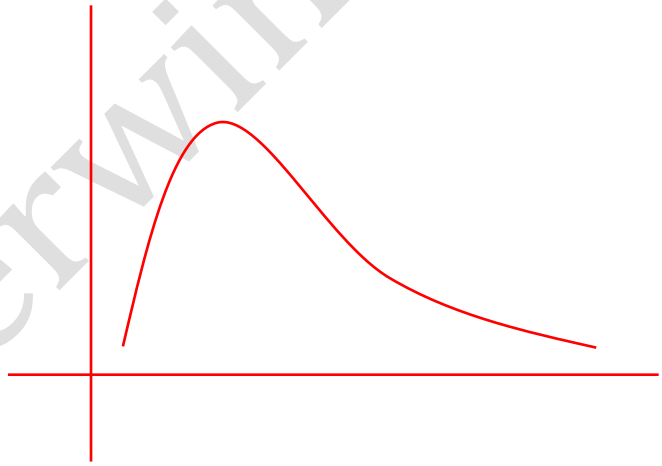
(iii) The semi-interquartile range of the scores [2]

$$\text{Interquartile range} = Q_3 - Q_1$$
$$= 17 - 7$$
$$= 10$$

$$\text{Semi-interquartile range} = \frac{IQR}{2}$$
$$= \frac{10}{2}$$
$$= 5$$

(iv) Comment on the shape of the distribution of the scores. [1]

The distribution of the scores is positively skewed.



A student wants to determine the mean score for the data set.

(v) State ONE reason why it would be impossible to determine the mean score from the box plot. [1]

It would be impossible to determine the mean score from the box plot since the individual scores were not given, and this is necessary in finding the mean.

(vi) State what additional piece of information would be needed to determine the mean score. [1]

To determine the mean score, we would also need the frequencies of the scores.

(vii) Given that the sum of the 30 scores for the class is 354 and the sum of the squares of the scores is 4994, determine the standard deviation for the data set. [3]

We are given that $\sum x = 354$ and $\sum x^2 = 4994$.

Mean, $\bar{x} = \frac{\sum x}{n}$

$= \frac{354}{30}$

$= 11.8$

The standard deviation for the data set is,

$$S = \sqrt{\frac{\sum x^2}{n} - (\overline{x})^2}$$

$$= \sqrt{\frac{4994}{30} - (11.8)^2}$$

$$= \sqrt{\frac{2042}{75}}$$

$= 5.22$ (to 3 significant figures)

∴ The standard deviation for the data set is 5.22.

Total: 20 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.