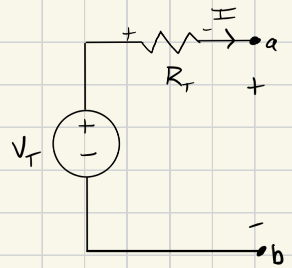
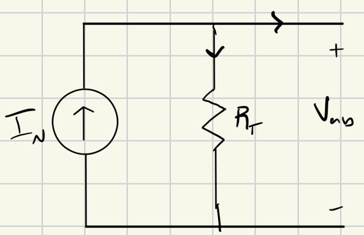
How to make thevenin and Norton circuits



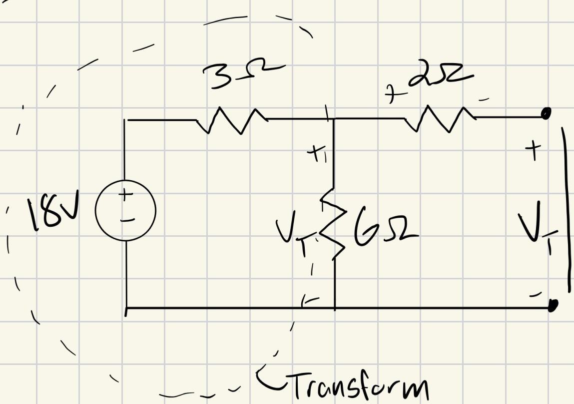
KVL: $-V_T + IR_T + V_{AB} = 0$

$$V_{AB} = \underbrace{V_T}_{\text{Open circuit voltage}} - R_T I$$



KCL: $\underbrace{I_N}_{\text{short-circuit current}} = \cancelto{0}{\frac{V_{ab}}{R_T}} + I$

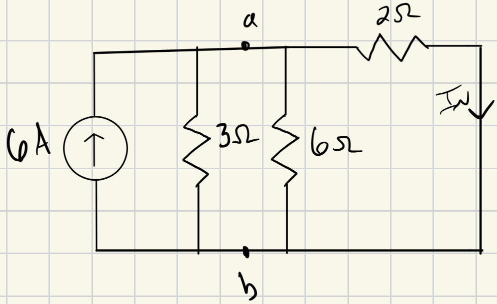
Example:



short the circuit for In

Voltage Division

$$V_T = 18\left(\frac{6}{6+3}\right) = 12v$$

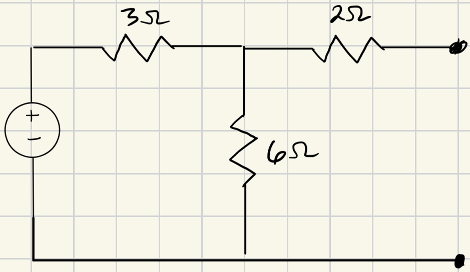


Current Division:

$$I_N = 6A\left(\frac{R_p}{2\Omega}\right) = 6\left(\frac{1}{2}\right) = 3A$$

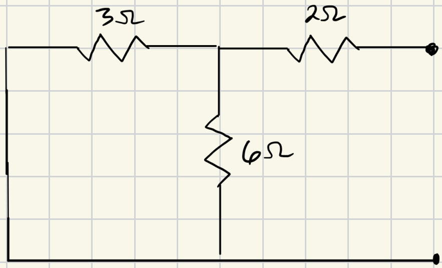
$$R_T = \frac{V_T}{I_N} = \frac{12v}{3A} = 4\Omega$$

How to get R_T directly:

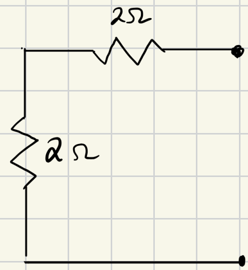


① Source suppression
- remove independent sources

② Find R_{eq}:



→



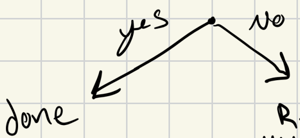
$R_T = 4\Omega$

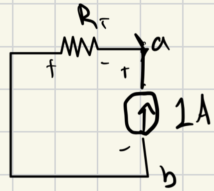
• Test Signal Method

- What if you cant simplify to a single resistor?

① Source suppression

② Check if we can reduce to single resistor





KVL: $R_T I + V_{ab} = 0$

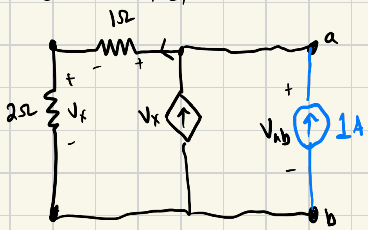
$$R_T = \frac{-V_{ab}}{I} = \frac{-V_{ab}}{-1} = V_{ab}$$

Example: Obtain V_T, I_N, R_T for the following circuit.

V_T: No independent sources: $V_T = 0\,V$

I_N " " $I_N = 0\,A$

Get R_T:



KCL:

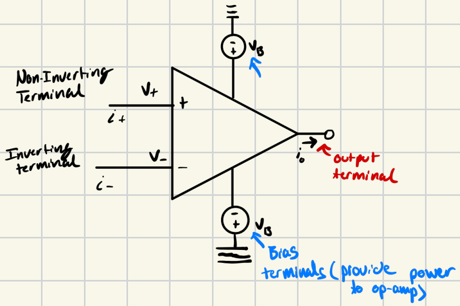
$$1 + V_x = I_{2\Omega} = \frac{V_x}{2} \Rightarrow V_x = -2V$$

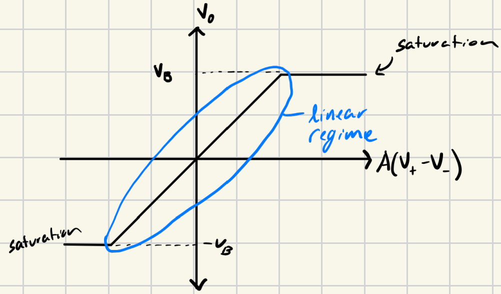
KVL:

$$-V_x - 1\left(\frac{V_x}{2}\right) + V_{AB} = 0$$

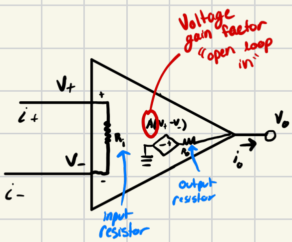
$$V_{ab} = \frac{3}{2}V_x = -3V \Rightarrow R_T = -3\,\Omega$$

- Operational Amplifier (Op-amp)





OP-Amp in linear regime:



- $|V_+ - V_-| \ll (< 10\,\mu V)$
- $R_i \gg (> 10^6\,\Omega)$
- $R_o \ll (< 10\,\Omega)$
- $A \gg (> 10^6)$

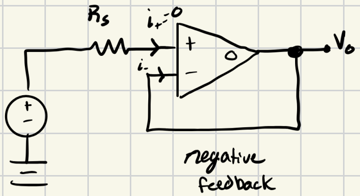
Ideal op-amp approximations

- $i_+ = i_- = 0$ → within pA
- $V_+ = V_-$ → within μV

Example 1: Voltage follower

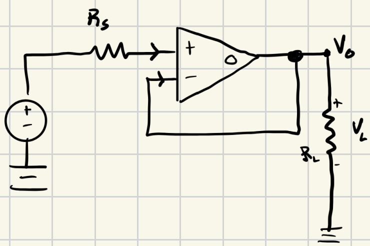
- Obtain V_o in the following circuit assuming the ideal op-amp approx:

  - $V_+ = V_-$
  - $i_+ = i_- = 0$



$V_o = V_- = V_+ = V_s$

Consider attaching a load

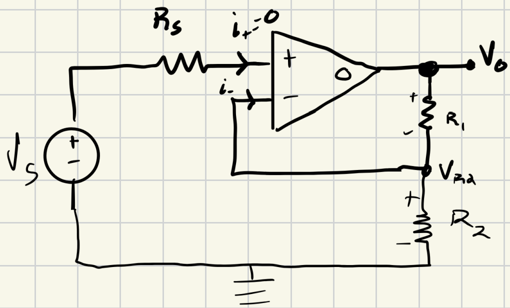


- $P_L = \dfrac{V_L^2}{R_L} = \dfrac{1}{R_L}\left(V_s \dfrac{R_L}{R_L+R_s}\right)^2 = \dfrac{V_s^2}{R_L}\left(\dfrac{R_L^2}{(R_L+R_s)^2}\right)$
  $=$

- Op-Amp acts as a buffer between the source and the load

Example 2: non-inverting amplifier

- Obtain V_o in the following circuit assuming the ideal op-amp aprox



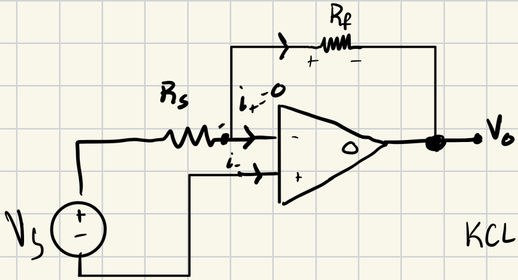
$V_o = V_{R_1} + V_{R2} = i_1 R_1 + V_s$

$V_{R2} = V_- = V_+ = V_s$

- Never do KCL @ ground in op-amps
- $i_o = 0$ (in general)

Example 3: Inverting Amplifier

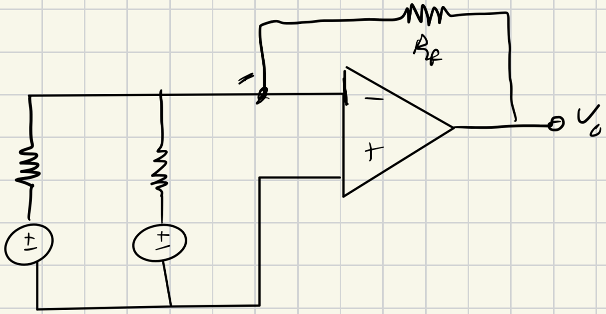
- Obtain V_o in the following circuit



KCL @ (−)

$$\frac{V_s}{R_s} = \frac{-V_o}{R_f} \Rightarrow -V_s\left(\frac{R_f}{R_s}\right) = V_o$$

Example 4: Adder



$$\frac{V_1}{R_1} + \frac{V_2}{R_2} = \frac{-V_o}{R_f}$$

Linearity, time-invarience and LTI systems

$$f(t) \longrightarrow \boxed{\text{system}} \longrightarrow y(t)$$

$$V_o(t) = -\frac{R}{L}\int_{-\infty}^{t} V_s(x)\,\partial x =$$

$$= \underbrace{-\frac{R}{L}\int_{-\infty}^{0} V_s(x)\,\partial x}_{V_o} - \frac{R}{L}\int_{0}^{t} V_s(x)\,\partial x$$

$$= V_0 - \frac{R}{L}\int_{0}^{t} V_s(x)\,\partial x$$

zero input response (when $V_s(t)=0$) (no input)

zero-state response (when $V_0=0$ (no initial states))

$$= V_{ZI}(t) + V_{ZS}(t)$$

zero state:

- no initial state
- a weighted sum of inputs produces a similarly weighted sum of corresponding zero-state outputs, consistent with superposition principle

if

$$f_1(t) \to \boxed{\text{zero state linear}} \to y_{zs,1}(t)$$

and

$$f_2(t) \to \boxed{\text{zero state linear}} \to y_{zs,2}(t)$$

then

$$f_3(t) = K_1 F_1(t) + K_2 f_2(t) \to \boxed{\text{zero state linear}} \to y_{zs,3} =$$

Example:

- Consider a system with input $f(t)$, initial state $y(0) = y_0$ and input-output rule given by

$$y(t) = y_0 e^{-t} - \int_0^t f(x)\,dx$$

- Determine if the system is linear or not

Q: Zero-state linear or not? $\Rightarrow y_0 = 0 \Rightarrow y_{zs}(t) = -\int_0^t f(x)\,dx$

$$f_1(t) \rightarrow \boxed{} \rightarrow y_1(t) = -\int_0^t f_1(x)\,dx$$

$$f_2(t) \rightarrow \boxed{} \rightarrow y_2(t) = -\int_0^t f_2(x)\,dx$$

$$K_1 f_1(t) + K_2 f_2(t) \rightarrow \boxed{} \rightarrow K_1 y_1(t) + K_2 y_2(t)$$

$$= -\int_0^t f_3(dx) = -\int_0^t (K_1 f_1 + K_2 f_2)\,dx =$$

$$K_1\underbrace{\left(-\int_0^t f_1\,dx\right)}_{y_1} + K_2\underbrace{\left(-\int_0^t f_2\,dx\right)}_{y_2}$$

Is linear

Q: zero input linear? $\Rightarrow f(t) = 0 \Rightarrow y_{ZI}(t) = y_0 e^{-t}$

$$\boxed{y_0} \rightarrow y_{ZI,0}(t) = y_0 e^{-t}$$

$$\boxed{y_1} \rightarrow y_{ZI,1}(t) = y_1 e^{-t}$$

$$\boxed{y_3 = K_1 y_0 + K_2 y_1} \rightarrow y_{ZI,3}(t) = K_1 y_{ZI,0}(t) + K_2 y_{ZI,1}(t)$$

Example: $y(0) = y_0$

$$y(t) = y_0^2 + f^2(t)$$

① zero-state linear? $\Rightarrow$ $y_0 = 0 \Rightarrow y_{zs}(t) = f^2(t)$

LTI - Linear + Time Invariant

Time Invariance: delayed inputs cause equally delayed outputs for zero initial state and any delay t_d. So

if

$$f_1(t) \rightarrow \boxed{\text{time-invariant}} \rightarrow y_1(t)$$

$$f_2(t) = f(t-t_d) \rightarrow \boxed{\text{time-invariant}} \rightarrow y_2(t) = y_1(t-t_d)$$

Example:

$$y(0) = y_0 \qquad y(t) = y_0^2 + 2f(t)$$

zero-state $\Rightarrow y_0 = 0 \Rightarrow y_{zs}(t) = 2f(t)$

$$f_1(t) \rightarrow \boxed{} \rightarrow y_1(t) = 2f_1(t)$$

$$f_2(t) \rightarrow \boxed{} \rightarrow y_2(t) \overset{?}{=} y_1(t-t_d)$$

$$f_2(t) = f_1(t-t_d)$$

$$y_2(t) = 2f_2(t) = 2f_1(t-t_d)$$

$$y_1(t-t_d) = 2f_1(t-t_d)$$

same TI system

Ex: $y(0) = y_0 \quad y(t) = y_0^2 + f(t^2)$

zero state $\Rightarrow y_0 = 0 \Rightarrow y_{zs}(t) = f(t^2)$

$$f_1(t) \rightarrow \boxed{} \rightarrow y_1(t) = f_1(t^2)$$

$t_d = 3s$ for simplicity. Must work for any t_d

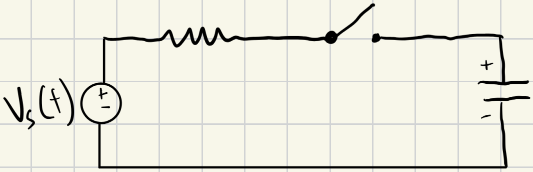
$$f_2(t) \rightarrow \boxed{} \rightarrow y_2(t) \overset{?}{=} y_1(t-\overset{t_d}{3})$$

$$f_2(t) = f_1(t-3)$$

$$y_2(t) = f_2(t^2) = f_1(t^2-\overset{t_d}{3}) \star$$

$$y_1(t-\overset{t_d}{3}) = f_1((t-\overset{t_d}{3})^2) \neq \text{TI}$$

- First Order RC and RL circuits



Initial Conditions:

$V_c(0^-) = V_c(0) = V_c(0^+)$ - I.C.

$t>0$

KVL: $-V_s(t) + i(t)\cdot R + V_c(t) = 0$

$i(t) = C\frac{dV_c}{dt}$

$RC\frac{dV_c}{dt} + V_c(t) = V_s(t)$

$$\frac{dV_c}{dt} + \frac{1}{RC}V_c(t) = \frac{V_s(t)}{RC}$$

First Order linear ordinary differential equation (ODE) with constant coefficients, which governs RC circuit for $t>0$

Can also use $\mu(t) = e^{\frac{t}{RC}}$

$$\left(e^{\frac{t}{RC}}V_c(t)\right)' = e^{\frac{t}{RC}}\frac{1}{RC}V_s(t)$$

How to solve:

- Start with a simple case: constant input

$f(t) = K \Rightarrow K = \frac{dy}{dt} + ay(t) \Rightarrow K = 0 + aB \Rightarrow B = \frac{K}{a}$

$y(t) = B$ - constant

$$K = \frac{dy}{dt} + ay(t)$$

$$y(t) = \underbrace{\frac{K}{a}}_{\text{particular solution}} + y_n(t)$$ homogeneous solution

$$K = \frac{d}{dt}\left(\frac{K}{a} + y_n(t)\right) + a\left(\frac{K}{a} + y_n(t)\right)$$

$$\cancel{K} = \frac{d}{dt}y_n(t) + \cancel{K} + ay_n(t)$$

$$\frac{dy_n(t)}{dt} + ay_n(t) = 0 \rightarrow -aAe^{-at} + aAe^{-at} = 0$$

$$y_n = Ae^{-at}$$

$$y(t) = \frac{K}{a} + \underbrace{Ae^{-at}}_{\text{1 when } t=0}$$

$$y(t) = \underbrace{\frac{K}{a}}_{y_p} + \underbrace{\left(y(0^+) - \frac{K}{a}\right)e^{-at}}_{y_n}$$

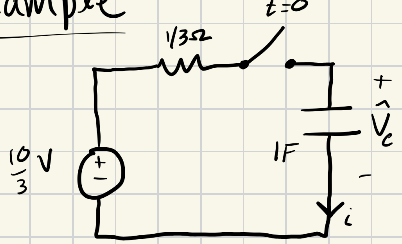
$$\Rightarrow \overset{K}{\frac{V_s}{RC}} = \frac{dV_c}{dt} + \overset{a}{\frac{1}{RC}}V_c(t)$$

$$y_p = \frac{K}{a} = V_s$$

$$y_n = \left(y(0^+) - \frac{K}{a}\right)e^{-at} = (V_c(0^-) - V_s)e^{-\frac{t}{RC}}$$

$$\boxed{V_c(t) = \left(V_c(0^-) - V_s\right)e^{-\frac{t}{RC}} + V_s} \quad t>0$$

Example



① $\frac{dV_c}{dt} + \underbrace{\frac{1}{RC}}_{a=3} V_c(t) = \frac{V_s}{RC} = 10$

② $\frac{dV_c}{dt} + 3v_c(t) = 10 \Rightarrow 0 + 3B = 10$

$B = \frac{10}{3}$

③ $y(t) = y_p + y_h = B + Ae^{-at}$

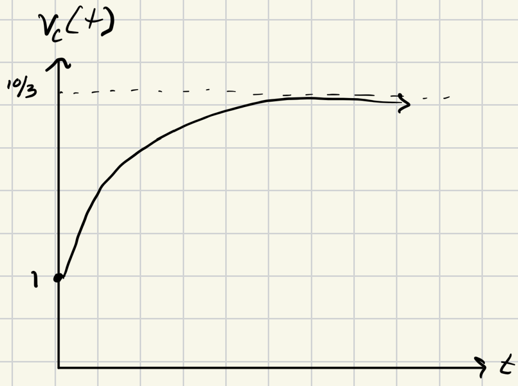
④ $V_c(t) = \frac{10}{3} + Ae^{-3t}$

⑤ Find A

$V_c(0^-) = V_c(0^+) = 1$

$1 = \frac{10}{3} + A \Rightarrow A = \frac{-7}{3}$

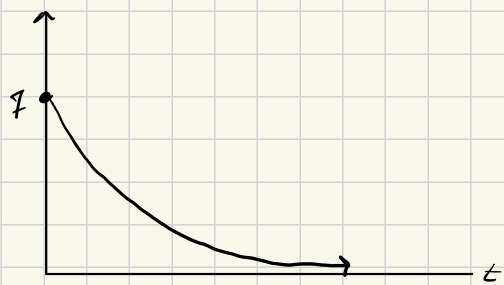
⑥ $V_c(t) = \left(\frac{10}{3} - \frac{7}{3}e^{-3t}\right)$ Volts



7) $i_c(t) = C\frac{dV_c}{dt} = 7e^{-7t}$ Amps

$i_c(0^+) = 7A$

$i_c(0^-) = 0A$



Time-Constant

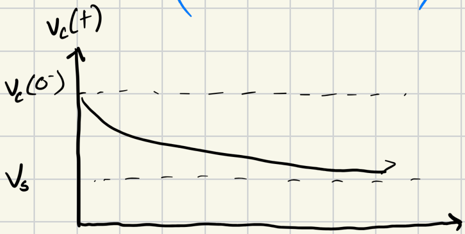
- Recall

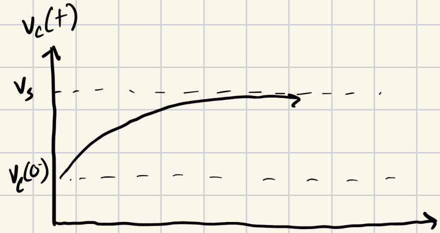
$$y(t) = \frac{K}{a} + \left(y(0^+) - \frac{K}{a}\right)e^{-at}$$

$$V_C(t) = \underbrace{(V_C(0^-) - V_s)e^{-\frac{t}{RC}}}_{\text{transient response}} + \underbrace{V_s}_{\text{steady state response}}$$

transient response (a part which goes to 0 when $t \to \infty$)

steady state response (which is left after transient response has vanished)





Rate of decay is controlled by a time const.

$\tau \uparrow \Rightarrow$ slower decay

$\tau = RC$ [s] ← unit = seconds

Simple Solution Method:

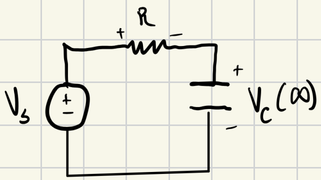
(only works for first order ode with const coefficients and const inputs)

$$y(t) = B + Ae^{-at} \qquad a = \frac{1}{RC} = \frac{1}{\tau}$$

$$V_c(t) = B + Ae^{-t/\tau}$$

Need to evaluate:

- $V_c(\infty) = B + \cancel{Ae^{-t/\tau}} \to 0$
- $V_c(0^-) = B + A$



$i_c(t) = C\frac{dV_c}{dt}$

⇓

at the DC steady-state ($t \to \infty$) capacitor acts as an open-circuit ($i_c = 0A$)

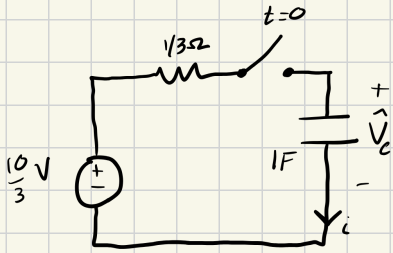
$\Delta \quad B = V_S$

$$A = V_C(0^-) - B = V_C(0^-) - V_S$$

$$V_C(t) = V_S + (V_C(0^-) - V_S)e^{-t/\tau}$$

Redo example w/ shortcut:

$V_C(0^-) = 1V$, determine $V_C(t)$ for $t > 0$



<u>$t > 0$</u>

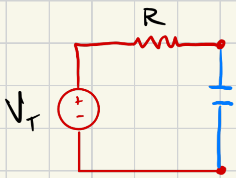
$$V_C(t) = B + Ae^{-t/\tau}$$

$$V_C(\infty) = \frac{10}{3}V = B$$

$$V_C(0^-) = B + A = 1 \Rightarrow A = \left(1 - \frac{10}{3}\right) = \frac{-7}{3}$$

$$V_C(t) = \frac{10}{3} - \frac{7}{3}e^{-3t}$$

If a resistive circuit w/ a single capacitor:



Find thevinin circuit <u>as seen by the capacitor</u>

• Zero-input (ZI), zero-state (ZS)

$$y(t) = \underbrace{\frac{K}{a}}_{\text{particular}} + \underbrace{\left(\overset{\to 0}{y(0^+)} - \frac{K}{a}\right)e^{-at}}_{\underline{\text{hom}}}$$

$$\frac{K}{a} - \frac{K}{a}e^{-at}$$

First order circuits are when there are only one power storage device in a circuit.

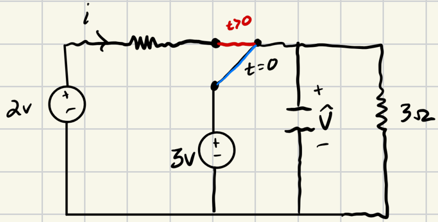
- Zero state is easiest to do.

Recall

$$y(t) = \frac{K}{a} + \left(y(0^+) - \frac{K}{a}\right)e^{-at}$$

with $y(0^+) \nearrow 0$

Zero state: $\frac{K}{a}$



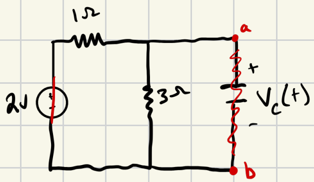
$t < 0$

capacitor acts as open circuit.

$V_c(0^-) = 3v$

$t > 0$

$$V_c(t) = B + Ae^{-t/\tau}$$



Find τ: $\tau = R_T C$ → Find R_T from perspective of the cap.

$R_T = \left(\frac{3}{3} + \frac{1}{3}\right)^{-1} = \frac{3}{4}\Omega$

Must find V_c at ∞ and initial conditions

$V_c(\infty) = B = 2v\left(\frac{3}{3+1}\right) = \frac{3}{2}v$

$V_c(0^+) = V_c(0^-) = B + A \Rightarrow A = (V_c(0^-) - \frac{3}{2})$

$$V_c(t) = B + Ae^{-t/\tau} =$$

$$\frac{3}{2} + \left(V_c(0^-) - \frac{3}{2}\right)e^{-t/3} = \frac{3}{2} + \underbrace{V_c(0^-)}_{0}e^{-t/3} - \frac{3}{2}e^{-t/3}$$

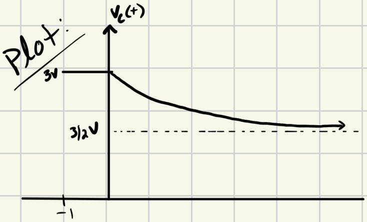
$$= y_{ZS}(t) = \frac{3}{2} - \frac{3}{2}e^{-t/3}\,v$$

$+$

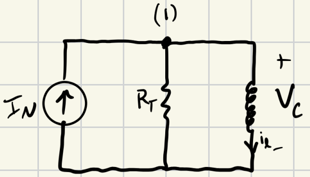
$$y_{ZI}(t) = V_c(0^-)e^{-t/3} = 3e^{-t/3}\,v$$

$$V_c(t) = \frac{3}{2} + \frac{3}{2}e^{-t/3}\,v$$

Plot:



Ex with inductor



KCL @ Node 1 : $I_N = \frac{V_L}{R_T} + i_L$

$= \frac{L}{R_T}\frac{di_L}{dt} + i_L \Big| \times \frac{R_T}{L}$

$\frac{di_L}{dt} + \frac{R_T}{L} i_L = \frac{R_T}{L} I_N$

$\boxed{V_L = L\frac{di}{dt}}$

↳ In DC-steady state, inductor acts as a short circuit.

$\boxed{\tau = \frac{L}{R_T}}$

$\boxed{a = \frac{1}{\tau}}$

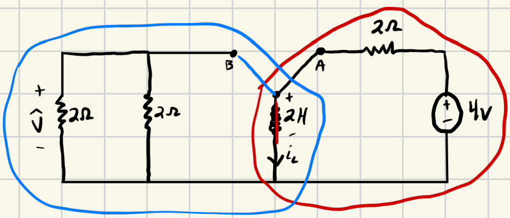
Solution should be in the form: $i_L(t) = B + Ae^{-at}$

$i_L(t) = B + Ae^{-t/\tau}$

$i_L(\infty) = B$

$i_L(0^+) = i_L(0^-) = B + A$

Example:



$t < 0$

$i_L(0^-) = \frac{4V}{2\Omega} = 2A$

$t > 0$

$i_L(t) = B + Ae^{-t/\tau}$

$\tau = \frac{L}{R_T}$

First Find τ: $\tau = \frac{L}{R_T}$

$R_T = 1\Omega$ (From perspective of the inductor)

$\tau = \frac{2H}{1\Omega} = 2s$

$i_L(\infty) = B = 0A$ (no source)

$i_L(0^+) = i_L(0^-) = \cancel{B} + A$ ($\cancel{B} \to 0$)

$i_L(t) = i_L(0^-)e^{-t/2}$ Amps

$i_{ZS}(t) = i_L(0^-)e^{-t/2}$ Amps

$i_{ZS}(t) = 0$ (force $i_L(0^-)$ to be 0)

$i_{ZI}(t) = i_L(0^-)e^{-t/2} = 2e^{-t/2}$ Amps

What if the circuit goes from $A \to B \to A$

$t < 0$

$i_L(0^-) = \frac{4}{2} = 2A$

$0 < t < 2$

$i_L(t) = B + Ae^{-t/\tau}$; $\tau = 2s$

$i_L(\infty) = B = 0$

$i_L(0^+) = i_L(0^-) = A = 2$

$i_L(t) = 2e^{-t/2}$ amps

Now going back to pos A →

$t > 2$

$i_L(t) = \hat{B} + \hat{A}e^{-t/\tau}$

Consider the particular solution:

$$y_p(t) = \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t) = H\cos(t+\psi)$$

$$= H\cos(\psi)\cos(t) - H\sin(\psi)\sin(t) \Rightarrow H\cos(\psi) = \frac{1}{2}$$

$$-H\sin(\psi) = \frac{1}{2}$$

Determine H and ψ

Get ψ:

$$\frac{\cancel{H}\sin(\psi)}{\cancel{H}\cos(\psi)} = \frac{-1/2}{1/2} = \tan(\psi) = -1 \Rightarrow$$

$$\psi = \frac{-\pi}{4} \text{ or } \frac{3\pi}{4}$$

Get H

$$H\cos(\psi) = 1/2$$

$$H\cos\left(\frac{3\pi}{4}\right) = \frac{1}{2}$$

$$H = \frac{\sqrt{2}}{2} \text{ or } \frac{-\sqrt{2}}{2}$$

$$y_p(t) = \frac{\sqrt{2}}{2}\cos\left(t - \frac{\pi}{4}\right)$$

Transient Response v.s. Steady-State Response

Transient: $y_{tr}(t)$ $\forall$:

$$\lim_{t\to\infty} y_{tr}(t) \to 0$$

Steady state: $y_{ss}(t)$ is what's left after $y_{tr} \to 0$

$$y(t) = y_{tr}(t) + y_{ss}(t)$$

· Recall

$$y(t) = \overbrace{Ae^{-at}}^{\text{transient}} + \underbrace{B}_{ss}$$

$$y(t) = \underbrace{Ae^{-at} + Be^{-pt}}_{\text{transient}}$$

$$y(t) = \underbrace{Ae^{-at}}_{\text{transient}} + \underbrace{H\cos(\omega t + \psi)}_{\text{steady state}}$$

Last Week:

$$\frac{\partial y}{\partial t} + \frac{y}{t} = C_1 \Rightarrow y(t) = K_1 e^{-t/\tau} + K_2$$

Lets make a particular solution table

What if:

$$\frac{\partial y}{\partial t} + \frac{y}{\tau} = f(t) \qquad (y = y_n(t) + y_p(t))$$

s.t. $\frac{\partial y_n}{\partial t} + \frac{y_n}{\tau} = 0$

$f(t)$	$y_p(t)$
C_1	C_2
$Kt + b$	$\alpha t + \beta$
polynomial of degree "n"	polynomial w/ same degree "n" with all other lower terms.
$\cos(\omega t)$ or $\sin(\omega t)$	$A\cos(\omega t) + B\sin(\omega t)$
Ae^{ct}	if $\begin{cases} c \neq \frac{-1}{\tau}; \ Be^{ct} \\ c = \frac{-1}{\tau}; \ Bte^{ct} \end{cases}$

* constants

Example:

$$\frac{\partial y}{\partial t} + y = e^{-2t} \qquad y_n = Ae^{-t/\tau} \ ; \ \tau = 1$$

$$= Ae^{-t}$$

y_p: $c \neq \frac{-1}{\tau}$

plug into original eq where $y = y_p$

$$\begin{cases} y_p = Be^{-2t} \\ \frac{dy_p}{dt} = -2Be^{-2t} \end{cases}$$

$$-2Be^{-2t} + Be^{-2t} = e^{-2t}$$

$$-Be^{-2t} = e^{-2t} \rightarrow B = -1$$

$$\Delta \ y_p = -e^{-2t}$$

$$y = Ae^{-t} - e^{-2t}$$

use I.C (initial conditions) to find A.

Example:

$$\frac{dy}{dt}+y=e^{-t}$$

$$y_p = Bte^{-t}$$

$$\frac{dy_h}{dt}+y_h=0 \quad \Rightarrow \quad \frac{dy_p}{dt}=B\left(-te^{-t}+e^{-t}\right)$$

$\tau=1$

$$-\cancel{Bte^{-t}}+Be^{-t}+\cancel{Bte^{-t}}=e^{-t}$$

$$Be^{-t}=e^{-t}$$

$$B=1$$

$$y_p=te^{-t}$$

$$\underbrace{y_h=Ae^{-t}}$$

Particular is at the same order as homogeneous we must multiply by t.

Δy:

$$y=y_h+y_p \quad \Rightarrow \quad y=Ae^{-t}+te^{-t}$$

Example:

$$\frac{dy}{dt}+y=t^2$$

$$y_h=Ae^{-t}$$

$\tau=1$

$$y_p=Bt^2+Ct+D$$

$$\frac{dy_p}{dt}=2Bt+C$$

plug in

$$2Bt+C+Bt^2+Ct+D=t^2$$

$$\underline{Bt^2}+\underline{2Bt+Ct}+\underline{C+D}=t^2$$

$B=1$

$2B+C=0$

$C+D=0$

$2+C=0 \Rightarrow C=-2$

$D=-C \Rightarrow D=2$

$$y_p=t^2-2t+2$$

$$y=Ae^{-t}+t^2-2t+2$$

Ex w/ Sinusoidal input

$$\frac{dy}{dt}+y=\cos(t)$$

$$y_h=Ae^{-t}\ ;\ \tau=1$$

$$y_p=B\cos(t)+C\sin(t)$$

$$\frac{dy_p}{dt}=-B\sin(t)+C\cos(t)$$

$$-B\sin(t)+C\cos(t)+B\cos(t)+C\sin(t)=\cos(t)$$

$C+B=1$

$C-B=0$

$C=B$

$C=\frac{1}{2}\quad B=\frac{1}{2}$

$$y_p=\frac{1}{2}\cos(t)+\frac{1}{2}\sin(t)$$

$$\left\{\begin{array}{l} C\cos(t)+D\sin(t)=K\cos(t+\theta) \\ K=\sqrt{C^2+D^2} \\ \theta=\tan^{-1}\left(\frac{-D}{C}\right)\end{array}\right.$$

$$K=\sqrt{\frac{1}{2}^2+\frac{1}{2}^2}=\sqrt{\frac{2}{4}}=\frac{\sqrt{2}}{2}$$

$$\theta=\tan^{-1}\left(\frac{-1}{1}\right)=\tan^{-1}(-1)=-\frac{\pi}{4}$$

$$y_p=\frac{\sqrt{2}}{2}\cos\left(t-\frac{\pi}{4}\right)$$

$$y=Ae^{-t}+\frac{\sqrt{2}}{2}\cos\left(t-\frac{\pi}{4}\right)$$

- Dissipative Systems

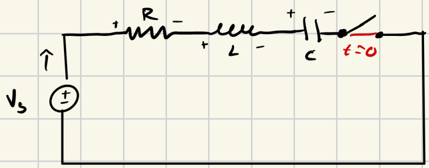
  - has a transient zero-state response

$$y_{zi}(t) \to 0 \quad t \to \infty$$

  - the ss response to a cosinusoidal input applied at $t = -\infty$ will be cosinusoidal independent of the initial state

$$y(t) = Ae^{-at} + \underbrace{H\cos(\omega t + \gamma)}_{y_{ss}(t)}$$

- n^{th} order LTI systems
  - Determine the ODE governing the capacitor voltage in the following circuit:



$$-V_s + V_R + V_C + V_L = 0$$

$$-V_s + iR + L\frac{di}{dt} + V_C = 0$$

$$-V_s + R\left(C\frac{dV_C}{dt}\right) + L\frac{d}{dt}\left(C\frac{dV_C}{dt}\right) + V_C = 0$$

- In general: $\frac{d^n y}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_n y = f(t)$

Where n-number of energy storage elements (L, C)

- Solution is

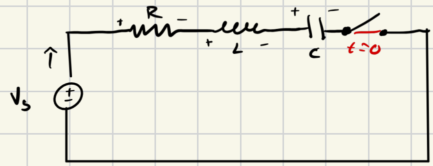
$$y(t) = \underbrace{y_p(t)}_{\text{same as in 1st order ODE}} + y_h(t)$$

if $P_1 = P_2$ $\quad A_1 e^{-P_1 t} + tA_2 e^{-P_2 t} + \dots$

Example:

- For $t>0$, obtain $v_c(t)$

$V_s = 0\,V$, $R = 0\,\Omega$, $i_L(0^-) = 0A$, $v_c(0^-) = 1v$, $L = 1H$, $C = 1F$



$$\frac{\cancel{V_s}}{LC} = \frac{d^2 V_c}{dt^2} + \frac{R}{L}\cancel{\frac{dV_c}{dt}} + \frac{V_c}{LC}$$

0 0 1

$$0 = \frac{d^2 V_c}{dt^2} + V_c$$

Solution:

$$\frac{d^2 V_c}{dt^2} + V_c = 0$$

$$y_n(t) = e^{st}$$

$$s^2 e^{st} + e^{st} = 0$$

$$s^2 + 1 = 0$$

$$s = \pm\sqrt{-1} = \pm j$$

$$y_n = A_1 e^{-jt} + A_2 e^{jt}$$

now we find A_1 and A_2:

$$V_c(0^-) = V_c(0^+) = 1v$$

$$1 = A_1 + A_2 \quad ①$$

$i_L(0^+) = 0A$

$i_c(t) = C\frac{dV_c}{dt} = -jA_1 e^{-jt} + jA_2 e^{jt}$

$i_c(0^+) = -jA_1 + jA_2 = 0 \quad ②$

Solving ① and ② $\Rightarrow A_1 = A_2 = \frac{1}{2}$

$$y_n = \frac{1}{2}e^{-jt} + \frac{1}{2}e^{jt}$$

$= \cos(t) \rightarrow$ zero-input response is not transient

$i_c(t) = -\sin(t)$ Amps

system is non-dissipative

Friday Lecture: to watch

Last time:

must convert sin into cos
- phasors are cos-based objects

Phasor Superposition

- The weighted superposition
- Don't neve to have same magnitude, must have same frequency

Derivative Principle

The derivative

$$g(t) = \frac{d}{dt} f(t)$$

of co-sinusoid $f(t) = Re\left[Fe^{j\omega t}\right]$ is also a co-sinusoid with phasor

$$G = j\omega F$$

Note: N^{th} derivative

$$\frac{d^n f}{dt^n} = (j\omega)^n F$$

Consider

$f(t) = 3\cos(2t + \frac{\pi}{4}) \rightarrow F = 3e^{j\frac{\pi}{4}} = 3\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)$

$= \frac{3}{\sqrt{2}} + j\frac{3}{\sqrt{2}}$

Determine $f'(t)$

$$g(t) = Re\left[Ge^{j\omega t}\right]$$

$G = j\omega F = j \cdot 2 \cdot \left(\frac{3}{\sqrt{2}} + j\frac{3}{\sqrt{2}}\right)$

$= j\frac{6}{\sqrt{2}} - \frac{6}{\sqrt{2}} =$

$A = \sqrt{\left(\frac{6}{\sqrt{2}}\right)^2 + \left(\frac{6}{\sqrt{2}}\right)^2} = 6$

$\theta = \pi + \tan^{-1}(-1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Determine the steady state solution to the following diff eq.

turn this to pos by shifting π
-then convert to cos by subtracting $\frac{\pi}{2}$

$$\frac{\partial^2 y}{\partial t^2} + 2\frac{\partial y}{\partial t} + y = -5\sin(2t)$$

- turn into phaser

$= 5\cos(2t + \frac{\pi}{2})$

$= 5e^{j\pi/2} = 5j$

↓ phasors

$$(j\omega)^2 + 2j\omega y + y = 5j$$

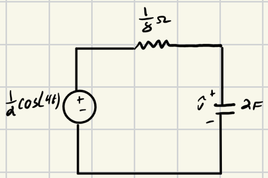
$$-4y + j4y + y = j5$$

$$y = \frac{j5}{-3+j4} = \frac{5e^{j\pi/2}}{5e^{j(\pi + \tan^{-1}(\frac{4}{-3}))}}$$

$$y_{ss} = 1\cos(2t + \tfrac{\pi}{2} - \pi - \tan^{-1}(\tfrac{4}{3})) =$$

$$= \cos(2t - \tfrac{\pi}{2} - \tan^{-1}(\tfrac{4}{3}))$$

Determine $\hat{v}_{ss}(t)$ using phasors



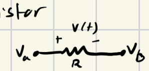
KVL: $-\frac{1}{2}\cos(4t) + \frac{1}{4}\frac{\partial \hat{v}}{\partial t} + \hat{v} = 0$

$\frac{1}{4}\frac{d\hat{v}}{dt} + \hat{v} = \frac{1}{2}\cos(4t)$

$\frac{1}{4}(j\omega)\hat{v} + \hat{v} = \frac{-1}{2}$ $\quad \frac{1}{2}e^{j\pi} = -\frac{1}{2}$

$j\hat{v} + \hat{v} = \frac{-1}{2}$

- V-I relationships in phasors
  - Resistor

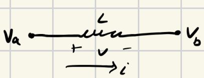


$v(t) = R \cdot i(t)$

$V = IR$

$|V|e^{j\theta_v} = R \cdot |I|e^{j\theta_i} \Rightarrow \theta_v = \theta_i$ (voltage and current are in-phase)

  - Inductor



$v(t) = L\frac{di}{\partial t}$

↓ phasors

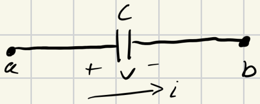
$$V = \underbrace{(Lj\omega)}_{Z\text{-impedence}[\Omega]} I = ZI = V$$

$$|V|e^{j\theta_v} = L\omega \underbrace{e^{j\frac{\pi}{2}}}_{j} |I|e^{j\theta_i} = \theta_v = \theta_i + \tfrac{\pi}{2}$$

$$\theta_i = \theta_v - \tfrac{\pi}{2}$$

Voltage leads current by $\frac{\pi}{2}$, or current lags voltage by $\frac{\pi}{2}$

- Capacitor



$$i = C\frac{dv}{dt}$$

↓ phasors

$$I = Cj\omega V \rightarrow \frac{1}{j\omega C} I = V$$

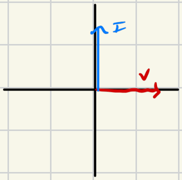
$$Z = \frac{1}{j\omega C} = \frac{-j}{\omega C}$$

$$V = Z \cdot I$$

$$|V|e^{j\theta_v} = \frac{1}{\omega C}e^{-j\frac{\pi}{2}}|I|e^{j\theta_i}$$

$$\theta_v = \theta_i - \frac{\pi}{2}$$

$$\theta_i = \theta_v + \frac{\pi}{2}$$



Current leads voltage by $\frac{\pi}{2}$

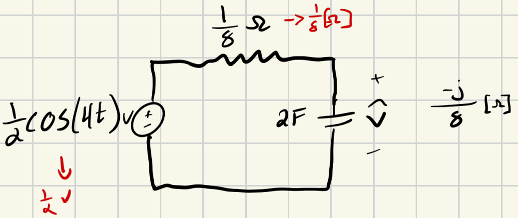
- V-I relationships in phasors

$$\text{Impedence, } Z = \begin{cases} R & \text{resistor} \\ j\omega L & \text{inductor} \\ \frac{1}{j\omega C} = \frac{-j}{\omega C} & \text{capacitor} \end{cases}$$

← purely resistive (resistor)

← purely reactive (inductor, capacitor)

- Determine $\hat{V}_{ss}(t)$ using phasors

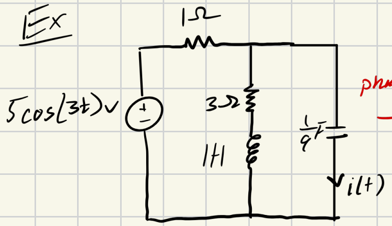


Voltage Division

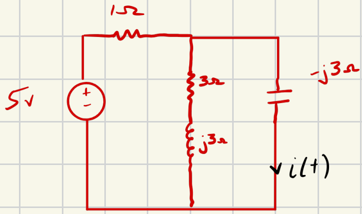
$$\hat{V} = \frac{1}{2}\left(\frac{-j/8}{-j/8 + \frac{1}{8}}\right) = \frac{1}{2}\left(\frac{-j}{-j+1}\right) = \frac{1}{2}\left(\frac{e^{-j\frac{\pi}{2}}}{\sqrt{1^2+(-1)^2}\,e^{-j\frac{\pi}{4}}}\right) = \frac{1}{2\sqrt{2}}e^{-j\frac{\pi}{4}}$$

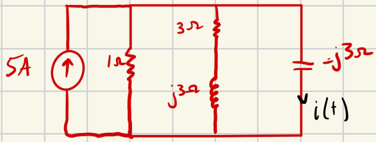
$$\hat{V}_{ss}(t) = \frac{1}{2\sqrt{2}}\cos\left(4t - \frac{\pi}{4}\right) [V]$$

Ex



phasors →

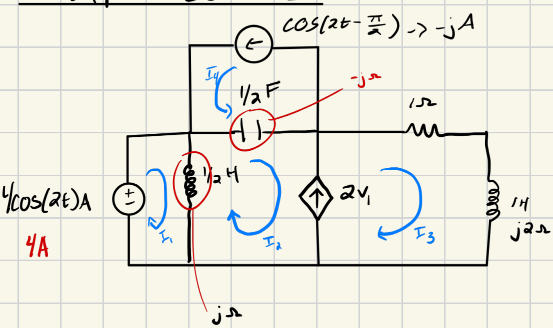




current division

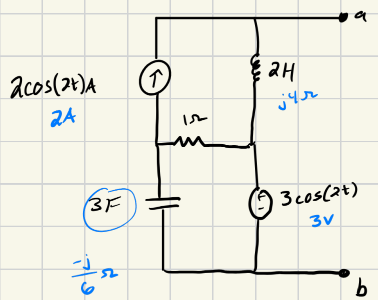
$$5A\left(\frac{Z_p}{-j3}\right) =$$

Multiple Sources



Can use Node-voltage method or loop current

- We can still find Thevinin equivalent s



$V_T = V_{j4} + 3 = 2(j4) + 3 = 3 + j8\,[V]$

Get Z_t

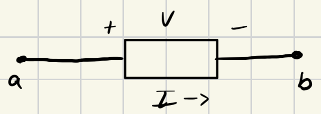
$j4\,[\Omega]$

Get I_N

$I_N = \frac{3+j8\,[V]}{j4\,[\Omega]}$

- Average power

  - Recall $P = IV$



  - If v and i are time varying, then the instantaneous power is

$$p(t) = v(t)\,i(t)$$

  - For periodic signals with period T, the average power is:

$$P = \frac{1}{T}\int_T p(t)\,dt = \frac{1}{T}\int_T \underbrace{v(t)i(t)}\,dt$$

$$\frac{1}{2}\left(Re\{VIe^{j2\omega t}\} + Re\{VI^*\}\right)$$

Average Power for Phasor

$$P = \frac{1}{T}\int_T p(t)\,dt = \frac{1}{T}\int_T v(t)\,i(t)\,dt$$

$$P = \frac{1}{T}\int_T \frac{1}{2}\left(\mathrm{Re}\{VIe^{j2\omega t}\} + \mathrm{Re}\{VI^*\}\right)dt =$$

$$= \frac{1}{2T}\int_T \mathrm{Re}\{VIe^{j2\omega t}\}\,dt + \frac{1}{2T}\int_T \mathrm{Re}\{VI^*\}\,dt$$

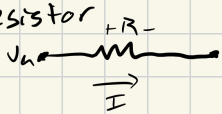
$$= \underbrace{\frac{1}{2T}\int_T |VI|\cos(2\omega t + \angle VI)\,dt}_{\text{Average of a cos over 2 cycles is zero}} + \frac{1}{2}\mathrm{Re}\{VI^*\}$$

angle of VI

$$P = \frac{1}{2}\mathrm{Re}\{VI^*\}$$

• Average Absorbed Power

- Resistor

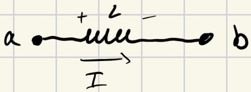


$$P = \frac{1}{2}\mathrm{Re}\{V I^*\} = \frac{1}{2}\left(\mathrm{Re}\{R \cdot I \cdot I^*\}\right)$$

$V = R \cdot I$

$$= \frac{1}{2}R(I^2) = \frac{1}{2}\frac{|V|^2}{R}$$

Inductor



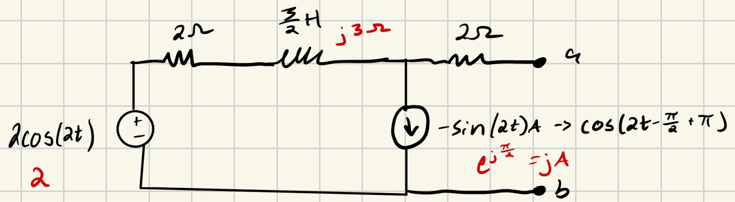
$$P = \frac{1}{2}\mathrm{Re}\{VI^*\} = \frac{1}{2}\mathrm{Re}\{j\omega L|I^2|\} = 0$$

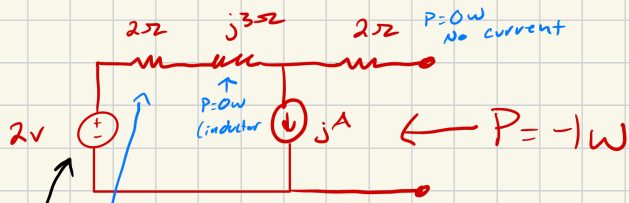
$V = j\omega L \cdot I$

Capacitor → P = 0W!

0W since they store energy then end up discharging it (1 cycle charge, 1 cycle discharge)

• Determine Avg Power for each element



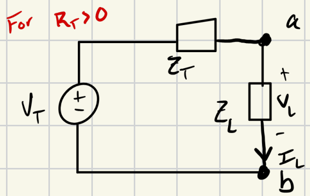


$$P = \frac{1}{2}|I^2| \cdot R$$
$$= \frac{1}{2}|j|^2 \cdot 2$$
$$= \frac{1}{2} \cdot 1^2 \cdot 2 = 1W$$

$$P = \frac{1}{2}Re\{VI^*\} = \frac{1}{2}Re\{2 \cdot (-j)^*\} = 0W$$

• Available Power

For $R_T > 0$



$$Z_T = R_T + jX_T$$
$$Z_L = R_L + jX_L$$

$$P_L = \frac{1}{2}Re\{V_L \cdot I_L^*\} = \frac{1}{2}Re\left\{\frac{V_T \cdot Z_L}{Z_T + Z_L} \cdot \frac{V_T^*}{(Z_T+Z_L)^*}\right\} =$$

$$I_L = \frac{V_T}{Z_T + Z_L}$$
$$V_L = V_T\left(\frac{Z_L}{Z_L + Z_T}\right)$$

$$= \frac{1}{2}Re\left\{\frac{|V_T|^2 Z_L}{|Z_T + Z_L|^2}\right\}$$
$$= \frac{1}{2}\frac{|V_T|^2 \cdot R_L}{|Z_T + Z_L|^2} =$$
$$= \frac{1}{2}\frac{|V_T|^2 \cdot R_L}{|(R_L + R_T) + j(X_L + X_T)|^2}$$

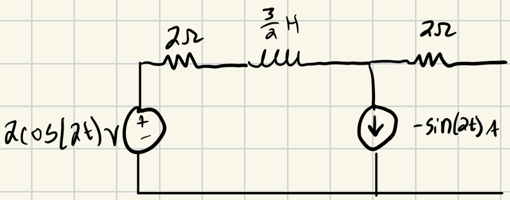
$$Z_L = R_T - jX_T = Z_T^* \qquad \begin{cases} X_L = -X_T \\ R_L = R_T \end{cases}$$

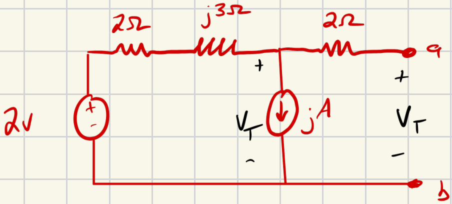
$$P_a = \frac{|V_T|^2}{8R_T} = \frac{1}{2}\frac{|V_T|^2}{4R_T}$$

From phasors

only real part of Z_T

Example:





Get V_T:

KVL: $-2+2j+j(j3)+V_T=0$

$V_T = 5-j2\,V$

Get Z_T

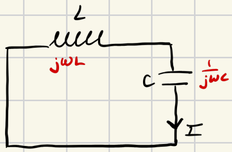
$Z_T = 2+j3+2 = 4+j3\,\Omega$ (R_T)

$$P_a = \frac{1}{2}\frac{|V_T|^2}{4R_T} = \frac{1}{2}\frac{\left(\sqrt{5^2+(-2)^2}\right)^2}{4\cdot 4} = \frac{29}{32}\,W \quad \text{only if}$$

$$Z_L = Z_T^* = 4-j3\,\Omega$$

- Resonance

  - Circuit below with $v_c(0^-)=1v$, $i_L(0^-)=0A$, $v_c(t)=A\cos(t+\theta)$



KVL: $V_L+V_c=0$

$j\omega L\cdot I+\frac{1}{j\omega C}I=0$

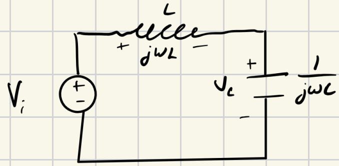
$I\underbrace{\left(j\omega L-\frac{j}{\omega C}\right)}_{\neq 0}=0$

can be satisfied by any I as long as $j\omega L-\frac{j}{\omega C}=0$

$\omega L=\frac{1}{\omega C}$

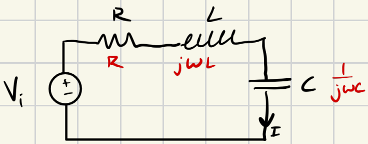
$\boxed{\omega_0=\frac{1}{\sqrt{LC}}}$ resonant frequency

What if we add a source



$$V_c = V_i\left(\frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C}+j\omega L}\right) = \frac{V_i}{\underbrace{-\omega^2 LC+1}_{0 \text{ if } \omega=\frac{1}{\sqrt{LC}}}} \Rightarrow V_c \to \infty$$

What if we also add a resistor?



$$V_c = V_i\left(\frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C}+j\omega L+R}\right) = \frac{V_i}{1-\omega^2LC+j\omega RC}$$

$$V_c = \frac{V_i}{j\omega RC} = -j\sqrt{\frac{L}{C}}\cdot\frac{1}{R}\cdot V_i$$

$$V_L = V_i\left(\frac{j\omega L}{j\omega L+\frac{1}{j\omega C}+R}\right) = j\sqrt{\frac{L}{C}}\cdot\frac{1}{R}V_i = -V_c$$

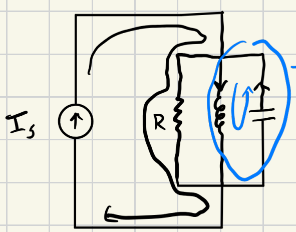
complex conjugate of one another

$V_c + V_L = 0$

(net zero voltage

$$Z_s = R + \underbrace{j\omega L - \frac{j}{\omega C}}_{=0 \text{ if } \omega=\frac{1}{\sqrt{LC}}}$$

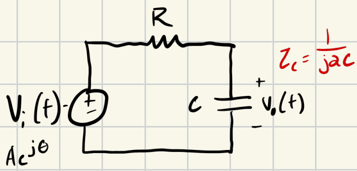
- What if the elements are in parallel : Parallel Resonance



they act as open circuit,
but they still have current.
It goes around just between L and C

Chapter 5 Frequency Response $H(\omega)$ of LTI Systems

Example:



$V_i = Ae^{j\theta}$

$$V_o = V_i\left(\frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}\right) = V_i\left(\frac{1}{j\omega RC + 1}\right)$$

$$= \frac{|V_i| e^{j\angle V_i}}{\sqrt{1^2 + (\omega RC)^2}\; e^{j\tan^{-1}\left(\frac{\omega RC}{1}\right)}}$$

$$V_o = \frac{|V_i|}{\sqrt{1 + (\omega RC)^2}} e^{j\left(\angle V_i - \tan^{-1}\left(\frac{\omega RC}{1}\right)\right)}$$

let $V_i(t) = A\cos(\omega t + \theta)$

Determine $V_{o,ss}(t)$

$$H(\omega) = |H(\omega)| e^{j\angle H(\omega)}$$

Amplitude response: $\frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} = |H(\omega)|$

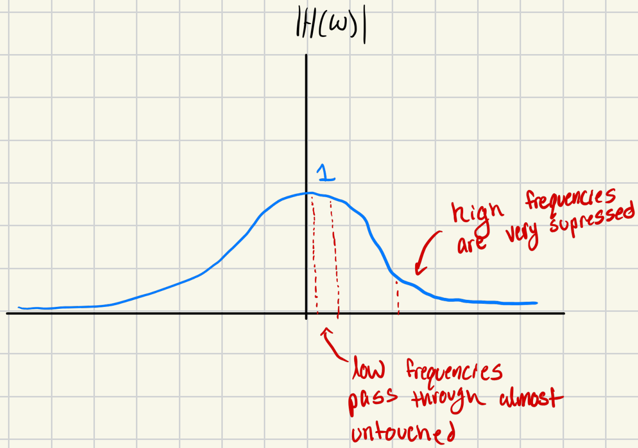
Phase response: $-\tan^{-1}(\omega RC) = \angle H(\omega)$

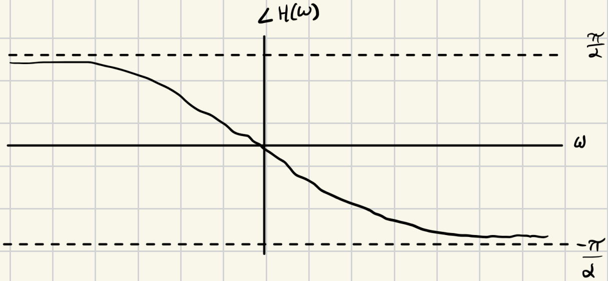
$$V_o = V_i\, H(\omega)$$

output phasor (V_o), input phasor (V_i)

$$V_i = |V_i| e^{j(\angle V_i)}$$

$$V_o = |V_i|\,|H(\omega)|\, e^{j(\angle V_i + \angle H(\omega))}$$





• Recall from example above:

$$|H(\omega)| = \frac{1}{\sqrt{1+(\omega RC)^2}}$$

$$\angle H(\omega) = -\tan^{-1}(\omega RC)$$

Find $V_o(t)$ if $V_i(t) = 2\cos(3t + \frac{\pi}{3})$

$$V_o = |V_i||H(3)|\cos(3t + \frac{\pi}{3} - \tan^{-1}(3RC))$$

Find $V_o(t)$ if $V_i(t) = 3\sin(6t + \frac{\pi}{6})$
$= 3\cos(6t - \frac{\pi}{3})$

$$V_o(t) = 3|H(6)|\cos(6t - \frac{\pi}{3} - \angle H(6))$$

• For LTI systems with
input $f(t) = \mathrm{Re}\{Fe^{j\omega t}\}$
output $y(t) = \mathrm{Re}\{Ye^{j\omega t}\}$

$f(t) \rightarrow \boxed{LTI} \rightarrow y(t)$

$F \rightarrow \boxed{H(\omega)} \rightarrow Y = F \cdot H(\omega)$

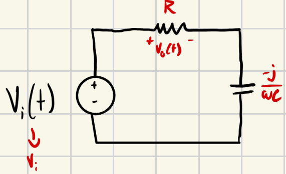
We have:

$$Y = FH(\omega)$$

$$F = |F|e^{j\angle F}$$

$$Y = |F||H(\omega)|e^{j(\angle F - \angle H(\omega))}$$

Example

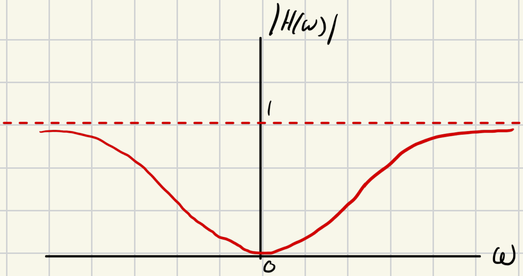


$$V_o = V_i\left(\frac{R}{R+\frac{1}{j\omega C}}\right) = V_i \underbrace{\frac{j\omega RC}{j\omega RC+1}}_{H(\omega)}$$

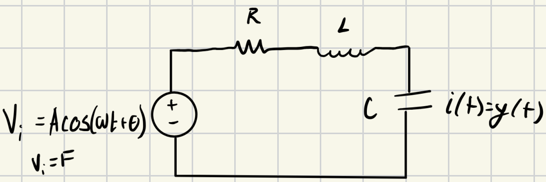
$$|H(\omega)| = \frac{|\omega| RC}{\sqrt{1+\omega^2R^2C^2}}$$

$$\angle H(\omega) = \begin{cases} \frac{\pi}{2} - \tan^{-1}(\omega RC) & \text{if } \omega > 0 \\ \frac{-\pi}{2} - \tan^{-1}(\omega RC) & \text{if } \omega < 0 \\ 0 & \text{if } \omega = 0 \end{cases}$$

Let $V_i(t) = A\cos(\omega t + \theta)$



Example:



$$H(\omega) = \frac{Y}{F} = \frac{I}{V_i}$$

$$I = \frac{V_i}{R + j\omega L + \frac{1}{j\omega C}}$$

Consider the ODE

$$\frac{dy}{dt} + Y = 3f(t) \quad ; \text{ let } f(t) = A\cos(\omega t + \theta)$$

$$j\omega Y + Y = 3F \rightarrow Y = \frac{3F}{j\omega + 1}$$

· Find $H(\omega)$, $|H(\omega)|$, and $\angle H(\omega)$

$$H(\omega) = \frac{Y}{F} \quad ; \quad Y = F H(\omega)$$

$$|H(\omega)| = \frac{3}{\sqrt{1+\omega^2}}$$

$$\underbrace{\angle H(\omega) = 0 - \tan^{-1}(\omega) = -\tan^{-1}(\omega)}$$

Comes from $\frac{Y}{F}$ ← angle of this ← minus angle of this

- General First Order filters and second order filters

  - Low pass

$$|H(0)| = c$$

$$|H(\infty)| = 0$$

$$|H(\omega)| \approx \frac{1}{\sqrt{1+\omega^2}}$$

  - High Pass

$$|H(0)| = 0$$

$$|H(\infty)| = c$$

$$|H(\omega)| \approx \frac{|\omega|}{\sqrt{1+\omega^2}}$$

  - Band Pass (only for disipative systems)

$$|H(0)| = 0$$

$$|H(\infty)| = 0$$

in between $H(t) \neq 0$

$$|H(\omega)| \approx \frac{|\omega|}{\sqrt{\omega^2 + (1-\omega^2)^2}}$$

Note: $H(0)$ is a real number

- Properties of $H(\omega)$ for real valued LTI systems

  - Conjugate Symmetry $H(\omega) = H^*(\omega)$
  - Even Amplitude Response $|H(\omega)| = |H(-\omega)|$
  - Odd phase Response $\angle H(\omega) = -\angle H(-\omega)$
  - Real-Valued DC Response $H(0) \in \mathbb{R}$
  - Steady-State response to complex exponential

$$f(t) = Ae^{j(\omega t+\theta)} \longrightarrow \boxed{H(\omega)} \longrightarrow H(\omega)Ae^{j(\omega t+\theta)}$$

- What if you have multiple frequencies

Let $f(t) = \cos\left(\frac{1}{2}t\right) + 3\cos\left(2t + \frac{\pi}{4}\right) + 2\sin(3t)$ $\qquad y(t) = \frac{\partial}{\partial t} f(t)$

$f(t) \downarrow$ input; ω_1, ω_2, ω_3

$Y = j\omega F$

- Find $y_{ss}(t)$

① $H(\omega)$, $|H(\omega)|$, $\angle H(\omega)$

$H(\omega) = \frac{Y}{F} = j\omega$

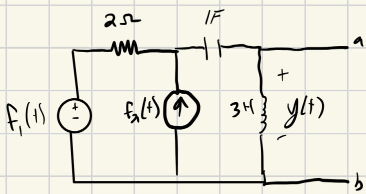
$|H(\omega)| = |\omega|$

② $y_{ss}(t) = 1 \cdot |H(\frac{1}{2})| \cos\left(\frac{1}{2}t + \angle H(\frac{1}{2})\right) + 3|H(2)|\cos\left(2t + \frac{\pi}{4} + \angle H(2)\right) + 2|H(3)|\sin\left(3t + \angle H(3)\right) =$

$= \frac{1}{2}\cos\left(\frac{1}{2}t + \frac{\pi}{2}\right) + 3 \cdot 2\cos\left(2t + \frac{\pi}{4} + \frac{\pi}{2}\right) + 2 \cdot 3\sin\left(3t + \frac{\pi}{2}\right)$

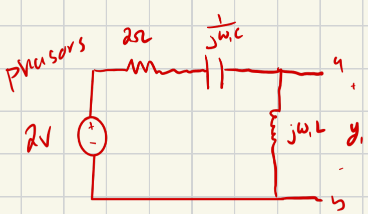
Example:

$f_1(t) = 2\cos\left(\frac{1}{3}t\right)$ V and $f_2(t) = 3\sin(t)$ A



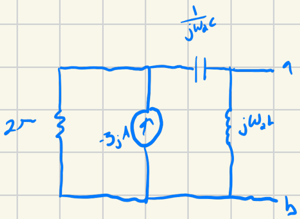
Look at each source individually and add in time domain

$y_{ss}(t) = y_{ss_1}(t) + y_{ss_2}(t)$



$y_1 = \frac{1}{\sqrt{2}} e^{j\frac{3\pi}{4}} \xrightarrow{\text{convert to time domain}} = \frac{1}{\sqrt{2}}\cos\left(\frac{1}{3}t + \frac{3\pi}{4}\right)$ V

Only $f_2(t)$



$Y_2 = \frac{9}{\sqrt{2}} e^{-j\frac{\pi}{4}} \rightarrow y_{ss_2} = \frac{9}{\sqrt{2}} \cos(t - \frac{\pi}{4}) V$

$$y_{ss} = y_{ss_1} + y_{ss_2} = \frac{1}{\sqrt{2}} \cos\left(\frac{1}{3}t + \frac{3\pi}{4}\right) + \frac{9}{\sqrt{2}} \cos\left(t - \frac{\pi}{4}\right)$$

Looking back at Schuh's lectures

Frequency Response

$$f(t) \rightarrow \boxed{\text{system}} \rightarrow y(t)$$

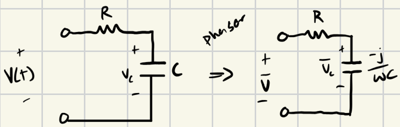
$$f(t) = F_p \cos(\omega t) \longleftrightarrow \bar{F} = F_p \angle 0^\circ$$

$$F_p \angle 0^\circ \rightarrow \boxed{\text{system}} \rightarrow Y_p \angle \theta$$

$H(\omega) \longleftarrow$ frequency response (complex) *

$$\bar{Y} = H(\omega)\bar{F} \Rightarrow H(\omega) = \frac{\bar{Y}}{\bar{F}}$$

Ex: RC Circuit



$$\bar{V}_c = \left(\frac{\frac{-j}{\omega C}}{R - j/\omega C}\right)\bar{V} \Rightarrow \left(\frac{\frac{-j}{\omega C}(R + j/\omega C)}{R^2 + (\frac{1}{\omega C})^2}\right)\bar{V} = \left(\frac{\frac{1}{(\omega C)^2} - j\frac{R}{\omega C}}{R^2 + \frac{1}{\omega^2}(\frac{1}{C^2})}\right)\bar{V} \cdot \frac{\omega^2}{\omega^2} \left(\frac{\frac{1}{C^2} - j\frac{\omega R}{C}}{\omega^2 R^2 + \frac{1}{C^2}}\right)\bar{V} =$$

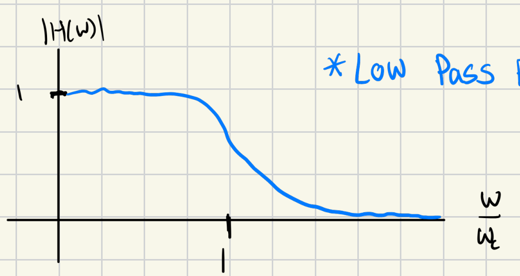
$$= \left(\frac{\frac{1}{R^2C^2} - \frac{j\omega}{RC}}{\omega^2 + \frac{1}{(RC)^2}}\right)\bar{V} \qquad \omega_c = \frac{1}{RC} \Rightarrow \underbrace{\left(\frac{\omega_c^2 - j\omega\omega_c}{\omega^2 + \omega_c^2}\right)}_{H(\omega)}\bar{V}$$

$$\rightarrow \quad H(\omega) = \frac{\omega_c^2 - j\omega\omega_c}{\omega_c^2 + \omega^2}$$

$$|H(\omega)| = \sqrt{Re^2 + Im^2} = \sqrt{\left(\frac{\omega_c^2}{\omega_c^2 + \omega^2}\right)^2 + \left(\frac{-\omega\omega_c}{\omega_c^2 + \omega^2}\right)^2} = \sqrt{\frac{\omega_c^4 + \omega^2\omega_c^2}{(\omega_c^2 + \omega^2)^2}} \sqrt{\frac{\omega_c^2(\omega^2 + \omega_c^2)}{(\omega_c^2 + \omega^2)^2}} = \sqrt{\frac{\omega_c^2}{\omega_c^2 + \omega^2}} =$$

$$|H(\omega)| = \sqrt{\frac{1}{1 + (\frac{\omega}{\omega_c})^2}}$$

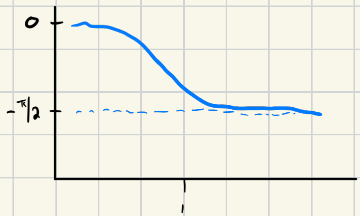
$$\frac{\omega}{\omega_c} \rightarrow 0, \quad |H(\omega)| \rightarrow 1 \qquad \frac{\omega}{\omega_c} \rightarrow \infty, \quad |H(\omega)| \rightarrow 0$$



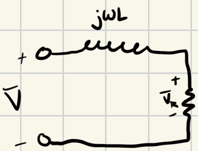
Low Pass Filter

$$\theta(\omega) = \tan^{-1}\left(\frac{Im}{Re}\right) \to \theta(\omega) = \tan^{-1}\left(\frac{-\omega\omega_c}{\omega_c^2}\right) = \tan^{-1}\left(\frac{-\omega}{\omega_c}\right)$$

$$\frac{\omega}{\omega_c} \to 0,\ \theta \to 0 \qquad \frac{\omega}{\omega_c} \to \infty,\ \theta \to \frac{-\pi}{2}$$

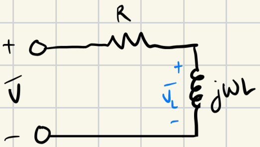


RL Low Pass



$\omega_c = \frac{R}{L}$

R-L High Pass



H(ω)

$$\bar{V}_L = \left(\frac{j\omega L}{R + j\omega L}\right)\bar{V}$$

$$= \left(\frac{j\omega L(R - j\omega L)}{R^2 + (\omega L)^2}\right)\bar{V} = \frac{j\omega LR + \omega^2 L^2}{R^2 + \omega^2 L^2}\bar{V} = \frac{j\omega\left(\frac{L}{R}\right) + \omega^2\left(\frac{L}{R}\right)^2}{1 + \omega^2\left(\frac{L}{R}\right)^2}\bar{V}$$

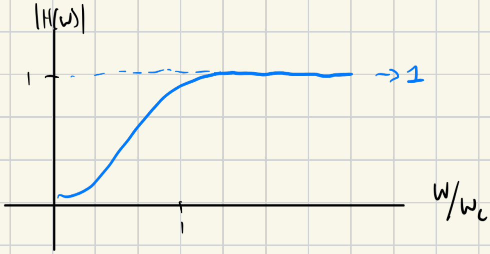
$$= \underbrace{\frac{j\left(\frac{\omega}{\omega_c}\right) + \left(\frac{\omega}{\omega_c}\right)^2}{1 + \left(\frac{\omega}{\omega_c}\right)^2}}_{H(\omega)}\bar{V}$$

$$|H(\omega)| = \sqrt{\left(\frac{\left(\frac{\omega}{\omega_c}\right)^2}{1 + \left(\frac{\omega}{\omega_c}\right)^2}\right)^2 + \left(\frac{\left(\frac{\omega}{\omega_c}\right)}{1 + \left(\frac{\omega}{\omega_c}\right)^2}\right)^2}$$

$$= \sqrt{\frac{\left(\frac{\omega}{\omega_c}\right)^2\left(1 + \left(\frac{\omega}{\omega_c}\right)^2\right)}{\left(1 + \left(\frac{\omega}{\omega_c}\right)^2\right)^2}} = \sqrt{\frac{\left(\frac{\omega}{\omega_c}\right)^2}{1 + \left(\frac{\omega}{\omega_c}\right)^2}}$$

$$\frac{\omega}{\omega_c} \to 0,\ |H(\omega)| \to 0$$

$$\frac{\omega}{\omega_c} \to \infty,\ |H(\omega)| \to 1$$



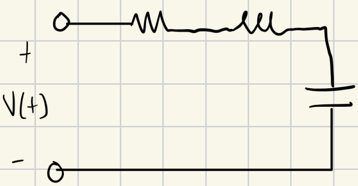
High Pass Filter

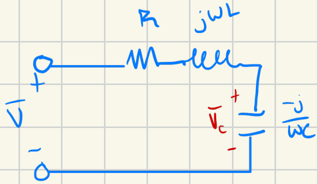
$$\theta(\omega) = \tan^{-1}\left(\frac{Im}{Re}\right) = \tan^{-1}\left(\frac{\omega/\omega_c}{(\omega/\omega_c)^2}\right) = \tan^{-1}\left(\frac{\omega_c}{\omega}\right)$$

$\omega \to \infty$; $\theta(\omega) \to 0$

$\omega \to 0$; $\theta(\omega) \to \frac{\pi}{2}$

RLC





$$H(\omega) = \frac{-j/\omega C}{R + j\omega L - \frac{j}{\omega C}}$$

* Band Pass Filter *

More Schun lectures:

- LTI system response to cosinusoids & Multifrequency inputs

Ex)

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \frac{df}{dt}$$

Assuming $f(t) = F_p\cos(\omega t)$

$$\bar{F} = (F_p e^{j0})e^{j\omega t}$$

phasors

$$(j\omega)^2\bar{Y} + j4\omega\bar{Y} + 4\bar{Y} = j\omega\bar{F}$$

$$\frac{d\bar{F}}{dt} = j\omega \underbrace{F_p e^{j0} e^{j\omega t}}_{\bar{F}}$$

$$= j\omega\bar{F}$$

$$\bar{Y}(4-\omega^2+j4\omega) = j\omega\bar{F}$$

$$\bar{Y} = \underbrace{\left(\frac{j\omega}{4-\omega^2+j4\omega}\right)}_{H(\omega)}\bar{F}$$

$$H(\omega) = \frac{j\omega}{4-\omega^2+j4\omega}$$

$$H(\omega) = \frac{j\omega}{4-\omega^2+j4\omega}\cdot\frac{(4-\omega^2-j4\omega)}{(4-\omega^2-j4\omega)} = \frac{4\omega^2+j\omega(4-\omega^2)}{(4-\omega^2)^2+16\omega^2}$$

$$|H(\omega)| = \sqrt{Re^2\cdot Im^2} = \sqrt{\frac{(4\omega^2)^2+\omega^2(4-\omega^2)^2}{((4-\omega^2)^2+16\omega^2)^2}} = \sqrt{\frac{16\omega^4+\omega^2(4-\omega^2)^2}{((4-\omega^2)^2+16\omega^2)^2}} = \sqrt{\frac{\omega^2(16\omega^2+(4-\omega^2)^2)}{(16\omega^2+(4-\omega^2))^2}} = \sqrt{\frac{\omega^2}{16\omega^2+(4-\omega^2)^2}} = |H(\omega)|$$

$\omega \to 0,\ |H(\omega)| \to 0$

$\omega \to \infty,\ |H(\omega)| \to 0$

* Band Pass Filter *

Decibels (dB)

$$|H(\omega)|_{dB} = 20\log_{10}(|H(\omega)|)$$

Low Pass Filter $|H(\omega)| = \frac{1}{\sqrt{1+\omega^2}}$

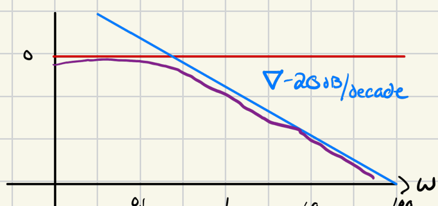
$$|H(\omega)|_{dB} = 20\log_{10}([1+\omega^2]^{-1/2})$$

$$= -10\log_{10}(1+\omega^2)$$

$\omega \to 0,\ |H(\omega)|_{dB} \to 0$

$\omega \to \infty,\ |H(\omega)|_{dB} \to -10\log_{10}(\omega^2) \to -20\log_{10}(\omega)$

For everytime ω increases by 10x $|H(\omega)|$ decreases by 20dB



High Pass Filter

Assume $|H(\omega)| = \sqrt{\dfrac{\omega^2}{1+\omega^2}}$

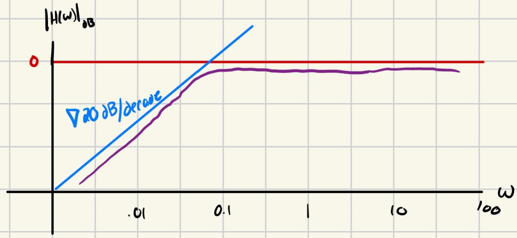
$$|H(\omega)|_{dB} = 20\log_{10}\left(\left[\frac{\omega^2}{1+\omega^2}\right]^{-1/2}\right)$$

$$= 10\log_{10}\left(\frac{\omega^2}{1+\omega^2}\right)$$

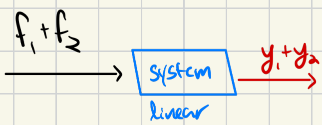
$$= 10\left(\log_{10}(\omega^2) - \log_{10}(1+\omega^2)\right)$$

$\omega \to \infty$; $|H(\omega)|_{dB} \to 10(\log(\omega^2) - \log(\omega^2)) \to 0$

$\omega \to 0$; $|H(\omega)|_{dB} \to 10(\log(\omega^2) - \log(1)) \to 10\log(\omega^2) \to 20\log_{10}(\omega)$ ($\log(1) \to 0$)



Multiple Frequencies



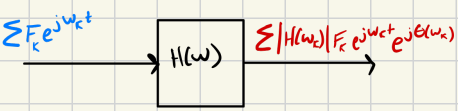
$\bar{F}_1 = F_{P_1} e^{j0} e^{j\omega_1 t}$

$\bar{F}_2 = F_{P_2} e^{j0} e^{j\omega_2 t}$

$\bar{Y}_1 = |H(\omega_1)| F_{P_1} e^{j0} e^{j\omega_1 t} e^{j\theta(\omega_1)}$

$\bar{Y}_2 = |H(\omega)| F_{P_2} e^{j0} e^{j\omega_2 t} e^{j\theta(\omega_2)}$

$Y = Y_1 + Y_2$



Ex) let $H(\omega) = \frac{1}{1+j\omega} = \frac{1-j\omega}{1+\omega^2} \rightarrow |H(\omega)| = \frac{1}{\sqrt{1+\omega^2}} \rightarrow \theta(\omega) = \tan^{-1}(-\omega)$

let $f(t) = 1\cos(t) + 0.1\cos(10t) + 0.01\cos(100t)$

ω	F	$\lvert H(\omega)\rvert$	$\theta(\omega_k)$
1	1	$\frac{\sqrt{2}}{2} \approx 0.707$	$\tan^{-1}(-1) = -45° = -\frac{\pi}{4}$
10	0.1	$\frac{1}{\sqrt{101}} \approx 0.1$	$\tan^{-1}(-10) = -84.29°$
100	0.1	$\frac{1}{\sqrt{10001}} \approx 0.03$	$\tan^{-1}(-100) = -89.43°$

$y(t) = 0.7(1)\cos(t - 45°) + 0.1(0.1)\cos(10t - 84.29°) + 0.03(0.01)\cos(100t - 89.43°)$

Periodic Signals

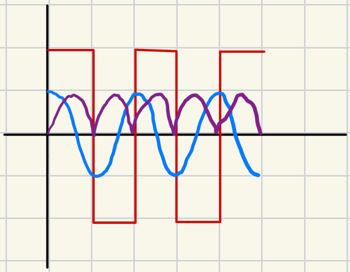
$\exists\ f(t) = f(t - t_0)$

$t \neq 0$, $f(t)$ is periodic

$K \in$ integers

$f(t) = f(t - Kt_0)$

periodic functions repeat in time



T: smallest t_0 for which a signal repeats

Ex) $\cos(t) \rightarrow T = 2\pi$ sec

$\cos(3t) \rightarrow T = \frac{2\pi}{\omega} \rightarrow T = \frac{2\pi}{3}$ sec

$f(t) = \cos(2t) + \cos(3t)$

$T_1 = \frac{2\pi}{2} = \pi$ sec $\quad T_2 = \frac{2\pi}{3}$ sec

Overall: LCM T_1 & T_2

$T_{tot} = 2\pi$

$T_1 \cdot (2) = 2\pi$

$T_2 \cdot (3) = 2\pi$

Test for if sum of periodic signals is periodic

$\frac{\omega_1}{\omega_2}$ = rational Number (Fraction) → will be periodic

Ex) $\omega_1 = 2$, $\omega_2 = 3$

$\frac{\omega_1}{\omega_2} = \frac{2}{3}$ periodic

Ex) $f(t) = \cos(t) + \cos(\sqrt{2}t)$

$\omega_1 = 1$ $\omega_2 = \sqrt{2}$

$\frac{\omega_1}{\omega_2} = \frac{1}{\sqrt{2}}$ non periodic

Fourier Series

* Periodic functions can be approximated by sums of sines + cosines *

Complex Exponential Form ($e^{j\theta} = \cos\theta + j\sin\theta$)

complex #

fundemental frequency

$$f(t) = \sum_{\substack{k=-\infty \\ k \in \text{integers}}}^{\infty} F_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T} \qquad k\omega_0 = k^{th} \text{ harmonic}$$

$F_k \in \mathbb{C}$ ($F_k \in$ complex #'s)

Trigonometric Form

$$F_k e^{jk\omega_0 t} = F_k \cos(k\omega_0 t) + jF_k \sin(k\omega_0 t)$$

$$f(t) = \sum_{k=-\infty}^{\infty} F_k \cos(k\omega_0 t) + jF_k \sin(k\omega_0 t)$$

$k=0$: $F_0 \cos(0) + jF_0 \sin(0)$

$= F_0$

$$f(t) = F_0 + \sum_{k=1}^{\infty} F_k \cos(k\omega_0 t) + jF_k \sin(k\omega_0 t) + F_{-k}\cos(-k\omega_0 t) + jF_{-k}\sin(-k\omega_0 t)$$

Since cos is an even function, positives and negatives are equal and sin functions are odd: $\sin(-1) = -\sin(1)$

$$= F_0 + \sum_{k=1}^{\infty} (F_k + F_{-k})\cos(k\omega_0 t) + j(F_k - F_{-k})\sin(k\omega_0 t)$$

We can define:

$a_0 = 2F_0$

$a_k = F_k + F_{-k}$

$b_n = j(F_k - F_{-k})$

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

Compact Form (non-complex functions only)

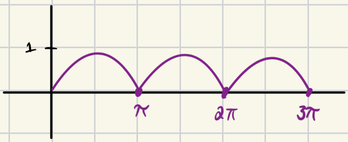
$C_0 = 2|F_0|$

$C_k = 2|F_k|$

$\theta_k = \tan^{-1}\left(\frac{Im(F_k)}{Re(F_k)}\right)$

$$f(t) = \frac{C_0}{2} + \sum_{k=1}^{\infty} C_k \cos(k\omega_0 t + \theta_k)$$

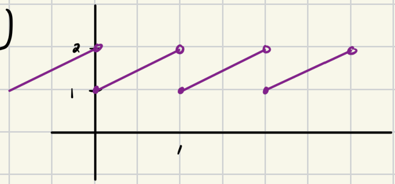
Ex) $f(t) = |\sin(t)|$



$T = \pi$

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2 \frac{rad}{sec}$

Ex)



$T = 1 sec$

$\omega_0 = \frac{2\pi}{T} \Rightarrow \omega_0 = 2\pi \ rad/sec$

Fourier Series + its forms

1) Sinusoidal input (Frequency Response)
2) multiple sinusoidal inputs (Frequency Response + Linearity)
3) Periodic Signals
4) Fourier Series - periodic signal writen as sines + cosines

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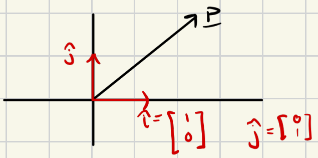
Linear Algebra



Basis: $[\hat{i}, \hat{j}]$ ← orthogonal (no mixing of information)

$$a = \frac{\underline{P} \cdot \hat{i}}{|\hat{i}|^2} \qquad b = \frac{\underline{P} \cdot \hat{j}}{|\hat{j}|^2}$$

$$\underline{P} = a\hat{i} + b\hat{j}$$

* functions are vectors in ∞ dimensional space

Inner Product

- Extension of the dot product to ∞-dimensional spaces

$$\underbrace{\langle f(t), g(t) \rangle}_{\text{inner product of } f(t) \text{ and } g(t)} = \int_0^T f(t) g^*(t)\, dt$$

$\langle f(t), g(t) \rangle = 0 \rightarrow$ the functions are orthogonal

Ex| $e^{jw_0 t}, e^{j2w_0 t}$ -> Find inner product

$$\langle e^{jw_0 t}, e^{j2w_0 t} \rangle = \int_0^T e^{jw_0 t} e^{-j2w_0 t}\, dt$$

$$= \int_0^T e^{-jw_0 t}\, dt$$

$$= \int_0^T \cos(w_0 t) - j\sin(w_0 t)\, dt$$

$$= \frac{1}{w_0}\sin(w_0 t) + j\frac{1}{w_0}\cos(w_0 t) \Big|_0^T$$

$$= \frac{1}{\omega_0}\left(\sin(\omega_0 T) - \sin(0) + j\left[\cos(\omega_0 T) - \cos(0)\right]\right)$$

$\omega_0 = \frac{2\pi}{T}$

$\omega_0 T = 2\pi$

$$= \frac{1}{\omega_0}\left(\sin(2\pi) - \sin(0) + j\left[\cos(2\pi) - \cos(0)\right]\right)$$

$$= \frac{1}{\omega_0}\left(0 + j(1-1)\right) = \frac{1}{\omega_0}(0 + j0) = 0$$

orthogonal

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 T}$$

$$\langle f(t), e^{jm\omega_0 T}\rangle = \int_0^T \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} e^{-jm\omega_0 t}\, dt$$

$$= \sum_{n=-\infty}^{\infty}\left(F_n \int_0^T e^{j\omega_0(n-m)t}\, dt\right)$$

whichever F_n we care about

$n \neq m$

$$\int_0^T e^{j(n-m)\omega_0 t}\, dt = 0$$

$n = m$

$$\int_0^T e^{j(n-n)\omega_0 t}\, dt = \int_0^T e^{j0}\, dt = \int_0^T dt = T$$

$n = m$

$$\langle f(t), e^{jm\omega_0 t}\rangle = T F_m$$

$$\boxed{F_m = \frac{1}{T}\int_0^T f(t) e^{-jm\omega_0 t}\, dt}$$

Ex) $f(t) = \cos^2(t) = \frac{1}{2}(1 + \cos(2t))$

$= \frac{1}{2} + \frac{1}{2}\cos(2t)$

① Start with finding F_0

$T = \pi$

$$F_0 = \frac{1}{T}\int_0^T f(t) e^{j0}\, dt = \underbrace{\frac{1}{T}\int_0^T f(t)\, dt}_{\text{Average value of } f(t)} = \frac{1}{\pi}\int_0^{\pi}\left(\frac{1}{2} + \frac{1}{2}\cos(2t)\right) dt = \frac{1}{\pi}\left[\frac{1}{2}t + \frac{1}{2}\left(\frac{1}{2}\sin(2t)\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{2}(\pi - 0) + \frac{1}{4}(\sin(2\pi) - \sin(0))\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2}\right] \Rightarrow \boxed{F_0 = \frac{1}{2}}$$

② Find F_n $(n \neq 0)$

$$F_n = \frac{1}{\pi}\int_0^{\pi} \cos^2(t) e^{jn\omega_0 t}\,\partial t \Rightarrow \frac{1}{\pi}\int_0^{\pi}\cos^2(t) e^{-j2nt}\,\partial t \Rightarrow \frac{1}{\pi}\int_0^{\pi}\left(\frac{1}{2}+\frac{1}{2}\cos(2t)\right)\left(\cos(2nt) - j\sin(2nt)\right)\partial t$$

$\omega_0 = \frac{2\pi}{T} = 2$

$$= \frac{1}{\pi}\int_0^{\pi}\Big(\underbrace{\tfrac{1}{2}\cos(2nt) - j\tfrac{1}{2}\sin(2nt)}_{=0 \text{ (after integration)}} + \tfrac{1}{2}\cos(2t)\cos(2nt) - j\tfrac{1}{2}\cos(2t)\sin(2nt)\Big)\partial t$$

$\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$

$\cos(A)\sin(B) = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$

$$= \frac{1}{\pi}\left(\int_0^{T}\frac{1}{4}\left(\cos(2(1+n)t) + \cos(2(1-n)t)\right) + \frac{1}{4}\left(\sin(2(1+n)t) + \sin(2(1-n)t)\right)\partial t\right.$$

only 2 values of n where the integral $\neq 0$

$n=-1 \Rightarrow \frac{1}{\pi}\int_0^{T}\frac{1}{4}(1)\,\partial t \Rightarrow \frac{1}{\pi}\left(\frac{1}{4}t\right)\Big|_0^{\pi} = \frac{1}{4}$

$n=1 \Rightarrow \frac{1}{\pi}\int_0^{T}\frac{1}{4}(1)\,\partial t \Rightarrow \frac{1}{\pi}\left(\frac{1}{4}t\right)\Big|_0^{\pi} = \frac{1}{4}$

$F_{-1} = \frac{1}{4}$ $\quad F_1 = \frac{1}{4}$

In terms of complex exp.

$$f(t) = \frac{1}{2} + \frac{1}{4}e^{-j2t} + \frac{1}{4}e^{j2t}$$

Trig Form:

$a_0 = 2F_0$ $\qquad a_0 = 2\left(\frac{1}{2}\right) = 1$

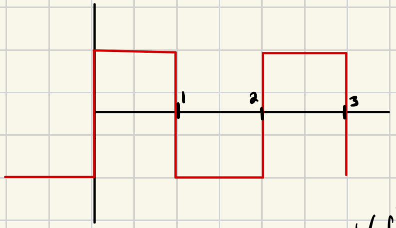
$a_n = F_n + F_{-n}$ $\qquad a_1 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$b_n = F_n - F_{-n}$ $\qquad b_1 = \frac{1}{4} - \frac{1}{4} = 0$

$$f(t) = \frac{1}{2} + \frac{1}{2}\cos(2t)$$

✓

Running through examples of Fourier



$T = 2s$

$W_0 = \frac{2\pi}{T} = \pi$

$F_0 = \frac{1}{T}\int_0^T f(t)\,dt \Rightarrow$

$\frac{1}{2}\int_0^2 f(t)\,dt$

must break this up because values are different @ dif locations

$= \frac{1}{2}\left(\int_0^1 1\,dt + \int_1^2 -1\,dt\right)$

$= \frac{1}{2}\left(t\Big|_0^1 - t\Big|_1^2\right) = \frac{1}{2}(1-(2-1)) = 0$ ✓ $\quad F_0 = 0$

$F_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}\,dt = \frac{1}{2}\int_0^2 f(t)e^{-jn\pi t}\,dt = \frac{1}{2}\left(\int_0^1 1e^{-jn\pi t}\,dt + \int_1^2 -1e^{-jn\pi t}\,dt\right) = \frac{1}{2}\left(\frac{1}{-jn\pi}e^{-jn\pi t}\Big|_0^1 + (-1)\frac{1}{-jn\pi}e^{-jn\pi t}\Big|_1^2\right)$

$\frac{1}{2}\left(\frac{1}{-jn\pi}\left(e^{-jn\pi}-1\right) - \frac{1}{-jn\pi}\left(e^{-j2\pi n} - e^{-jn\pi}\right)\right) = \frac{1}{2}\left(\frac{1}{jn\pi}\left(e^{-jn\pi}-1\right) + \frac{1}{-jn\pi}\left(e^{-jn\pi} - \underbrace{e^{-j2\pi n}}_{1}\right)\right)$

$\frac{1}{2}\left(\frac{1}{-jn\pi}\left(e^{-jn\pi}-1\right) + \frac{1}{-jn\pi}\left(e^{-jn\pi}-1\right)\right) = \frac{1}{-jn\pi}\left(e^{-jn\pi}-1\right)$

$e^{-jn\pi} = \cos(n\pi) - j\sin(n\pi)$
$n \in \mathbb{Z}$

$\frac{1}{-jn\pi}\left((-1)^n - 1\right)$ $\quad n$: even $\Rightarrow F_n = 0$

odd $\Rightarrow F_n = \frac{-2}{jn\pi}\cdot\frac{j}{j} = \frac{-j2}{n\pi}$

n	$\cos(n\pi)$
1	-1
2	1
3	-1

if n is odd: $\cos(n\pi) = -1$
" " even: " $= 1$

$\cos(n\pi) = (-1)^n$

n: even, $F_n = 0$
n: odd, $F_n = \frac{-j2}{n\pi}$

$$\boxed{f(t) = 0 + \sum_{\substack{n=-\infty \\ n\neq 0 \\ n \text{ odd}}}^{\infty} \frac{-j2}{n\pi}e^{jn\pi t}}$$

Ex)

$$f(t) = \frac{t}{2} \quad 0 \le t < 2$$

$T = 2s$

$\omega_0 = \frac{2\pi}{2s} = \pi$

① Find F_0

$$F_0 = \frac{1}{2}\int_0^2 \frac{t}{2}\,dt \Rightarrow \frac{1}{2}\left(\frac{t^2}{4}\Big|_0^2\right) = \frac{1}{2}\left(\frac{4}{4} - 0\right) = \frac{1}{2}$$

② Find F_n

$e^{-jn2\pi} = 1$

$$F_n = \frac{1}{2}\int_0^2 \frac{t}{2} e^{-jn\pi t}\,dt = \frac{1}{4}\int_0^2 t e^{-jn\pi t}\,dt = \frac{1}{4}\left(\frac{t}{-jn\pi}e^{-jn\pi t}\Big|_0^2 - \underbrace{\int_0^2 \frac{1}{-jn\pi}e^{-jn\pi t}\,dt}_{=0}\right) = \frac{1}{4}\left(\frac{2}{-jn\pi}e^{-j2\pi n} - 0\right) = \frac{1}{2}\left(\frac{1}{-jn\pi}\right)e^{-jn2\pi}$$

let $u = t$ $\quad v = \frac{1}{-jn\pi}e^{-jn\pi t}$

$du = dt$ $\quad dv = e^{-jn\pi t}$

$$= \frac{1}{2}\left(\frac{1}{-jn\pi}\right) \Rightarrow \frac{j}{2n\pi} = F_n$$

$$f(t) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{j}{2n\pi} e^{jn\pi t}$$

Ex) $f(t) = |\sin(t)|$

$T = \pi s$

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2 \text{ rad/sec}$

① Find F_0

$$\frac{1}{\pi}\int_0^{\pi} \sin(t)\,dt = \frac{1}{\pi}\left(-\cos(t)\Big|_0^{\pi}\right) = \frac{1}{\pi}(-(-1)+1) = \frac{2}{\pi} = F_0$$

② Find F_n

$$F_n = \frac{1}{\pi}\int_0^{\pi} \sin(t) e^{-jn2t}\,dt = \frac{1}{\pi}\int_0^{\pi}\left(\frac{e^{jt}-e^{-jt}}{j2}\right)e^{-j2nt}\,dt = \frac{1}{j2\pi}\int_0^{\pi} e^{jt}e^{-j2nt} - e^{-jt}e^{-j2nt}\,dt = \frac{1}{j2\pi}\int_0^{\pi} e^{j(1-2n)t} - e^{-j(1+2n)t}\,dt$$

$$F_n = \frac{2}{\pi}\left[\frac{1}{1-4n^2}\right]$$

$$\boxed{f(t) = \frac{2}{\pi} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{2}{\pi}\left(\frac{1}{1-4n^2}\right)e^{j2nt}}$$

- Exponential Fourier-Series

· Let $f(t) = \underset{\text{DC offset}}{1} + 2\sin(t) + \sin^2(\frac{5}{4}t)$

· Determine its exponential Fourier Series

Find signal must look like $\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$

$\sin^2(x) = \frac{1}{2}(1-\cos(2x))$

$f(t) = 1 + 2\sin(t) + \frac{1}{2} - \frac{1}{2}\cos(\frac{5}{2}t) = \frac{3}{2} + 2\sin(t) - \frac{1}{2}\cos(\frac{5}{2}t)$

$\omega_0 = GCD(1, \frac{5}{2}) = \frac{1}{2}$ rad/s

$$= \frac{3}{2} + 2\left(\frac{e^{jt} - e^{-jt}}{2j}\right) - \frac{1}{2}\left(\frac{e^{j\frac{5}{2}t} + e^{-j\frac{5}{2}t}}{2}\right) =$$

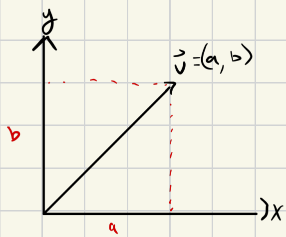
$$\frac{3}{2} + \frac{1}{j}e^{jt} - \frac{1}{j}e^{-jt} - \frac{1}{4}e^{j\frac{5}{2}t} - \frac{1}{4}e^{-j\frac{5}{2}t} = f(t)$$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\overset{1/2}{\omega_0} t} = \underset{3/2}{F_0} + \underset{0}{F_1} e^{j\frac{1}{2}t} + \underset{0}{F_{-1}} e^{-j\frac{1}{2}t} + \underset{0}{F_2} e^{jt} + \underset{0}{F_{-2}} e^{-jt} + \underset{0}{F_3} e^{j\frac{3}{2}t} + \underset{0}{F_{-3}} e^{-j\frac{3}{2}t} + \dots + \underset{1/4}{F_5} e^{j\frac{5}{2}t} + \underset{1/4}{F_{-5}} e^{-j\frac{5}{2}t} + \dots$$

· What if $f(t)$ is cosinusoidal or a complex exponential? Just match coefficients

· If not, but $f(t)$ is periodic, will use $\{e^{jn\omega_0 t}\}$ as a basis to project onto.

a is a projection of $\vec{v}$ onto $\vec{x}$
b is a projection of $\vec{v}$ onto $\vec{y}$.



$$F_n = \langle f(t), e^{jn\omega_0 t} \rangle = \frac{1}{T}\int_T f(t) e^{-jn\omega_0 t}\, dt$$

0 to T
or $-\frac{T}{2}$ to $\frac{T}{2}$

- $\{e^{jn\omega_0 t}\}$ is an orthonormal basis:

$$\langle e^{jn\omega_0 t}, e^{jm\omega_0 t}\rangle = \begin{cases} 1 & n=m \\ 0 & \text{else} \end{cases}$$

$$\langle e^{jn\omega_0 t}, e^{jm\omega_0 t}\rangle = \frac{1}{T}\int_T e^{jn\omega_0 t}\cdot e^{-jm\omega_0 t}\,dt = \frac{1}{T}\int_T e^{j(n-m)\omega_0 t}\,dt =$$

$$\frac{1}{T}\frac{e^{j(n-m)\omega_0 t}}{j(n-m)\omega_0}\Bigg|_0^T = \frac{1}{T}\frac{(e^{j(n-m)\overset{\to 2\pi}{\omega_0 T}}-1)}{j(n-m)\omega_0} = \frac{1}{T}\left(\frac{\overset{\to 0}{e^{j(n-m)2\pi}}-1}{j(n-m)\omega_0}\right) = 0$$

if $n \neq m$:

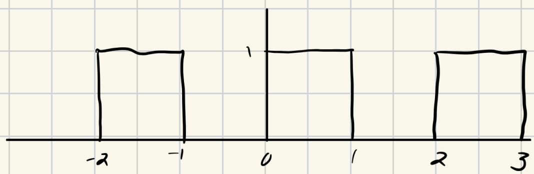
if $n=m$:

$$\langle e^{jn\omega_0 t}, e^{jn\omega_0 t}\rangle = \frac{1}{T}\int_0^T e^{jn\omega_0 t}\cdot e^{-jn\omega_0 t}\,dt = \frac{1}{T}\int_0^T dt = 1$$

Ex:

- Let $f(t) = \begin{cases} 1 & t\in[0,1) \\ 0 & t\in[1,2) \end{cases}$

have period $= 2s$



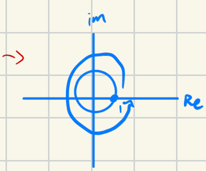
Determine exponential Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\overset{\pi}{\omega_0} t}$$

$$F_n = \frac{1}{T}\int_T f(t)e^{-jn\pi t}\,dt = \frac{1}{2}\left[\int_0^1 (1)e^{-jn\pi t}\,dt + \int_1^2 \overset{0}{(0)}e^{-jn\pi t}\,dt\right] \quad (n \text{ is an integer})$$

$$= \frac{1}{2}\frac{e^{-jn\pi t}}{-jn\pi}\Bigg|_0^1 = \frac{-1}{2jn\pi}\left(e^{-jn\pi}-1\right)$$

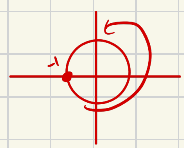
$e^{-jn\pi}$ = 1 if n is even, -1 if n is odd



$n \neq 0$

$$F_0 = \frac{1}{T}\int_T f(t)e^{-j0\pi t} = \frac{1}{2}\int_T f(t)\,dt = \frac{1}{2} \leftarrow \text{average of } f(t)$$

DC Term



$$F_n = \begin{cases} \frac{-1}{jn2\pi}(1-1) = 0 & n \text{ even} \\ & n \neq 0 \\ \frac{-1}{jn2\pi}(-1-1) = \frac{1}{jn\pi} & n \text{ is odd} \\ \frac{1}{2} & n = 0 \end{cases}$$

$$f(t) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{1}{jn\pi} e^{jn\pi t}$$

Terminology:

F_0 is the DC component

F_1 corresponds to the fundamental frequency term

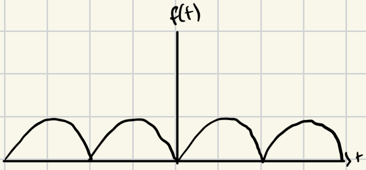
F_n corresponds to the n^{th} harmonic

- Existence of Fourier series representation

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

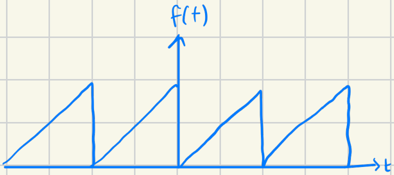
$$f(t) = F_0 + F_1 e^{j\omega_0 t} + F_{-1} e^{-j\omega_0 t} + \dots$$

① $f(t) = |\sin(t)|$



> Note: With Fourier transform, the transform does not know where discontinuities exist. This is known as Gibb's phenomenon

②



- For existence of the Fourier series, we need:

  ① Absolute integrability of f(t) over period T

  $\int_T |f(t)|dt < \infty$ has to be finite

  ② Ploftability

  over one period
  - Finite # of maxima and minima
  - Finite number of finite discontinuities
    (series converges to midpoint at discontinuity)

- Fourier Series Forms
  - Exponential Fourier series

    $$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

  - Trigonometric

    $$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$a_0 = 2F_0 \qquad a_n = F_n + F_{-n} \qquad b_n = j(F_n - F_{-n})$

$F_0 = \frac{a_0}{2} \qquad F_n = \frac{a_n - jb_n}{2} \qquad F_{-n} = \frac{a_n + jb_n}{2}$

- Compact Fourier Series, for real valued functions only

$$f(t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n(\cos(n\omega_0 t + \theta_n))$$

$C_0 = 2F_0 \qquad C_n = 2|F_n| \qquad \theta_n = \angle F_n$

$F_0 = \frac{C_0}{2} \qquad F_n = \frac{C_n}{}$

$$f(t) = \frac{1}{2} + \sum_{n=-\infty}^{\infty} \frac{1}{jn\pi} e^{jn\pi t} \longrightarrow \text{trig FS of } f(t)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{1}{jn\pi} e^{jn\pi t} + \frac{1}{j(-n)\pi} e^{j(-n)\pi t} \right) =$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi} \left(\frac{e^{jn\pi t} - e^{-jn\pi t}}{j} \right) =$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \left(\frac{e^{jn\pi t} - e^{-jn\pi t}}{j2} \right) =$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(n\pi t) \longrightarrow$$

$$a_n = \begin{cases} 1 & n=0 \\ 0 & \text{else} \end{cases}$$

$$b_n = \begin{cases} 0 & n=\text{even} \\ \frac{2}{n\pi} & n=\text{odd} \end{cases}$$

Exponential F.S. $\rightarrow$ compact F.S.

$$F_n = \begin{cases} \frac{1}{2} & n=0 \\ \frac{1}{jn\pi} & n \text{ odd} \\ 0 & \text{else} \end{cases}$$

$$C_0 = 2F_0 = 1$$

$$C_n = 2|F_n| = 2\left|\frac{1}{jn\pi}\right|$$

$$= \begin{cases} \frac{2}{n\pi} & n \text{ odd} \\ 0 & \text{even}, \ n \neq 0 \end{cases}$$

$$\theta_n = \angle F_n = \begin{cases} -\pi/2 & n \text{ odd} \\ 0 & \text{else} \end{cases}$$

$$f(t) = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{n\pi} \cos\left(n\pi t - \frac{\pi}{2}\right)$$

• Properties of Exp. F.S.

Consider

$$g(t) = \begin{cases} 2 & t \in [0,1) \\ 0 & t \in [1,2) \end{cases} \text{ with period } 2s, \text{ and } f(t) = \begin{cases} 1 & t \in [0,1) \\ 0 & t \in [1,2) \end{cases} = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{1}{jn\pi} e^{jn\pi t}$$

What are the exponential Fourier series coefficients G_n

$$g(t) = 2f(t)$$

$$2(f(t)) = 2\left(\frac{1}{2} + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{1}{jn\pi} e^{jn\pi t}\right)$$

(Amplitude scaling property)
only the amplitude is affected

Consider:

$$f(t) = \begin{cases} 1 & t\in[0,1) \\ 0 & t\in[1,2) \end{cases} = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{1}{jn\pi} e^{jn\pi t}$$

and

$$h(t) = \begin{cases} 0 & t\in[0,1) \\ 1 & t\in[1,2) \end{cases}$$

$$h(t) = f(t-1) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0(t-1)} = \sum_{n=-\infty}^{\infty} F_n e^{-jn\omega_0} \cdot e^{jn\omega_0 t}$$

$$h(t) = \sum_{n=-\infty}^{\infty} H_n e^{jn\omega_0 t}$$

$$h(t) = \sum_{n=-\infty}^{\infty} H_n e^{jn\omega_0 t} \; [v]$$

$$H_n = F_n e^{-jn\omega_0 t_0}$$

$$h(t) = f(t-1) \Rightarrow H_n = F_n e^{-jn\omega_0} = F_n e^{-jn\pi}$$

- Time shift property:

$$h(t) = f(t-t_0) \qquad H_n = F_n e^{-jn\omega_0 t_0}$$

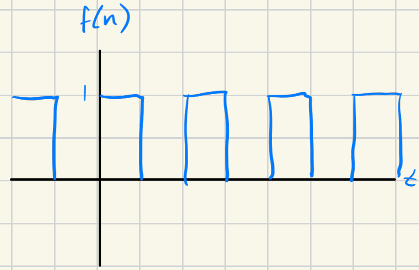
$$F_n = \begin{cases} \frac{1}{jn\pi} & n \text{ odd} \\ \frac{1}{2} & n=0 \\ 0 & n \text{ even}, \; n\neq 0 \end{cases} \qquad H_n = \begin{cases} \frac{1}{jn\pi}(-1) = \frac{-1}{jn\pi} & n \text{ odd} \\ \frac{1}{2}(1) & n=0 \\ 0 & n \text{ even}, \; n\neq 0 \end{cases}$$

• Addition Property: If $f(t)$ and $h(t)$ have same w_o, then

$$X(t) = f(t) + h(t) \quad ; \quad X_n = F_n + H_n$$

Let:

$$x(t) = \begin{cases} 1 & t \in [0,1) \\ -1 & t \in [1,2) \end{cases} \quad T = 2s$$



Recall:

$$g(t) = \begin{cases} 2 & t \in [0,1) \\ 0 & t \in [1,2) \end{cases} \qquad G_n = \begin{cases} \frac{2}{jn\pi} & n \text{ odd} \\ 1 & n = 0 \\ 0 & n \text{ even} \neq 0 \end{cases}$$

$$X(t) = \begin{cases} 1 & t \in [0,1) \\ -1 & t \in [1,2) \end{cases} \quad T = 2s$$

• Recall that in an LTI system

$$e^{jwt} \rightarrow \underset{H(w)}{\boxed{LTI}} \rightarrow H(w)\, e^{jwt}$$

$$\cos(wt + \theta) \rightarrow \underset{H(w)}{\boxed{LTI}} \rightarrow |H(w)| \cos(wt + \theta + \angle H(w))$$

So that

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jnw_o t} \rightarrow \underset{H(w)}{\boxed{LTI}} \rightarrow y(t) = \sum_{n=-\infty}^{\infty} H(nw_o) F_n e^{jnw_o t}$$

or

$$f(t) = \frac{C_o}{2} + \sum_{n=1}^{\infty} C_n \cos(nw_o t + \theta_n) \rightarrow \boxed{LTI} \rightarrow y(t) = H(0)\frac{C_o}{2} + \sum_{n=1}^{\infty} |H(nw_o)| C_n \cos(nw_o t + \theta_n + \angle H(nw_o))$$

Ex:

$$f(t) = \begin{cases} 1 & t \in [0,1) \\ 0 & t \in [1,2) \end{cases} = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{1}{jn\pi} e^{jn\pi t} \qquad W_0 = \pi \text{ rad}$$

Let

$$H(\omega) = \frac{1}{2+j\omega} \qquad |H(\omega)| = \frac{1}{\sqrt{4+\omega^2}} \qquad \angle H(\omega) = -\tan\left(\frac{\omega}{2}\right)$$

- Determine the steady state output, $y(t)$

$$y(t) = \underline{\frac{1}{2} H(0)} + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} H(n\omega_0) \frac{1}{jn\pi} e^{jn\pi t} =$$

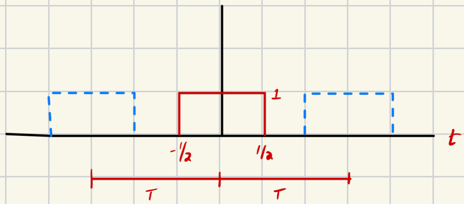
$$\frac{1}{2}\left(\frac{1}{2}\right) + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \left(\frac{1}{2+jn\pi}\right) \frac{1}{jn\pi} e^{jn\pi t}$$

Chapter 7

Fourier Transform and LTI System Response to Energy Signals

Recall that in an LTI system, the output to a periodic input is also periodic

What if $f(t)$ is $\underbrace{\text{not periodic}}_{\text{aperiodic}}$



· Fourier Transform

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{T} \int_{-T/2}^{T/2} f(s) e^{-jn\omega_0 s} ds \right) e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{\omega_0}{2\pi} \int_{-\pi/\omega_0}^{\pi/\omega_0} f(s) e^{-jn\omega_0 s} ds \cdot e^{jn\omega_0 t} =$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \Bigg(\underbrace{\int_{-\pi/\omega_0}^{\pi/\omega_0} f(s) e^{-jn\omega_0 s} ds}_{\text{define as } F(n\omega_0)} \Bigg) e^{jn\omega_0 t} \cdot \omega_0 =$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(n\omega_0) e^{jn\omega_0 t} \cdot \omega_0 \xrightarrow{\omega_0 \to 0}$$

$$\to \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega = f(t)$$

Fourier Transform Pairs

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \quad \in \mathbb{C} \qquad \text{Fourier Transform}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \qquad \text{inverse Fourier Transform}$$

time domain

$f(t) \longleftrightarrow F(\omega)$ unique transform pair

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \rightarrow \overset{H(\omega)}{\boxed{LTI}} \qquad y(t) = \frac{1}{2\pi}\ \ldots$$

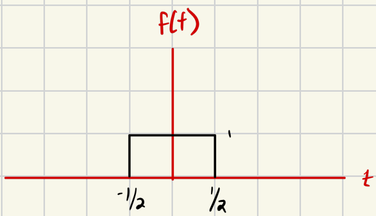
We can represent any function in terms of a Fourier Transform if $f(t)$ is absolutely integrable

i.e.

$$\int_{-\infty}^{\infty} |f(t)|\,dt < \infty$$

Ex)

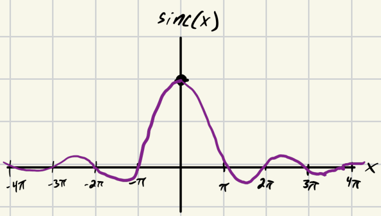
$$f(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & \text{else} \end{cases} = \text{rect}(t)$$

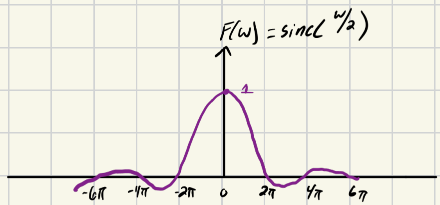


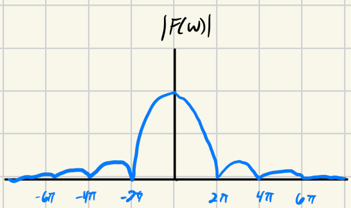
$\text{sinc}(x) = \frac{\sin(x)}{x}$

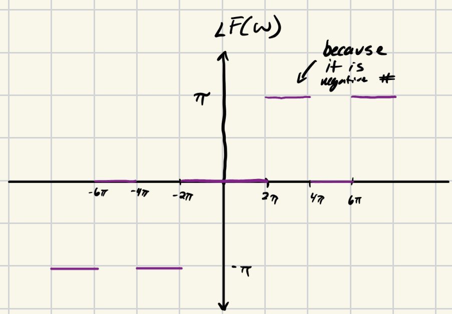
$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \int_{-1/2}^{1/2} (1)e^{-j\omega t}\,dt =$$

$$= \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-1/2}^{1/2} = \frac{e^{-j\omega(1/2)} - e^{-j\omega(-1/2)}}{-j\omega} = \left(\frac{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}{j\omega}\right)\frac{2}{2} = \frac{2\sin\left(\frac{\omega}{2}\right)}{\omega} = \frac{\sin\left(\frac{\omega}{2}\right)}{\omega/2} = \text{sinc}\left(\frac{\omega}{2}\right)$$









Doing Prof Schuh's lecture 3

$$P_{avg} = \frac{1}{T}\int_0^T |f(t)|^2 dt \quad \text{s.t. } |f(t)|^2 = f(t) * f^*(t)$$

$$= \left(\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}\right)\left(\sum_{m=-\infty}^{\infty} F_m^* e^{-jm\omega_0 t}\right)$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} F_n F_m^* e^{j(n-m)\omega_0 t}$$

$$P_{avg} = \frac{1}{T}\int_0^T \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} F_n F_m^* e^{j(n-m)\omega_0 t} dt$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} F_n F_m^* \cdot \int_0^T e^{j(n-m)\omega_0 t} dt$$

← n must $= m$, because otherwise you're integrating over sines + cosines which would be zero.

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} F_n F_n^* (T)$$

$$= \sum_{n=-\infty}^{\infty} |F_n|^2$$

Real Valued f(t)

$$C_0 = 2|F_0| \Rightarrow |F_0| = \frac{c_0}{2}$$

$$C_n = 2|F_n| \Rightarrow |F_n| = \frac{c_n}{2}$$

changing bounds

$$P_{avg} = |F_0|^2 + \sum_{n=1}^{\infty} |F_n|^2 + |F_{-n}|^2$$

$$= |F_0|^2 + 2\sum_{n=1}^{\infty} |F_n|^2$$

$$= \left(\frac{c_0}{2}\right)^2 + 2\sum_{n=1}^{\infty}\left(\frac{c_n}{2}\right)^2$$

$$P_{avg} = \frac{c_0^2}{4} + \sum_{n=1}^{\infty}\frac{c_n^2}{2}$$

↑ DC power

↖ similar to $P_{ac} = \frac{1}{2}V_p^2$

Distortion

Non-linear system: $y(t) = Af(t) + \epsilon f(t)^2$, $\epsilon \ll 1$

Let's say $f(t) = \cos(\omega t)$

$$y(t) = A\cos(\omega t) + \epsilon\cos^2(\omega t)$$

$$= A\cos(\omega t) + \epsilon\left(\frac{1}{2} + \frac{1}{2}\cos(2\omega t)\right)$$

$$= \frac{1}{2} + A\cos(\omega t) + \frac{\epsilon}{2}\cos(2\omega t)$$

1st harmonic power: $\frac{1}{2}A^2$

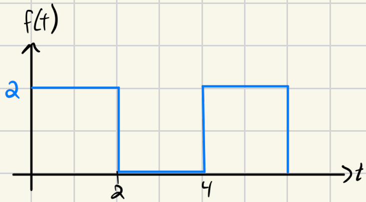
2nd harmonic power: $\frac{\epsilon^2}{8}$

2nd Harmonic Distortion: $\frac{\frac{\epsilon^2}{8}}{\frac{A^2}{2}} = \frac{\epsilon^2}{4A^2}$

Total Harmonic Distortion

$$THD = \frac{\sum_{n=2}^{\infty} \frac{1}{2} C_n^2}{\frac{1}{2} C_1^2} = \frac{\sum_{n=2}^{\infty} C_n^2}{C_1^2}$$

Ex



$$P_{avg} = \frac{1}{4}\int_0^4 f(t)^2 \partial t = \frac{1}{4}\left(\int_0^2 2^2 \partial t + \int_2^4 0 \partial t\right)$$
$$= \frac{1}{4}\int_0^2 4 \partial t$$
$$= t\Big|_0^2$$
$$\boxed{= 2w}$$

$$f(t) = 1 + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi t}{2}\right)$$

$$P_1 = \frac{1}{2} C_1^2 = \frac{1}{2}\left(\frac{4}{\pi}\right)^2 = \frac{16}{2\pi^2} = \frac{8}{\pi^2} = 0.8106w$$

$$P_{2-\infty} = \sum_{n=2}^{\infty} \frac{1}{2} C_n^2 \approx 0.1878w$$

$$THD = \frac{P_{2-\infty}}{P_1} = \frac{0.1878}{0.8106} = 23.17\%$$

$$H(w) = \frac{1}{1+jw} \Rightarrow |H(w)| = \frac{1}{\sqrt{1+w^2}}$$

$$y(t) = \underbrace{|H(0)|(1)}_{1} + \sum_{n=1}^{\infty} \left|H\left(\frac{n\pi}{2}\right)\right| \frac{4}{n\pi} \sin\left(\frac{n\pi t}{2} + \theta\right)$$

$$Y_n = \frac{4}{n\pi}\left(\frac{1}{\sqrt{1+\left(\frac{n\pi}{2}\right)^2}}\right) \qquad Y_0 = 2$$

$$P_{avg} = \frac{Y_0^2}{4} + \sum_{n=1}^{\infty} \frac{1}{2} Y_n^2 = 1.238w$$

$$P_1 = \frac{1}{2} Y_1^2 \approx 0.2338w$$

$$P_{2-\infty} = \sum_{n=2}^{\infty} \frac{1}{2} Y_n^2 = 0.00468w$$

$$THD = \frac{P_{2-\infty}^2}{P_1^2} \Rightarrow 1.984\%$$

Review of periodic signals

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T}$$

$$F_n \longrightarrow \boxed{H(\omega)} \longrightarrow Y_n = H(n\omega_0) F_n$$

Fourier Transform

is essentially converting from time domain to frequency domain

Fourier Transform is the combination of sines and cosines of all possible frequencies for aperiodic signals

* What if we are no longer periodic *

* Fourier Transform *

Assume $f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \Rightarrow F_n = F(n\omega_0)$ $\quad \frac{\omega_0}{2\pi}\hat{F}(n\omega_0) = F(n\omega_0)$

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{\omega_0}{2\pi}\hat{F}(n\omega_0) e^{jn\omega_0 t}$$

$n = m$ for $\int_{-T/2}^{T/2} e^{j(n-m)\omega_0 t} dt \neq 0$

$$\int_{-T/2}^{T/2} f(t) e^{-jm\omega_0 t} dt = \sum_{n=-\infty}^{\infty} \frac{\omega_0}{2\pi}\hat{F}(n\omega_0) \int_{-T/2}^{T/2} e^{j(n-m)\omega_0 t} dt$$

since it's the only one non-zero

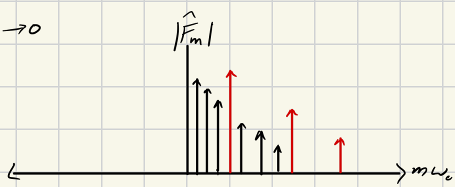
$$= \frac{\omega_0}{2\pi}\hat{F}(n\omega_0) T \quad \rightarrow T = \frac{2\pi}{\omega_0}$$

$$\hat{F}(m\omega_0) = \int_{-T/2}^{T/2} f(t) e^{-jm\omega_0 t} dt$$

For nonperiodic signals, $T \to \infty$, $\omega_0 \to 0$

$m\omega_0 \Rightarrow m \ldots -2, -1, 0, 1, 2, \ldots$

$-\infty \leq m \leq \infty$



$$\boxed{\hat{F}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt}$$ — Fourier Transform ★

$\forall \omega$

$m\omega_0 \rightarrow \omega$

discrete $\quad$ continuous

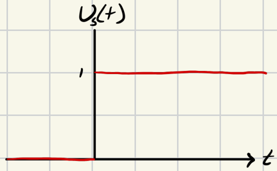
standard notation:

$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$ = Forward Fourier Transform

$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$ = inverse Fourier Transform

Ex) Unit step function

$$U_s(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$



$$f(t) = e^{-t}U_s(t)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \cancelto{0}{\int_{-\infty}^{0} f(t)e^{-j\omega t}\,dt} + \int_{0}^{\infty} f(t)e^{-j\omega t}\,dt$$

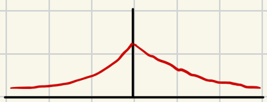
$$= \int_0^\infty e^{-t}e^{-j\omega t}\,dt$$

$$= \int_0^\infty e^{-(1+j\omega)t}\,dt$$

$$= \left.\frac{e^{-(1+j\omega)t}}{-(1+j\omega)}\right|_0^\infty$$

$$F(\omega) = \frac{1}{1+j\omega}$$

Ex) $f(t) = e^{-|t|}$



$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0} f(t)e^{-j\omega t}\,dt + \int_{0}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{t}e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-t}e^{-j\omega t}\,dt$$

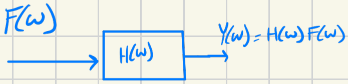
$$= \int_{-\infty}^{0} e^{(1-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(1+j\omega)t}\,dt$$

$$= \left.\frac{e^{(1-j\omega)t}}{1-j\omega}\right|_{-\infty}^{0} + \frac{1}{1+j\omega}$$

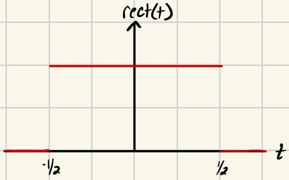
$$= \frac{1}{1-j\omega} + \frac{1}{1+j\omega}$$

$$= \frac{1+j\omega+1-j\omega}{1+\omega^2}$$

$$\boxed{F(\omega) = \frac{2}{1+\omega^2}}$$



Ex $\text{rect}(t) \rightarrow \begin{cases} 0 & |t|>\frac{1}{2} \\ 1 & |t|<\frac{1}{2} \end{cases}$



$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

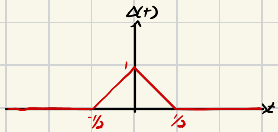
$$= \int_{-1/2}^{1/2} e^{-j\omega t}\, dt$$

$\sin(x) = \dfrac{e^{jx} - e^{-jx}}{j2} \Rightarrow j2\sin(x) = e^{jx} - e^{-jx}$

$$F(\omega) = \frac{e^{-j\omega t}}{-j\omega}\Bigg|_{1/2}^{1/2} = \frac{1}{-j\omega}\left(e^{-j\frac{\omega}{2}} - e^{j\frac{\omega}{2}}\right) = \frac{1}{-j\omega}\left(-j2\sin\left(\frac{\omega}{2}\right)\right) = \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right) = \frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}} = \text{sinc}\left(\frac{\omega}{2}\right)$$

$$\text{rect}(t) \longleftrightarrow \text{sinc}\left(\frac{\omega}{2}\right)$$

Ex $\Delta(t) = \begin{cases} 1-2|t| & |t|\le\frac{1}{2} \\ 0 & |t|>\frac{1}{2} \end{cases}$



$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-1/2}^{0} (1+2t)\, e^{-j\omega t}\, dt + \int_{0}^{1/2} (1-2t)\, e^{-j\omega t}\, dt$$

$\dots$

$$= \frac{2}{(-j\omega)^2}\left[2\cos\left(\frac{\omega}{2}\right) - 2\right]$$

$$= \frac{4}{\omega^2}\left(\cos\left(\frac{\omega}{2}\right) - 1\right)$$

$$= \frac{-4}{\omega^2}\left(1 - \cos\left(\frac{\omega}{2}\right)\right)$$

$$= \frac{8}{\omega^2}\sin^2\left(\frac{\omega}{4}\right)$$

$$= \frac{1}{2}\left(\frac{16}{\omega^2}\right)\sin^2\left(\frac{\omega}{4}\right)$$

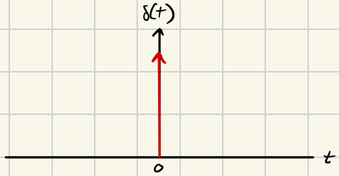
$$= \frac{1}{2}\left(\frac{\sin^2\left(\frac{\omega}{4}\right)}{\left(\frac{\omega}{4}\right)^2}\right) =$$

$$= \frac{1}{2}\text{sinc}^2\left(\frac{\omega}{4}\right)$$

$$\Delta(t) \longleftrightarrow \frac{1}{2}\text{sinc}^2\left(\frac{\omega}{4}\right)$$

Ex] $\delta(t)$ (Dirac Delta function)

$$\delta(t) = \begin{cases} \infty & t=0 \\ 0 & \text{otherwise} \end{cases}$$



Properties

1) linearity $\quad f_1 + f_2 \longleftrightarrow F_1(\omega) + F_2(\omega)$
2) time-shift $\quad f(t-t_0) \longleftrightarrow e^{-j\omega t_0} F(\omega)$
3) Freq.-Shift $\quad F(\omega-\omega_0) \longleftrightarrow e^{j\omega_0 t} f(t)$
4) Time scaling $\quad f(ct) \longleftrightarrow \frac{1}{|c|} F\left(\frac{\omega}{c}\right)$
5) Time derivative $\quad \frac{\partial f}{\partial t} \longleftrightarrow j\omega F(\omega)$

Ex] $f(t) = \Delta(t-\frac{1}{2})$

$\Delta(t) \longleftrightarrow \frac{1}{2}\operatorname{sinc}^2\left(\frac{\omega}{4}\right)$

$f(t) = \Delta(t-\frac{1}{2}) \longleftrightarrow \quad F(\omega) = \frac{1}{2}\operatorname{sinc}^2\left(\frac{\omega}{4}\right) e^{-j\omega/2}$

Ex] $f(t) = \cos(\omega_0 t)$

use inverse

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} dt$$

$$e^{j\omega_0 t} + e^{-j\omega_0 t} = \frac{1}{\pi}\int F(\omega) e^{j\omega t} dt$$

$$F(\omega) = \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$$

Periodic Signals $\Rightarrow f(t) \longleftrightarrow$ Fourier Series

F_n used to desribe a signal in Fourier domain

Discrete frequencies, $n\omega_0$

Aperiodic Signals $\Rightarrow g(t) \longleftrightarrow$ Fourier Transform

$G(\omega)$

All frequencies

How do we determine what freq. is important in Fourier Transform?

$$W = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} f(t) f^*(t) dt$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \qquad f^*(t) = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{-j\omega t} d\omega\right)^* = \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega) e^{-j\omega t} d\omega$$

$$W = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$ Rayleigh's Theorem

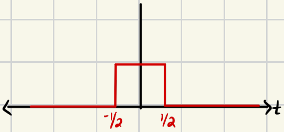
$$P = \sum_{n=-\infty}^{\infty} |F_n|^2$$ ← Parseval's Theorem

$|F(\omega)|^2$ = energy spectrum

signal has finite $|F(\omega)|^2$, "energy signal"

Ex

rect(t)



$$f(t) = \text{rect}(t) \overset{F}{\longleftrightarrow} \text{sinc}\left(\frac{\omega}{2}\right) = F(\omega)$$

Since integrating $\text{sinc}(\frac{\omega}{2})$ may be difficult, we will use time domain

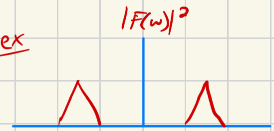
$$W = \int_{-\infty}^{\infty} |\text{rect}(t)|^2 dt = \int_{-\infty}^{\infty} \text{rect}^2(t) dt = \int_{-1/2}^{1/2} 1^2 dt = t\Big|_{-1/2}^{1/2} = 1\,J$$

Signal Classification

* $|F(\omega)|^2$ is concentrated around low frequencies
  $\Rightarrow$ low pass signal

* $|F(\omega)|^2$ is concentrated around a band of frequencies
  = band pass signal

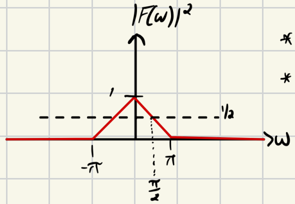
ex



Signal Bandwidth – width of the energy spectrum

low pass bandwidth $\rightarrow$ $\Omega = 2\pi \underbrace{B}_{\text{Hz, Bandwidth}}$

3dB bandwidth: $\dfrac{|F(\Omega)|^2}{|F(0)|^2} = \dfrac{1}{2}$



* Ω is positive
* total width is $2 \times \Omega$

$$\frac{|F(\Omega)|^2}{|F(0)|^2} = \frac{1}{2} \Rightarrow |F(\Omega)|^2 = \frac{1}{2}$$

$$\Omega = \frac{\pi}{2} \text{ rad/s} \Rightarrow \frac{\pi}{2} = 2\pi B \Rightarrow \boxed{B = \frac{1}{4} \text{Hz}}$$

Percentage Bandwith

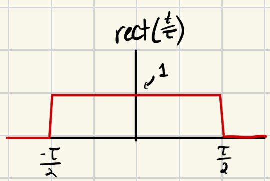
$$rW = \frac{1}{2}\int_{-\Omega}^{\Omega} |F(\omega)|^2 d\omega$$

$r \in [0,1]$

Ex| $f(t)$ $\text{rect}\left(\frac{t}{\tau}\right)$

$\frac{t}{\tau} = \frac{-1}{2}$ $\quad$ $\frac{t}{\tau} = \frac{1}{2}$

$t = \frac{-\tau}{2}$ $\quad$ $t = \frac{\tau}{2}$



$$W = \int_{-\infty}^{\infty} f^2(t)\,dt = \int_{-\tau/2}^{\tau/2} (1)^2 dt \Rightarrow W = \tau J$$

Let's say our bandwidth is $\Omega = \frac{2\pi}{\tau}$

$$\text{rect}\left(\frac{t}{\tau}\right) \longleftrightarrow \tau\,\text{sinc}\left(\frac{\omega\tau}{2}\right) = \tau\left(\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}}\right)$$

$$r(W) = \frac{1}{2\pi}\int_{-\Omega=\frac{-2\pi}{\tau}}^{\Omega=\frac{2\pi}{\tau}} \tau^2 \text{sinc}^2\left(\frac{\omega\tau}{2}\right) d\omega$$

let $x = \frac{\omega\tau}{2}$

$\frac{2}{\tau}dx = d\omega$

$$r\tau = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tau^2 \text{sinc}^2(x)\left(\frac{2}{\tau}\right)dx =$$

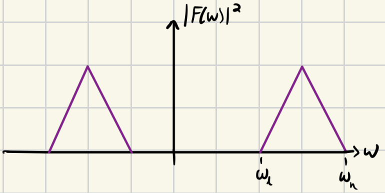
$$r\cancel{\tau} = \frac{\cancel{\tau}}{\pi}\int_{-\pi}^{\pi} \text{sinc}^2(x)dx$$

$r = 0.903$

$\Omega = \frac{2\pi}{\tau}$ is the 90.3% bandwidth

$B = \frac{\Omega}{2\pi} \Rightarrow \boxed{B = \frac{1}{\tau}\text{ Hz}}$

Bandpass Bandwidth

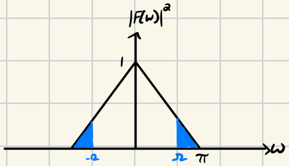


$\Omega = \omega_{upper} - \omega_{lower}$ (positive values)

if $f(t)$ is real:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{0}^{\infty} 2|F(\omega)|\cos(\omega t + \overbrace{\theta}^{\text{the phase shift}})d\omega$$

Ex)



Ω is the 95% bandwidth

$$W = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2 d\omega = \frac{1}{2\pi}\left(\frac{1}{2}(2\pi)(1)\right) \Rightarrow \boxed{W = \frac{1}{2}J}$$

We need the area of our triangles to be 5% of W

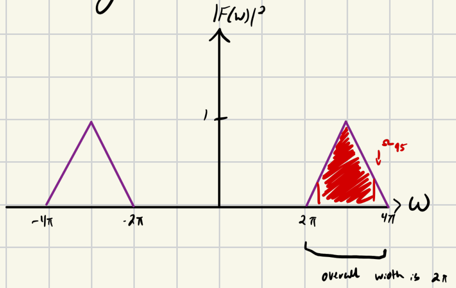
$$\frac{1}{2\pi}\left(\frac{1}{2}(\pi - \Omega)\left[\frac{\pi-\Omega}{\pi}\right]\right) = 0.025\left(\frac{1}{2}\right)$$

$$\frac{1}{4\pi^2}(\pi - \Omega)^2 = 0.025\left(\frac{1}{2}\right)$$

$$(\pi - \Omega)^2 = 0.05\pi^2$$

$$\boxed{\Omega = \pi(1 - \sqrt{0.05})\text{ rad/sec}}$$

Let's say new:

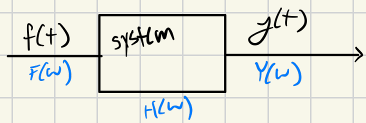


We want to find 95% bandwidth

We only shifted our bandwidth by 2, so our 95% bandwith is twice our previous i.e.

$$\Omega_{95,new} = 2(\Omega_{95})$$
$$= 2\pi(1-\sqrt{0.05}$$

$Y(\omega) = H(\omega)F(\omega)$ is the <u>entire</u> 0 state response not just steady state



Ex)
$$\frac{d^2y}{dt^2} + \frac{4dy}{dt} + 3y = u_s(t)$$

$$(j\omega)^2 Y(\omega) + 4j\omega Y + 3Y = \frac{1}{j\omega} + \pi\delta(\omega)$$

$$\frac{dg}{dt} \longleftrightarrow j\omega G(\omega)$$

$$(3-\omega^2 + j4\omega)Y(\omega) = \frac{1}{j\omega}\left[1 + j\omega\pi\delta(\omega)\right]$$

$$3 + j4\omega - \omega^2 = (1+j\omega)(3+j\omega)$$

$$Y(\omega) = \frac{1+j\omega\pi\delta(\omega)}{j\omega(3-\omega^2+j4\omega)} \Rightarrow Y(\omega) = \frac{1+j\omega\pi\delta(\omega)}{j\omega(1+j\omega)(3+j\omega)} = \frac{A}{j\omega} + \frac{B}{1+j\omega} + \frac{C}{3+j\omega} = \frac{\frac{1}{3}}{j\omega} + \frac{\frac{\pi}{2}\delta(\omega) - \frac{1}{2}}{1+j\omega} + \frac{\frac{1}{6} - \frac{\pi}{2}\delta(\omega)}{3+j\omega}$$

$$Y(\omega) = \frac{\frac{1}{3}}{j\omega} + \frac{\frac{\pi}{2}\delta(\omega)}{1+j\omega} - \frac{\frac{1}{2}}{1+j\omega} + \frac{\frac{1}{6}}{3+j\omega} - \frac{\frac{\pi}{2}\delta(\omega)}{3+j\omega}$$

$\updownarrow F^{-1}$ $\updownarrow F^{-1}$ $\updownarrow F^{-1}$ $\updownarrow F^{-1}$ $\updownarrow F^{-1}$

$\frac{1}{6}\text{sign}(t) +$ $\quad -\frac{1}{2}e^{-t}u_s(t) + \frac{1}{6}e^{-3t}u_s(t)$

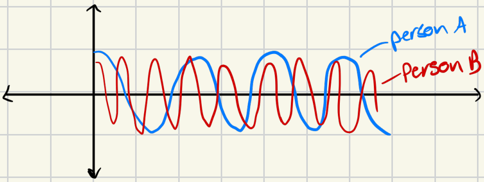
$$y(t) = \frac{1}{6}\text{sign}(t) - \frac{1}{2}e^{-t}u_s(t) + \frac{1}{6}e^{-3t}u_s(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\frac{\pi}{2}\delta(\omega)}{1+j\omega} - \frac{\frac{\pi}{2}\delta(\omega)}{3+j\omega}\right)e^{j\omega t}\,d\omega$$

$$+\frac{1}{2\pi}\left(\frac{\pi}{2}\left(\frac{1}{1}\right)e^{j0} - \frac{\pi}{2}\left(\frac{1}{3}\right)e^{j0}\right)$$

$$y(t) = \frac{1}{6}\text{sign}(t) + \frac{1}{6} - \frac{1}{2}e^{-t}u_s(t) + \frac{1}{6}e^{-3t}u_s(t) \qquad + \frac{1}{2\pi}\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$$

$$y_{zs}(t) = \frac{1}{3}u_s(t) + \frac{1}{6}e^{-3t}u_s(t) - \frac{1}{2}e^{-t}u_s(t) \qquad + \frac{1}{6}$$

Modulation property, AM signal, coherent demodulation



How do we get the information?

- occupy same frequency band + can't get information

Amplitude Modulation

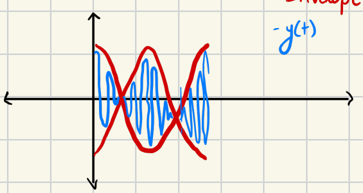
$$y = \underbrace{f(t)}_{\text{info}}\overbrace{\cos(\omega_c t)}^{\text{carrier wave}}$$

output

carrier frequency

In the United States, AM radio stations use 540-1600 KHz for f_c at 10 KHz intervals

Ex) $y(t) = \underbrace{\cos(t)}_{\text{info}}\underbrace{\cos(10t)}_{\text{carrier wave}}$



- Envelope
- $y(t)$

Frequency Modulation

$$y(t) = f(t)\cos(\omega_c t) = \frac{1}{2}f(t)\left[e^{j\omega_c t} + e^{-j\omega_c t}\right]$$

$$y(t) = \frac{1}{2}\left(f(t)e^{j\omega_c t} + f(t)e^{-j\omega_c t}\right)$$

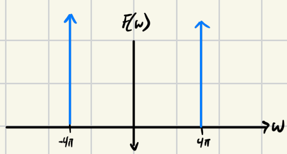
$$\updownarrow$$

$$Y(\omega) = \frac{1}{2}\left(F(\omega-\omega_c) + F(\omega+\omega_c)\right)$$

Modulation is not LTI

Ex) $f(t) = \cos(4\pi t)$

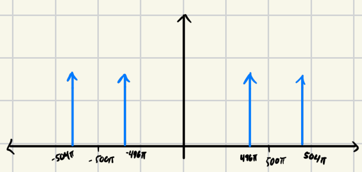
$F(\omega) = \pi\left[\delta(\omega-4\pi) + \delta(\omega+4\pi)\right]$



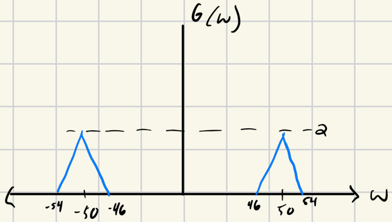
$y(t) = f(t)\cos(500\pi t)$

$\updownarrow$ $\qquad \omega_c = 500\pi$

$Y(\omega) = \frac{1}{2}\left[F(\omega-500\pi) + F(\omega+500\pi)\right]$

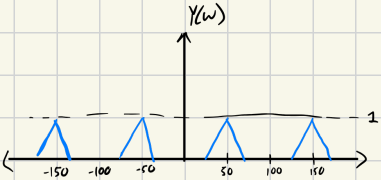


Ex)



$y(t) = g(t)\cos(100t)$

$Y(w) = \frac{1}{2}\left[G(w-100) + G(w+100)\right]$

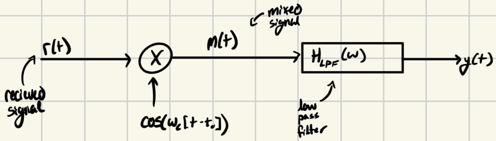


How to reconstruct a signal and get the information?

- Coherent Demodulation

* Recieving many signals at once *

* An adjustable band-pass filter to "tune in" the correct station *



$f(t) \xleftrightarrow{F} F(w)$

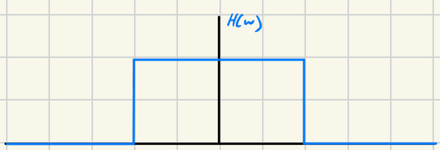
$f(t)\cos(w_c t) \xleftrightarrow[\text{Freq. Modulation}]{F} \frac{1}{2}\left[F(w-w_c) + F(w+w_c)\right]$

$r(t) \xleftrightarrow[\text{time delay}]{F} \frac{1}{2}\left(F(w-w_c)e^{jwt_0} + F(w-w_c)e^{jwt_0}\right)$

$r(t) = f(t-t_0)\cos(w_c(t-t_0))$

$m(t) = f(t-t_0)\cos^2(w_c(t-t_0))$

$m(t) = f(t-t_0)\left[\frac{1}{2} + \frac{1}{2}\cos(2w_c(t-t_0))\right] \longleftrightarrow M(w) = \frac{1}{2}F(w)e^{jwt_0} + \frac{1}{4}\left[F(w-2w_c)e^{jwt_0} + F(w+2w_c)e^{jwt_0}\right]$



$Y(w) = M(w)H(w)$

$= \frac{1}{2}F(w)e^{jwt_0}$

$\updownarrow$

$y(t) = \frac{1}{2}f(t-t_0)$

* when implementing this, must know W_c AND at the recieving end to use this *

Super heterodyne

Envelope Detection

Desired: $f(t)$

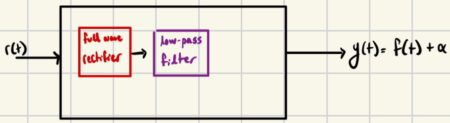
at send: $f(t) + \underset{\text{DC bias}}{\alpha} > 0 \quad \forall t$

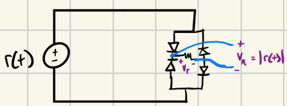
Modulation: $s(t) = (f(t)+\alpha)\cos(\underbrace{\omega_c}_{\text{carrier frequency}} t)$

Recieving End: $r(t) = (f(t)+\alpha)\cos(\omega_c t)$ * there will be t-delay but we are ignoring it for simplicity *

Envelope: $|f(t)+\alpha| = f(t)+\alpha$ because of init. cond.

Envelope Detector





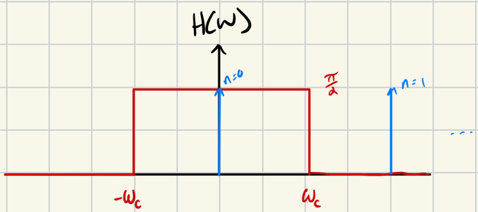
$$|r(t)| = |(f(t)+\alpha)\cos(\omega_c t)| = |f(t)+\alpha|\,|\cos(\omega_c t)|$$

$$= (f(t)+\alpha)\,\underbrace{|\cos(\omega_c t)|}_{T = \frac{1}{2}T_c = \frac{\frac{2\pi}{\omega_c}}{2} = \frac{\pi}{\omega_c}}$$

$$\to \omega_0 = \frac{2\pi}{\frac{\pi}{\omega_c}} = 2\omega_c \quad \to |\cos(\omega_c t)| = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(4n^2-1)}\cos(2n\omega_c t)$$

$$|r(t)| = (f(t)+\alpha)\left(\frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(4n^2-1)}\cos(2n\omega_c t)\right)$$

Design a filter that looks like this:

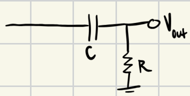


By designing the filter like this, we eliminate all higher frequencies of ω

$$y(t) = H(\omega)|r(t)|$$

$$= \left[\frac{2}{\pi}(f(t)+\alpha)\right]\left(\frac{\pi}{2}\right)$$

$$y(t) = f(t)+\alpha$$



* This requires a tuneable low pass filter *

Superheterodyne

1) Mix $r(t) = (f(t)+\alpha)\cos(\omega_c t)$ with $\underbrace{\cos(\omega_{LO} t)}_{\text{Local Oscillator}}$

$$m(t) = (f(t)+\alpha)\cos(\omega_c t)\cos(\omega_{LO} t)$$

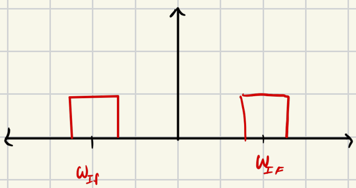
$$m(t) = \frac{(f(t)+\alpha)}{2}\left[\cos([\omega_c-\omega_{LO}]t)+\cos([\omega_c+\omega_{LO}]t)\right]$$

$\omega_{LO} = \omega_c + \omega_{IF}$

center frequency f_{IF} is fixed. f_{IF} tends to be $= 455\,kHz$

$$m(t) = \frac{f(t)+\alpha}{2}\left[\cos(-\omega_{IF}t)+\cos((2\omega_c+\omega_{IF})t)\right]$$

2) Send through a fixed BPF



$$\underbrace{y(t) = \frac{f(t)+\alpha}{2}\cos(\omega_{IF}t)}$$

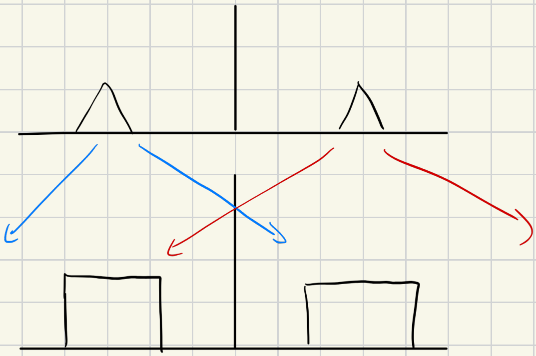
send through envelope detector

$\frac{f(t)+\alpha}{2}$ = output

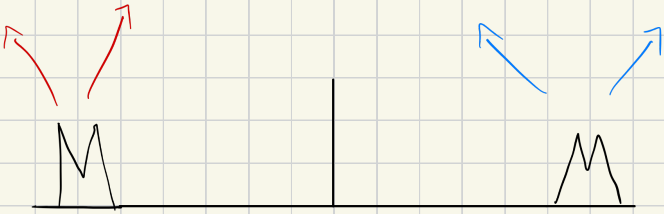
ω_{LO}: Local LC Circuit that we tune by adjusting the capacitance of the capacitor

$$\omega_{LO} = \frac{1}{\sqrt{LC}}$$

Image Station Problem



⟵ Mixing



$$W_{c2} = W_{c1} + 2W_{IF}$$

Image Station

$$m(t) = \frac{f(t) + \alpha}{2}\left(\cos((\omega_c - \omega_{c2})t) + \cos((\omega_c + \omega_{c2})t)\right)$$

$$W_{LO} = W_{c1} + W_{IF}$$

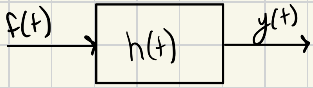
$$= W_{c2} - W_{LO}$$
$$= (W_{c1} + 2W_{IF}) - (W_{c1} + W_{IF})$$
$$= W_{IF}$$

← Will interfere w actual desired freq.

we can use pre-selector p. 276 in textbook

Convolution & FT convolution properties

Convolution: how the shape of one function is modified by another



Convolution is communitive

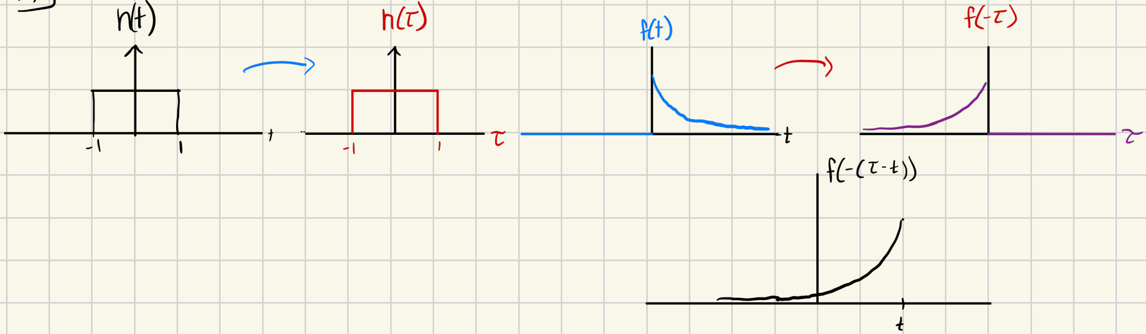
$$y(t) = f(t) * h(t)$$

convolved with

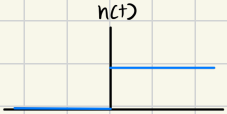
$$y(t) = \int_{-\infty}^{\infty} h(\tau) f(t-\tau)\, d\tau$$

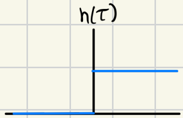
dummy variable

Ex)

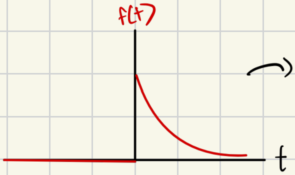


Ex) $h(t) = u_s(t)$



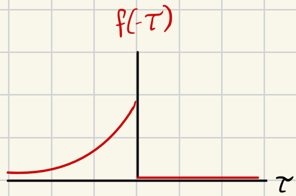


$f(t) = e^{-t}u_s(t)$

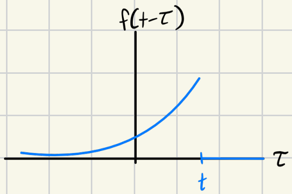


$f(-\tau) = e^{-(-\tau)}u_s(-\tau)$

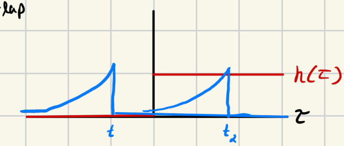
$= e^{\tau}u_s(-\tau)$



$f(t-\tau) = e^{-(t-\tau)}u_s(t-\tau) = e^{\tau-t}u_s(-(\tau-t))$



$y(t) = \int_{-\infty}^{\infty} h(\tau)f(t-\tau)\,d\tau \rightarrow$ look when the functions overlap



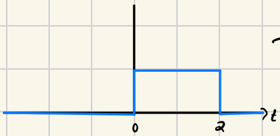
t can be any value.
For what value of t do we see the functions begin to overlap?

$t-\tau=0$: overlap

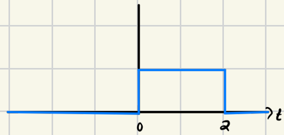
$$y(t) = \int_0^t (1)e^{-(t-\tau)}d\tau = \int_0^t e^{-t+\tau}d\tau = \int_0^t e^{-t}e^{\tau}d\tau = e^{-t}\int_0^t e^{\tau}d\tau = e^{-t}e^{\tau}\Big|_0^t = e^{-t}(e^{t}-1) = (1-e^{-t})u_s(t)$$

Ex)

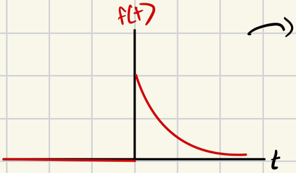
$h(t) = \text{rect}\left(\frac{t-1}{2}\right)$



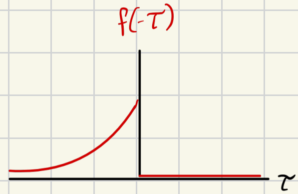
$h(t) = \text{rect}\left(\frac{\tau-1}{2}\right)$



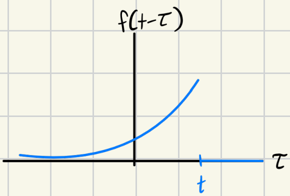
$f(t) = e^{-t}u_s(t)$



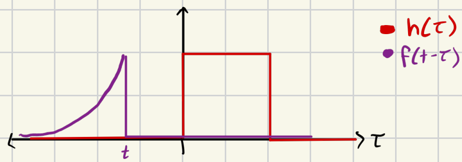
$$\to \quad f(-\tau) = e^{-(-\tau)} u_s(-\tau) = e^{\tau} u_s(-\tau)$$



$$f(t-\tau) = e^{-(t-\tau)} u_s(t-\tau) = e^{\tau - t} u_s(-(\tau - t))$$



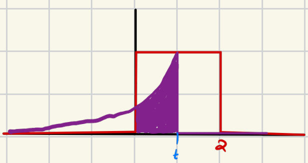
Again, t is arbitrary



$t=0$: overlap

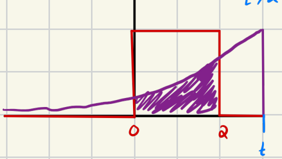
$t=2$: other consideration

$0 \le t \le 2$



$$\int_0^t e^{-(t-\tau)} \, d\tau = (1-e^{-t}) u_s(t)$$

$t > 2$



$$y(t) = \int_0^2 1 e^{-(t-\tau)} \, d\tau = e^{-t}\left(e^{\tau}\Big|_0^2\right) = e^{-t}(e^2 - 1) \Rightarrow \left(e^{-(t-2)} - e^{-t}\right) u(t-2)$$

$$y(t) = \begin{cases} 0 & t < 0 \\ (1-e^{-t}) u_s(t) & 0 \le t < 2 \\ (e^{-(t-2)} - e^{-t}) u_s(t-2) & 2 \le t \end{cases}$$

Properties of Convolution

1) $y(t) = f(t) * h(t) = h(t) * f(t)$

2) $y(t) = f(t) * h(t)$

$y(t-t_0) = f(t-t_0) * h(t) = f(t) * h(t-t_0)$

3) $y(t) = h(t) * f(t)$

$\frac{dy}{dt} = \frac{dh}{dt} * f(t)$

4) $y(t-(t_1+t_2)) = h(t-t_1) * f(t-t_2)$

5) $(f_1 + f_2) * h(t) = f_1 * h(t) + f_2 * h(t)$

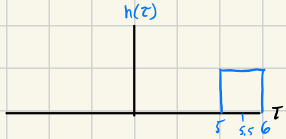
Graphical Convolution

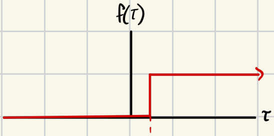
Ex) $\text{rect}\left(\frac{t-1}{2}\right)$ $\qquad f(t) = e^{-t}u_s(t)$

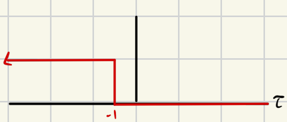
$$\text{rect}\left(\frac{t-1}{2}\right) = u_s(t) - u_s(t-2)$$

$$y(t) = \text{rect}\left(\frac{t-1}{2}\right) * e^{-t}u_s(t)$$
$$= \left(u_s(t) - u_s(t-2)\right) * e^{-t}u_s(t)$$
$$= u_s(t) * e^{-t}u_s(t) - u_s(t-2) * e^{-t}u_s(t)$$
$$= u_s(t) * e^{-t}u_s(t) - u_s(t) * e^{-(t-2)}u_s(t-2)$$
$$= (1-e^{-t})u_s(t) - (1-e^{-(t-2)})u_s(t-2)$$

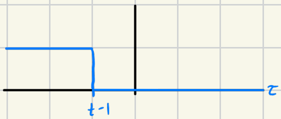
Ex) $y(t) = \underbrace{\text{rect}(t-5.5)}_{h(t)} * u_s(t-1)$

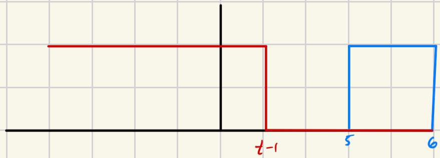






$f(t-\tau) = u_s((t-\tau)-1) = u_s((t-1)-\tau)$
$= u_s(-(\tau-(t-1)))$

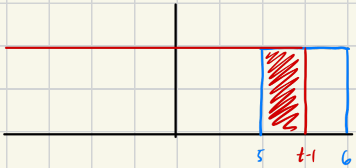




$t-1=5 \qquad t-1=6$

$t=6s \qquad t=7s$

$6 \le t < 7$



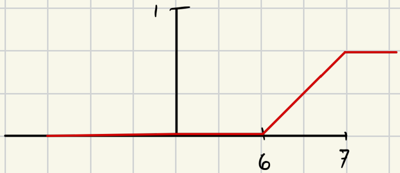
$6 \le t < 7$

$$y(t) = \int_5^{t-1} (1)(1)\,d\tau = \int_5^{t-1} d\tau = \tau\Big|_5^{t-1} = t-1-5 = \boxed{t-6 = y(t)}$$

$$y(t) = \begin{cases} 0 & t<6 \\ t-6 & 6\le t<7 \\ 1 & t\ge 7 \end{cases}$$

$t \ge 7$

$y(t) = 1$

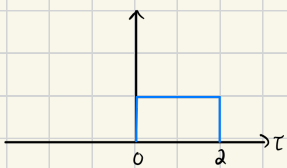


Convolution gives the zero-state solution
so does solving a system in frequency domain.

Convolution Examples

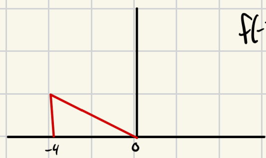
Ex) $\text{rect}\left(\frac{t-1}{2}\right) * f(t)$ st $f(t) = \begin{cases} \frac{1}{4}t & 0 \le t \le 4 \\ 0 & \text{otherwise} \end{cases}$

$h(\tau)$



$f(\tau)$

$f(-\tau)$



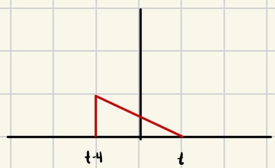
$f(-\tau) = \begin{cases} -\frac{1}{4}\tau & -4 < \tau \le 0 \\ 0 & \text{else} \end{cases}$

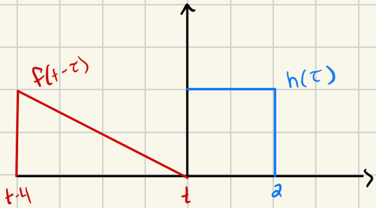
$$f(t-\tau) = \begin{cases} \frac{1}{4}(t-\tau) & 0 \le t-\tau < 4 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \frac{-1}{4}(\tau-t) & -t \le -\tau < 4-t \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \frac{-1}{4}(\tau-t) & t \ge \tau > t-4 \\ 0 & \text{else} \end{cases}$$

$f(t-\tau)$





$t<0$

$y(t)=0$

$t=0$: overlap

$t=2$: overlap 2

$t-4=2$

$t=6$; end of overlap

$0 \le t \le 2$

$$y(t)=\int_0^t (1)\left(\tfrac{-1}{4}(\tau-t)\right)d\tau \Rightarrow \tfrac{-1}{4}\left(\tfrac{1}{2}\tau^2-\tau t\right)\Big|_0^t = \frac{t^2}{8}$$

$2 \le t < 4$

$$y(t)=\int_0^2 (1)\left(\tfrac{-1}{4}(\tau-t)\right)d\tau \Rightarrow \tfrac{-1}{4}\left(\tfrac{1}{2}\tau^2-\tau t\right)\Big|_0^2 = \tfrac{1}{2}(t-1)$$

$4 \le t < 6$

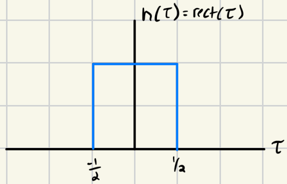
$$y(t)=\int_{t-4}^2 (1)\left(-\tfrac{1}{4}(\tau-t)\right)d\tau = \tfrac{-1}{4}\left(\tfrac{1}{2}\tau^2-\tau t\right)\Big|_{t-4}^2 \Rightarrow y(t)=\tfrac{1}{8}(t+2)(6-t)$$

$$y(t)=\begin{cases}\frac{1}{8}t^2 & 0 \le t < 2\\ \frac{1}{2}(t-1) & 2 \le t < 4\\ \frac{1}{8}(t+2)(6-t) & 4 \le t < 6\\ 0 & \text{else}\end{cases}$$

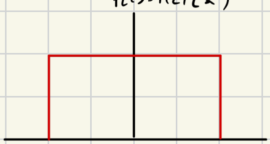
Ex

$\mathrm{rect}\left(\frac{t}{2}\right) * \mathrm{rect}(t)$

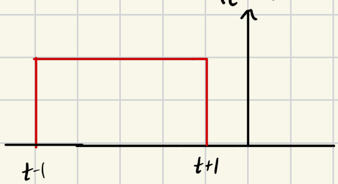
$h(\tau)=\mathrm{rect}(\tau)$



$f(\tau)=\mathrm{rect}\left(\frac{\tau}{2}\right)$



$f(t-\tau)$



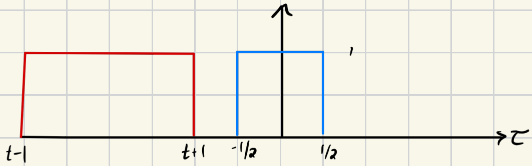
$$f(t-\tau)=\mathrm{rect}\left(\frac{t-\tau}{2}\right)$$
$$=\mathrm{rect}\left(-\left(\frac{\tau-t}{2}\right)\right)$$
$$=\mathrm{rect}\left(\frac{\tau-t}{2}\right)$$

$\frac{\tau-t}{2}=\frac{1}{2} \Rightarrow \tau-t=1$

$\tau=t+1$

$+$

$\tau=t-1$



$\underset{-1}{t+1} = \underset{-1}{\frac{-1}{2}} \Rightarrow t = \frac{-3}{2}$ (start of overlap)

$t-1 = \frac{1}{2} \Rightarrow t = \frac{3}{2}$ (end of overlap)

$t+1 = \frac{1}{2} \Rightarrow t = \frac{-1}{2}$ (blue fully enclosed by red)

$t-1 = \frac{-1}{2} \Rightarrow t = \frac{1}{2}$ (blue no longer fully enclosed by red)

$\underline{-\frac{3}{2} \le t < \frac{-1}{2}}$

$$y(t) = \int_{-1/2}^{t+1} (1)(1)\,d\tau = \tau\Big|_{-1/2}^{t+1} = t+1+\frac{1}{2} = t+\frac{3}{2}$$

$\underline{\frac{-1}{2} \le t < \frac{1}{2}}$

$$y(t) = \int_{-\frac{1}{2}}^{1/2} d\tau = \tau\Big|_{-1/2}^{1/2} = \frac{1}{2}+\frac{1}{2} = 1$$

$\underline{\frac{1}{2} \le t < \frac{3}{2}}$

$$y(t) = \int_{t-1}^{1/2} (1)(1)\,d\tau = \tau\Big|_{t-1}^{1/2} = \frac{1}{2}-t+1 = \frac{3}{2}-t$$

$$y(t) = \begin{cases} t+\frac{3}{2} & \frac{-3}{2} \le t < \frac{-1}{2} \\ 1 & \frac{-1}{2} \le t < \frac{1}{2} \\ \frac{3}{2}-t & \frac{1}{2} \le t < \frac{3}{2} \\ 0 & \text{else} \end{cases}$$

Properties

$$y(t) = \text{rect}\left(\frac{t}{2}\right) * \text{rect}(t) \Rightarrow \left(\text{rect}\left(t+\frac{1}{2}\right) + \text{rect}\left(t-\frac{1}{2}\right)\right) * \text{rect}(t)$$

$$= \text{rect}\left(t+\frac{1}{2}\right) * \text{rect}(t) + \text{rect}\left(t-\frac{1}{2}\right) * \text{rect}(t)$$

$y(t-t_0) = f(t-t_0) * h(t)$

$\text{rect}(t) * \text{rect}(t) = \Delta\left(\frac{t}{2}\right)$

$$= \Delta\left(\frac{t+\frac{1}{2}}{2}\right) + \Delta\left(\frac{t-\frac{1}{2}}{2}\right)$$

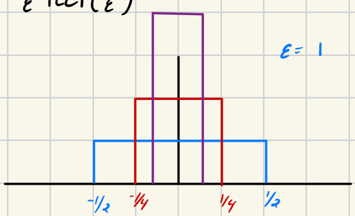
$$\text{rect}\left(\frac{t}{B}\right) * \text{rect}\left(\frac{t}{B}\right) = B \cdot \Delta\left(\frac{t}{2B}\right)$$

Impulse and its properties

* $a(1) = a$ ← scalar multiplication
* $MI = M$ ← matrix multiplication
* $f(t) * g(t) = f(t)$

Approximate Function

$$P_\varepsilon(t) = \frac{1}{\varepsilon}\,\text{rect}\left(\frac{t}{\varepsilon}\right)$$



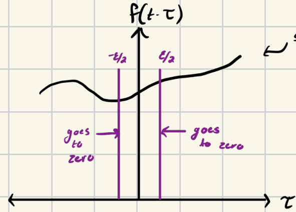
$$y(t) = f(t) * P_\varepsilon(t)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\varepsilon}\,\text{rect}\left(\frac{\tau}{\varepsilon}\right) f(t-\tau)\,\partial\tau$$

$$= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\varepsilon/2} (1)\, f(t-\tau)\,\partial\tau$$

$$= \frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2} f(t-\tau)\,\partial\tau$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2} f(t-\tau)\,\partial\tau \Rightarrow f(t)$$



Identity for convolution

$$f(t) * \delta(t) = f(t)$$

Properties of $\delta(t)$

1) $\delta(t-t_0) * f(t) = f(t-t_0)$

2) $\delta(t) = \frac{\partial u}{\partial t}$

$$u(t) = \int_{-\infty}^{t} \delta(x)\,\partial x$$

3) $\frac{\partial}{\partial t}\left(\delta(t)\right) * f(t) = \frac{\partial f}{\partial t}$

4) $\delta(t) \xleftrightarrow{F} 1$

$\boxed{\text{* Convolution in the time domain is the same as multiplication in the frequency domain *}}$

5) $\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = f(t_0)$

6) $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

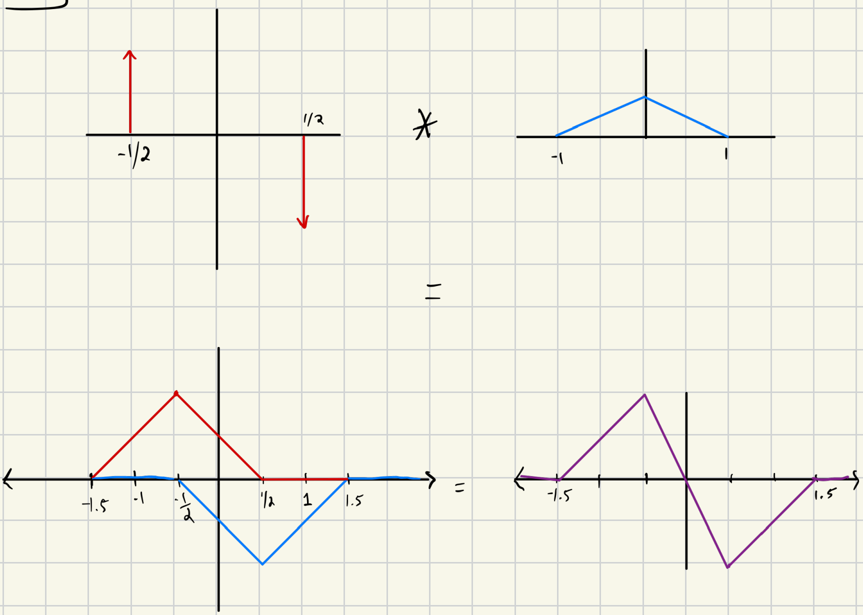
Ex) $f(t) = \text{rect}(t - \frac{1}{2}) \qquad h(t) = \delta(t-1) - \delta(t-3)$

$y(t) = \text{rect}(t-\frac{1}{2}) * (\delta(t-1) - \delta(t-3))$

$= \text{rect}(t-\frac{1}{2}) * \delta(t-1) - \text{rect}(t-\frac{1}{2}) * \delta(t-3)$

$= \text{rect}(t-1.5) - \text{rect}(t-3.5)$

Ex)



Ex $H(\omega) = \frac{j\omega}{1+j\omega}$ $f(t) = rect(t)$

$$\mathcal{F} \updownarrow$$

$$F(\omega) = sinc\left(\frac{\omega}{2}\right)$$

$$Y(\omega) = \frac{j\omega}{1+j\omega}\left(sinc\left(\frac{\omega}{2}\right)\right) \longleftarrow$$ Not useful
Not helpful

$$f(t) = rect(t)$$

$$H(\omega) = \frac{j\omega}{1+j\omega} = \frac{j\omega+1-1}{1+j\omega} = \frac{1+j\omega}{1+j\omega} - \frac{1}{1+j\omega} = 1 - \frac{1}{1+j\omega}$$

$$h(t) = \delta(t) - e^{-t}u_s(t)$$

$$y(t) = \left(\delta(t) - e^{-t}u_s(t)\right) * \left(u_s(t+\tfrac{1}{2}) - u_s(t-\tfrac{1}{2})\right)$$

$$= \delta(t) * u_s(t+\tfrac{1}{2}) - \delta(t) * u_s(t-\tfrac{1}{2}) - e^{-t}u_s(t) * u_s(t+\tfrac{1}{2}) + e^{-t}u_s(t) * u_s(t-\tfrac{1}{2})$$

$$= u_s(t+\tfrac{1}{2}) - u_s(t-\tfrac{1}{2}) - (1-e^{-(t+1/2)})u_s(t+1/2) + (1-e^{-(t-\frac{1}{2})})u_s(t-1/2)$$

$$= rect(t) - \left(1-e^{-(t+\frac{1}{2})}\right)u_s(t+\tfrac{1}{2}) + \left(1-e^{-(t-1/2)}\right)u_s(t-\tfrac{1}{2})$$

FT of power signals

What is a power signal?

Energy Signals : $W = \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$

What about signals s.t.

$\underbrace{|f(t)|^2 < \infty}_{\text{instantaneous power}}$, $W = \int_{-\infty}^{\infty} |f(t)|^2 dt \to \infty$

* Power Signal : $|f(t)|^2 < \infty$, $W \to \infty$

Ex, $\cos(w_0 t)$, $\sin(w_0 t)$, $u_s(t)$

Fourier Transform of Power Signals

$f(t) = \cos(w_0 t)$

$$F(w) = \int_{-\infty}^{\infty} \cos(w_0 t) e^{-jwt} dt = \frac{1}{2}\int_{-\infty}^{\infty} (e^{jw_0 t} + e^{-jw_0 t}) e^{-jwt} dt = \frac{1}{2}\int_{-\infty}^{\infty} e^{j(w_0 - w)t} + e^{-j(w_0 + w)t} dt$$

Value is unhelpful since

$\lim_{t\to\infty} e^{jwt} = \infty$

Solve the inverse problem instead.

We know we want $f(t) = \cos(w_0 t) = \frac{1}{2}(e^{jw_0 t} + e^{-jw_0 t})$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w) e^{jwt} dw$$

$$\frac{1}{2}\left[e^{jw_0 t} + e^{-jw_0 t}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w) e^{jwt} dw$$

$$\pi\underline{\left[e^{jw_0 t} + e^{-jw_0 t}\right]} = \int_{-\infty}^{\infty} F(w) e^{jwt} dw$$

← By anylzing this, we see only two frequency values survive: w_0 and $-w_0$

$$\boxed{F(w) = \pi\left[\delta(w - w_0) + \delta(w + w_0)\right]}$$

Ex) $f(t) = \sin(\omega_0 t) = \frac{-j}{2}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$

$$f(t) = \frac{-j}{2}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$$

$$j\pi\left(e^{-j\omega_0 t} - e^{j\omega_0 t}\right) = \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$$

$$F(\omega) = j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

Ex) $f(t) = 1$

$$1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$$

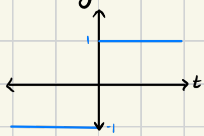
$$2\pi = \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$$

$\omega = 0$ rad/sec

$F(\omega) = 2\pi\,\delta(\omega)$

Ex) $f(t) = u_s(t) = \frac{1}{2}(\mathrm{sgn}(t) + 1)$

sgn(t)



$1 \overset{F}{\longleftrightarrow} 2\pi\,\delta(\omega)$

$\mathrm{sgn}(t) \overset{F}{\longleftrightarrow} \frac{2}{j\omega}$

$$f(t) = u_s(t) = \frac{1}{2}(1 + \mathrm{sgn}(t)) \longleftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

Ex) Modulation

$$f(t)g(t) \overset{F}{\longleftrightarrow} \frac{1}{2\pi} F(\omega) * G(\omega)$$

$$f(t) * g(t) \longleftrightarrow F(\omega)\cdot G(\omega)$$

let $g(t) = \cos(\omega_c t)$

$$f(t)\cos(\omega_c t) \overset{F}{\longleftrightarrow} \frac{1}{2\pi}\left(F(\omega) * \pi\left[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)\right]\right)$$

$$\longleftrightarrow \frac{1}{2}\left(F(\omega) * \delta(\omega-\omega_c) + F(\omega) * \delta(\omega+\omega_c)\right)$$

$$\longleftrightarrow \frac{1}{2}\left(F(\omega-\omega_c) + F(\omega+\omega_c)\right)$$

Let's now try the Fourier Transform of a generic periodic function

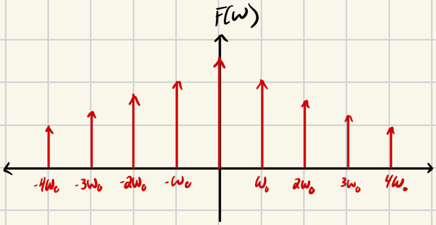
<u>periodic Signal</u>

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

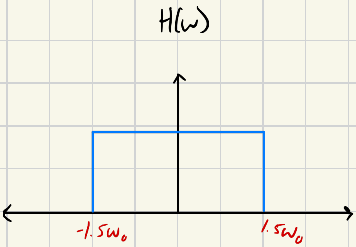
$$\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

integer multiples of some base ω

$$F(\omega) = \sum_{n=-\infty}^{\infty} 2\pi F_n \delta(\omega - n\omega_0)$$



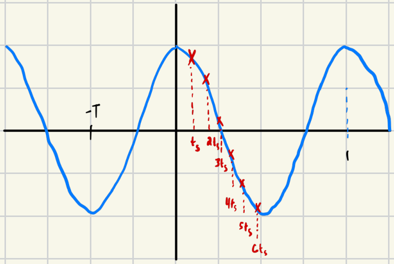
* Impulse Train: sum of delta functions a fixed interval apart *



Sampling and analog signal reconstruction

$f(t) = \cos(2\pi f t)$ ← f is a frequency in Hz

$f = \frac{1}{T}$



f_s = sampling rate

t_s = sampling time

$t = \frac{1}{f_s}$

$F[n] = f(nt_s) = \cos(2\pi f \underset{\uparrow}{n} t_s) = [f(t_s), f(2t_s), f(3t_s), \ldots]$

some integer

Reconstruction:

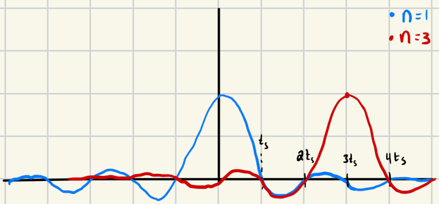
$f_r(t) = \sum_n F[n] \underbrace{\phi_n(t)}_{\text{basis function}}$

In this class, $\phi_n(t)$ will be $= \operatorname{sinc}\left(\frac{\pi}{t_s}[t - nt_s]\right)$

$\Rightarrow \sum_n F[n] \operatorname{sinc}\left(\frac{\pi}{t_s}[t - nt_s]\right)$

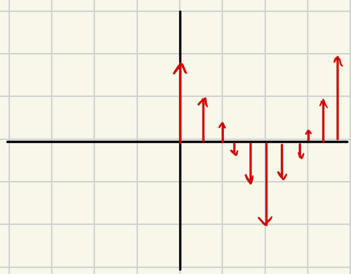
$\operatorname{sinc}(x) = \frac{\sin(x)}{x}$

$\phi_n = \begin{cases} 1 & t = nt_s \\ 0 & \text{all other vals of } t_s \end{cases}$



Now from continuous to discrete

$F[n] = \sum_n f(t)\delta(t - nt_s)$



We can also take the Fourier Transform of a discrete signal

$$\sum_{n} f(t)\delta(t-nt_s) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}\left(F(\omega) * \sum_{n=-\infty}^{\infty} \frac{2\pi}{t_s}\delta\left(\omega-n\frac{2\pi}{t_s}\right)\right)$$

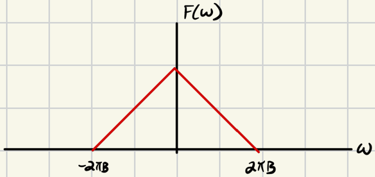
$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F(\omega) * \frac{2\pi}{t_s}\delta\left(\omega-n\frac{2\pi}{t_s}\right)$$

$$= \frac{1}{t_s}\sum_{n=-\infty}^{\infty} F\left(\omega-n\frac{2\pi}{t_s}\right)$$

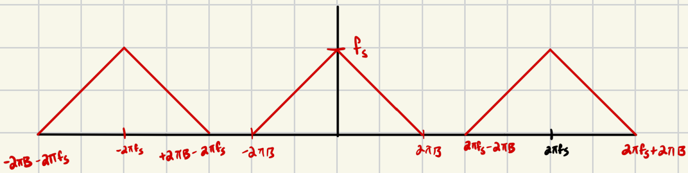
$$= f_s\sum_{n=-\infty}^{\infty} F(\omega-n2\pi f_s)$$

Let's say $f(t) \rightarrow$ Bandwith B

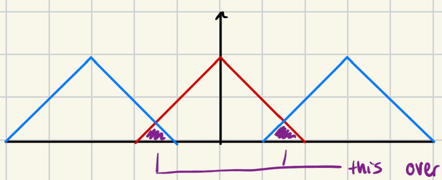
$\Omega = 2\pi B$



$\downarrow$ Sampling



for some sampling rate:

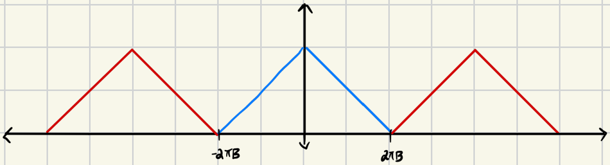


this overlap will cause noise.

$\therefore$ the minimum: $2\pi f_s - 2\pi B = 2\pi B$

$\downarrow$

what this means



$\underbrace{f_s = 2B}$ — Nyquist Frequency

Nyquist's Sampling Theorem:

$f_s > 2B$ to reconstruct $f(t)$ from sampled signal

If $f_s < 2B$: aliasing (projecting higher frequency signal onto a lower frequency)

DAC: Digital to analog converter

$$f(nt_s) \longrightarrow \otimes \xrightarrow{m(t)}$$

$$\uparrow \sum_n p(t - nt_s) \quad \rightarrow \quad p(t-nt_s) = \mathrm{rect}\left(\frac{t-nt_s}{t_s}\right)$$