"Shark-A"

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OH: Monday, 3-4pm & Friday 2-3pm in Computing Applications Bu

Lecture 1

Jan 18, 2023

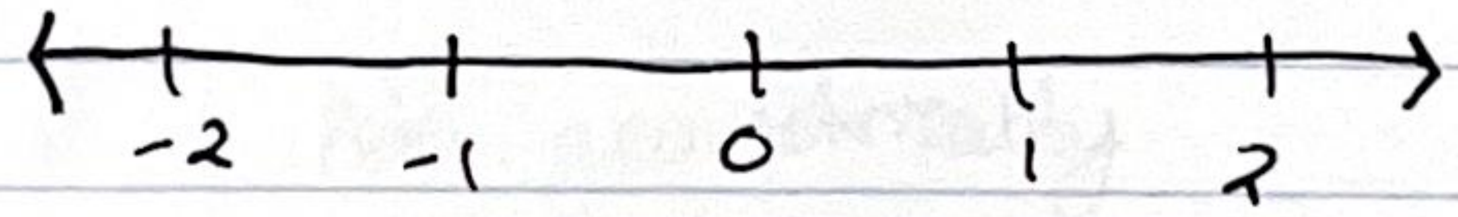
Tests: Feb 17
Mar 10
Apr 7
Apr 28

During lecture time

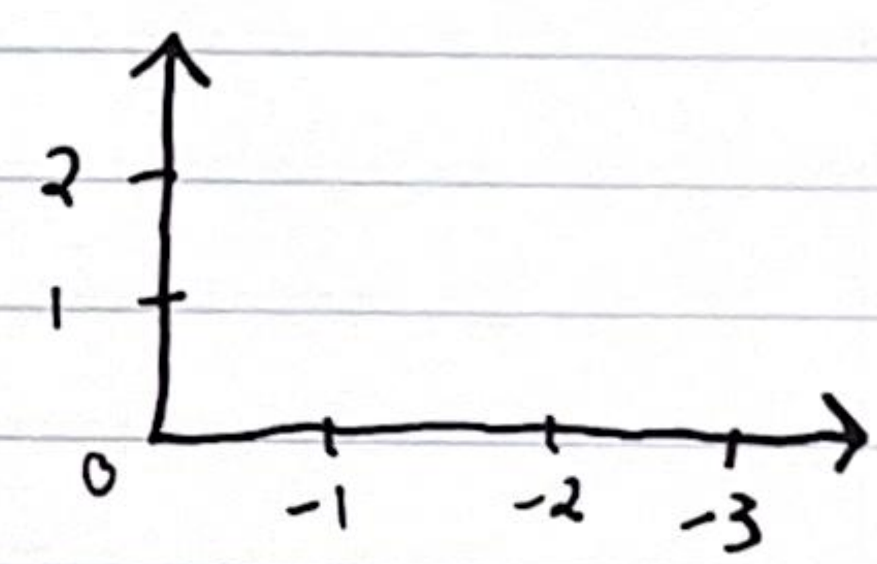
If you are unconfident in any way, shape, or form PLEASE do yourself a favor, Jack, and go to office hours.

12.1 Three Dimensional Cordinate systems

$\mathbb{R}$ = real numbers

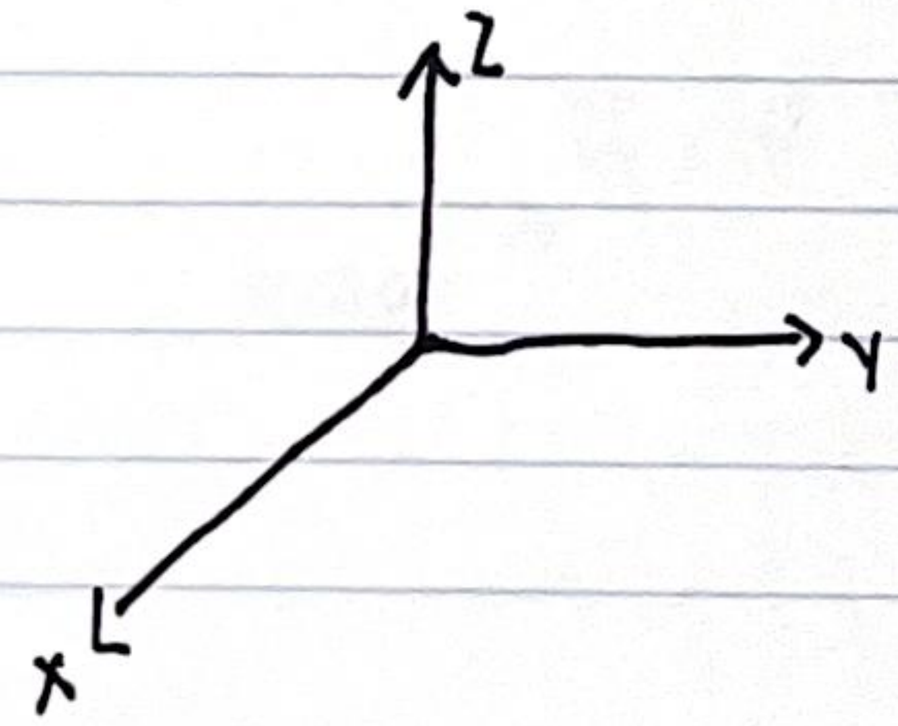


$\mathbb{R}^2$ = ordered pairs of real numbers = $\mathbb{R} * \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$



$\mathbb{R}^3$ = ordered triples (x, y, z) of real numbers =

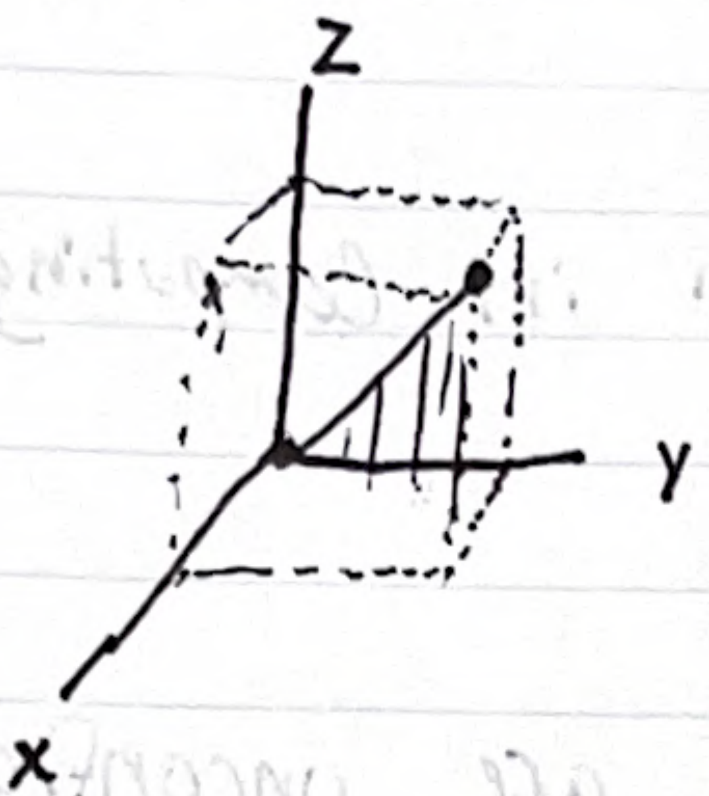
$$\mathbb{R} * \mathbb{R} * \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$



(must be like this)

xy plane, yz plane, xz plane

Distance in $\mathbb{R}^3$:



$$\sqrt{x^2+y^2+z^2} = d$$

Lecture 2

Jan 20, 2023

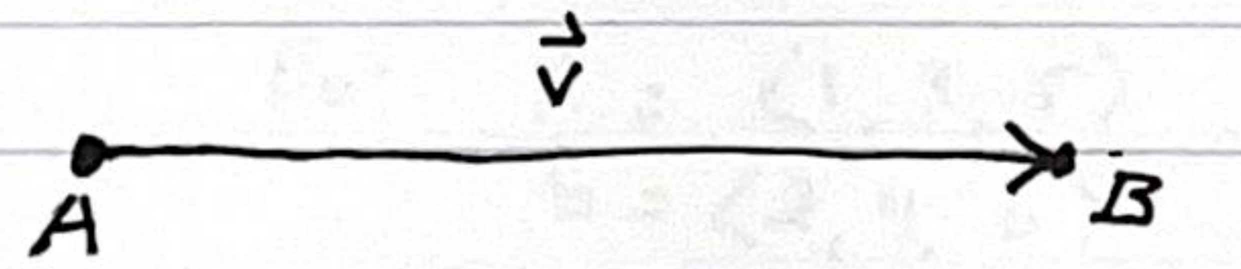
~~Spher~~ sphere: Set of all points (x, y, z) of distance r from $C(h, k, \ell)$

$$(x-h)^2 + (y-k)^2 + (z-\ell)^2 = r^2$$

12.2 Vectors

scalar: magnitude

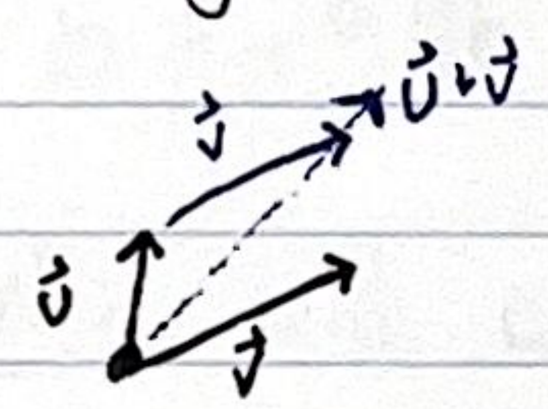
vector: magnitude + direction



$\vec{v} = \vec{AB}$

initial point (tail) → displacement vector from A to B ← terminal (head/tip)

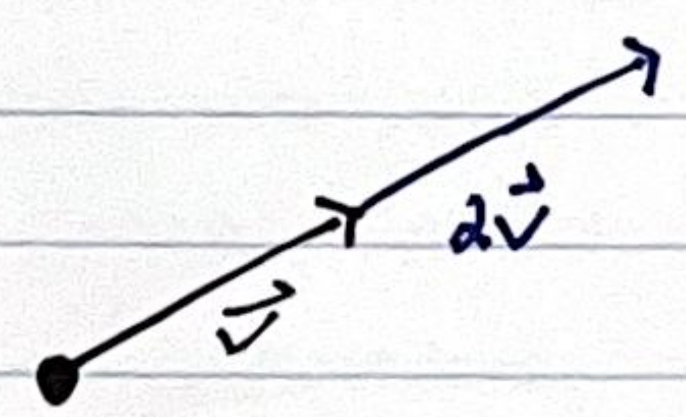
$\vec{u} + \vec{v}$: Add geometrically



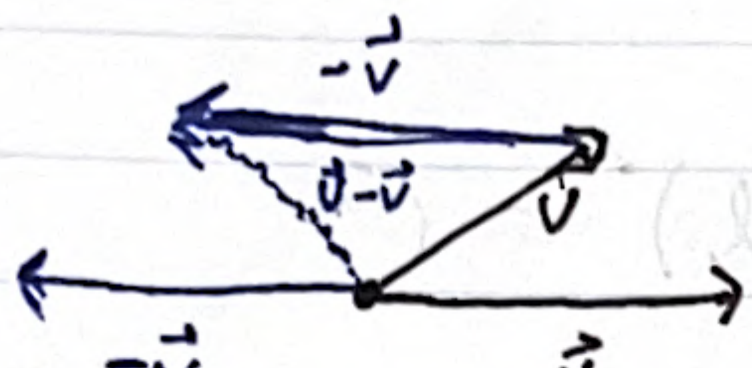
~~Scalar~~ vector multiple

$k \cdot \vec{v}$

scalar →



$\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v} = \vec{u} + (-\vec{v})$



Do they lay on the same line?

Lecture 3 Jan , 2023

Multiply vectors:

Dot Product -

$\vec{a} = \langle a_1, a_2, a_3 \rangle$
$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \cdot \vec{b} = \underbrace{a_1 b_1 + a_2 b_2 + a_3 b_3}_{\text{number}}$$

Ex: $\vec{a} = \langle 1, 3, 5 \rangle$
$\vec{b} = \langle 2, 4, 6 \rangle$

$$\vec{a} \cdot \vec{b} = 2 + 12 + 30 = 44$$

Angle between two vectors

lengths of $\vec{a}$ or $\vec{b}$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

Theta will always be $[0 \le \theta \le \pi]$

Lecture 4

Jan 25, 2023

Dot Product $\cdot$

- Valid in any dimension
- Result = scalar
- Application: work = $\vec{F} \cdot \vec{d}$

$\langle 1, -1, 2 \rangle \cdot \langle 0, 3, -4 \rangle = 0 \cdot 1 + -1(3) + 2(-4) = -11$

Cross Product $\times$

- Valid only in $\mathbb{R}^3$
- Result = vector
- Application: torque

$\langle 1, -1, 2 \rangle \times \langle 0, 3, -4 \rangle = \langle ?, ?, ? \rangle$

Determinate
~~Dot product~~ of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Determinate of order 2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$

Def: $\vec{a} \times \vec{b} =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{I} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{J} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{K} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \hat{I}(a_2 b_3 - b_2 a_3) - \hat{J}(a_1 b_3 - b_1 a_3) + \hat{K}(a_1 b_2 - b_1 a_2) =$$

$$\langle a_2 b_3 - b_2 a_3, a_1 b_3 - b_1 a_3, a_1 b_2 - b_1 a_2 \rangle$$

Properties of $\times$:

- $\vec{a} \times \vec{b}$ is a vector
- $\vec{a} \times \vec{b}$ is orthoganal to both $\vec{a}$ and $\vec{b}$
- $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$
- $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$
- $\vec{b} \times \vec{a}$ is also orthoganal to both $\vec{b}$ and $\vec{a}$ but in the opposite direction
- $\vec{a} \times \vec{a} = \vec{0}$
- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $(K\vec{a}) \times \vec{b} = K(\vec{a} \times b)$

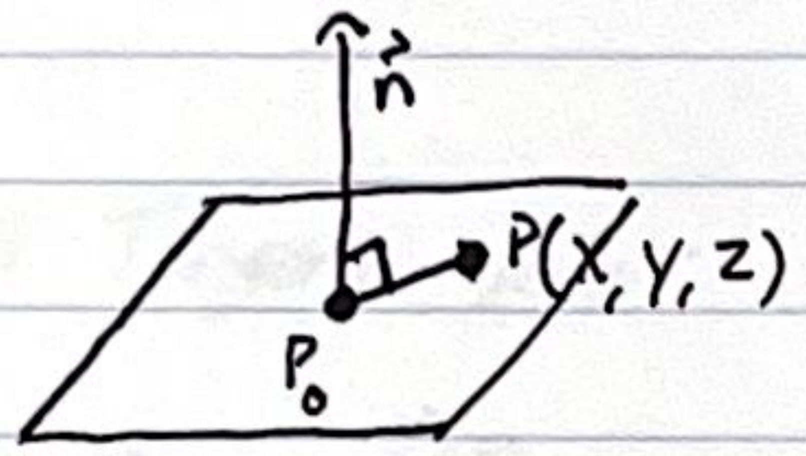
Theorem: $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

Lecture 6

Planes in $\mathbb{R}^3$

How to specify a plane?

- point $P_0(x_0, y_0, z_0)$
- normal vector $\vec{n} = \langle a, b, c \rangle$ (perpendicular to plane)



$P(x, y, z)$ is in the plane $\Leftrightarrow$

$$\Leftrightarrow \vec{n} \perp \overrightarrow{P_0 P}$$
$$\Leftrightarrow \vec{n} \cdot \overrightarrow{P_0 P} = 0$$
$$\Leftrightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Angle between two planes = angle between their normal vectors. $\vec{n}_1, \vec{n}_2$

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

Ex: find an equation of a plane that is parallel to the plane $8x - y - 3z + 3 = 0$ and goes through $(5, 6, 7)$

$$8(x - 5) + (-1)(y - 6) - 3(z - 7) = 0$$

Two planes are either parallel or intersecting

Find the line in which two planes intersect:

Plane 1: $8x - y - 3z + 3 = 0$

Plane 2: $-4x + y + 2z + 5 = 0$

- Find one point on both planes
  - $z = 0$ $\qquad$ $A(-2, -13, 0)$

$$\left.\begin{aligned} 8x - y + 3 &= 0 \\ -4x + y + 5 &= 0 \end{aligned}\right\} \text{add together}$$

$$4x + 8 = 0$$

$$x = -2 \qquad y = -13$$

- Find another point
  - $x = 0$ $\qquad$ $B(0, -21, 8)$

$$-y - 3z + 3 = 0$$
$$y + 2z + 5 = 0$$
$$-z + 8 = 0$$
$$z = 8$$
$$y = -21$$

Line:

$$\overrightarrow{AB} = \langle 2, -8, 8 \rangle$$

$$L \Rightarrow \begin{aligned} x &= -2 + 2t \\ y &= -13 - 8t \\ z &= 8t \end{aligned}$$

lecture 8

$$L = \int_a^b |\vec{r}(t)|\,dt$$

Arc length Function

$$s(t) = \int_a^t |\vec{r}'(u)|\,du$$

Unit Tangent vector

$$\vec{T}(t) = \frac{\vec{r}(t)}{|\vec{r}'(t)|}$$

Curvature = measure the rate of change of $\vec{T}(t)$ $\underbrace{\text{with respect to arclength}}_{|\vec{r}'(s)| = 1}$

Kappa $\rightarrow$ $$\kappa(s) = \left|\frac{d\vec{T}}{ds}\right|$$

if $|\vec{r}(s)| = 1$, then

$$\kappa(s) = \left|\frac{d\vec{r}(s)}{ds}\right| = |\vec{r}''(s)|$$

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

Example: Helix

$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$

$\vec{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle \Rightarrow |\vec{r}'(t)| = \sqrt{2}$

$\vec{T}(t) = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$

$\vec{T}'(t) = \frac{1}{\sqrt{2}} \langle -\cos(t), -\sin(t), 0 \rangle \Rightarrow |\vec{T}'(t)| = \frac{1}{\sqrt{2}}$

$$K(t) = \frac{|\vec{T}'|}{|\vec{r}'(t)|} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}$$

Velocity and Acceleration:

$\vec{r}(t)$ = position = $\vec{r}(t)$

$\vec{r}'(t)$ = velocity = $\vec{v}(t)$

$\vec{r}''(t)$ = accleration = $\vec{a}(t)$

Ex: $\vec{a}(t) = \langle t, 0, e^{-t} \rangle$

$\vec{r}(0) = \langle 0, 0, 1 \rangle$ Find $\vec{r}(t)$

$\vec{v}(0) = \langle 0, 1, 1 \rangle$

$\vec{a}(t) = \vec{v}'(t) \Rightarrow \int \vec{a}(t)\,dt = \vec{v}(t) + c$

$\int_0^t \vec{a}(u)\,du = \vec{v}(t) - \vec{v}(0)$

$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)\,du$

$\int_0^t \vec{a}(u)\,du = \int_0^t \langle u, 0, e^{-u} \rangle\,du = \left[\langle \frac{u^2}{2}, 0, \frac{e^{-u}}{-1} \rangle \right]_0^t =$

$\vec{v}(t) = \langle 0, 1, 1 \rangle + \langle t^2/2, 0, 1 - e^{-t} \rangle =$

$\langle \frac{t^2}{2}, 1, 2 - e^{-t} \rangle$

Lecture

Find critical points of :

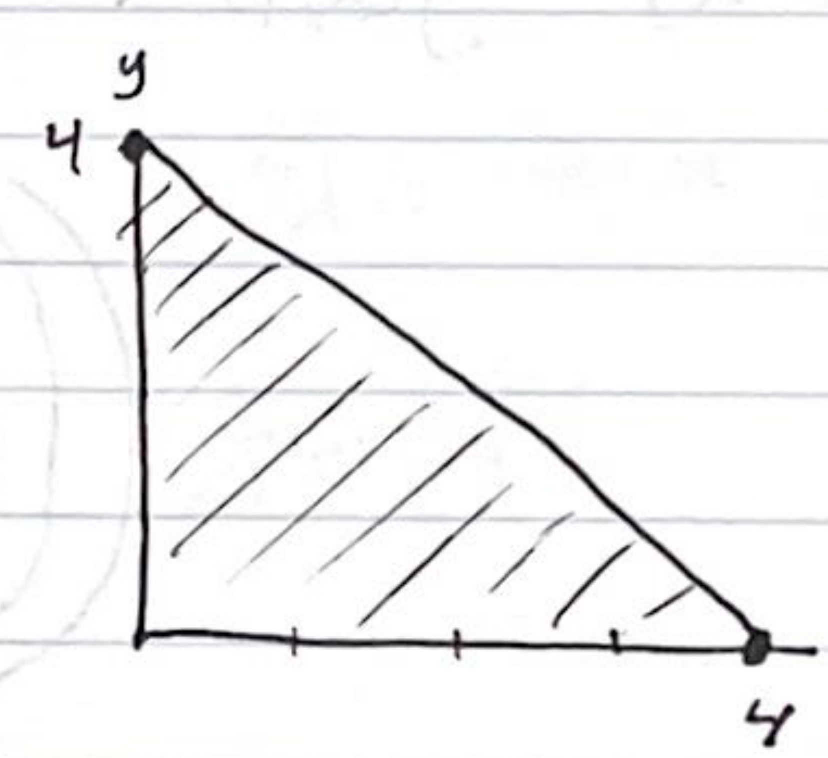
$$f(x,y) = y - xy$$

① Find critical values in D

$$\left.\begin{matrix} f_x = -y \\ f_y = 1-x \end{matrix}\right\} \left.\begin{matrix} -y=0 \\ 1-x=0 \end{matrix}\right\} = (1,0)$$

↑ critical point (saddle point)

② along $y=0,\ 0 \le x \le 4$



$$f(x,0) = 0$$

along $x=0,\ 0 \le y \le 4$

$$f(0,y) = y$$

Minimum when $y=0$ and max when $y=4$

along $y=-x+4$, $x=-y+4$ $\succ$ $h(y) = f(4-y, y) = y-(4-y)y$

$$= y - 4y + y^2$$

$$= y^2 - 3y$$

$$h'(y) = 2y - 3$$

$y = \frac{3}{2}$ ↳ plug into $h(y)$

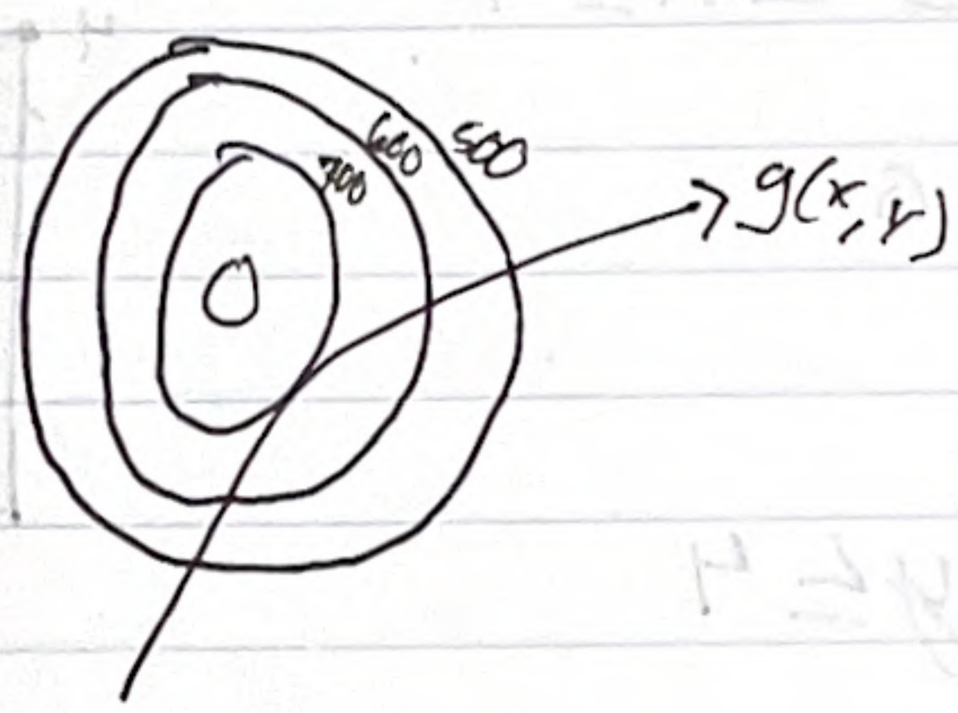
Lagrange Multipliers

Goal: Maximize (or minimize) $f(x,y)$ subject to a constraint $g(x,y)=K$

Ex: $f(x,y)$ represents trail
$g(x,y)=K$ represents trail

What is the max height along the trail

Ie find the largest c such that the curves $g(x,y)=K$ and $f(x,y)=c$ intersect



So $f(x,y)=c$ and $g(x,y)=K$ have a common tangent line at the point (x,y) where c is the maximum

$\vec{\nabla} f$ is orthogonal to level curve $f(x,y)=c$
$\vec{\nabla} g$ is orthogonal to

Lecture

Mar 1, 2023

Ex: maximize/minimize $f(x,y,z)=z$

subject to constraints $g(x,y,z)=4x-3y+8z=5$
$h(x,y,z)=x^2+y^2-z^2=0$

Lagrange eqn:

$$\vec{\nabla} f=\lambda\vec{\nabla} g+\mu\vec{\nabla} h$$

$$\langle 0,0,1\rangle=\lambda\langle 4,-3,8\rangle+\mu\langle 2x,2y,-2z\rangle$$

① $0=4\lambda+2x\mu \Rightarrow 4\lambda=-2x\mu \Rightarrow \lambda=\frac{-x\mu}{2}$

② $0=-3\lambda+2y\mu \quad = \lambda=\frac{\mu 2y}{3}$

③ $1=8\lambda+-2z\mu$

cancel as long as it's not 0

① + ②) $\frac{2y\mu}{3}=\frac{-x\mu}{2} \Rightarrow \frac{3}{2}\cdot\frac{2y}{3}=\frac{-x}{2}\cdot\frac{3}{2} \Rightarrow y=\frac{-3x}{4} \Rightarrow x=\frac{-4}{3}y$

④ $z^2=x^2+y^2=\left(\frac{-4}{3}y\right)^2+y^2=\frac{16}{9}y^2+\frac{9}{9}y^2=\frac{25}{9}y^2 \Rightarrow z=\pm\frac{5}{3}y$

⑤ $4x-3y+8z=5$

Case 1: $x=\frac{-4}{3}y,\ z=\frac{5}{3}y$

$4\left(\frac{-4}{3}y\right)-3y+8\left(\frac{5}{3}y\right)=5 \Rightarrow$

$y=1$

Case 2: $x = -\frac{4}{3}y,\ z = -\frac{5}{3}y$

$$4\left(\frac{-4}{3}y\right) - 3y + 8\left(-\frac{5}{3}y\right) = 5$$

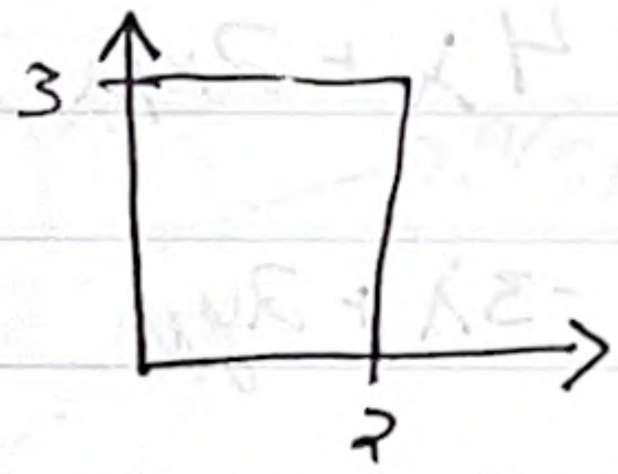
$$y = \frac{-3}{13}$$

MAX: $(x,y,z) = \left(\frac{-4}{3}, 1, \frac{5}{3}\right)$

MIN: $(x,y,z) = \left(\frac{4}{13}, \frac{-3}{13}, \frac{5}{13}\right)$

15.1 Double integral

$$\iint_R 5\,dA$$



$R = [0,2] \times [0,3]$

$$\iint 5\,dA = 5 \cdot 2 \cdot 3 = 30$$

Definitions:

$R_{i,j} =$

$\Delta A = \Delta x \Delta y$

~~Volume of prism~~ =

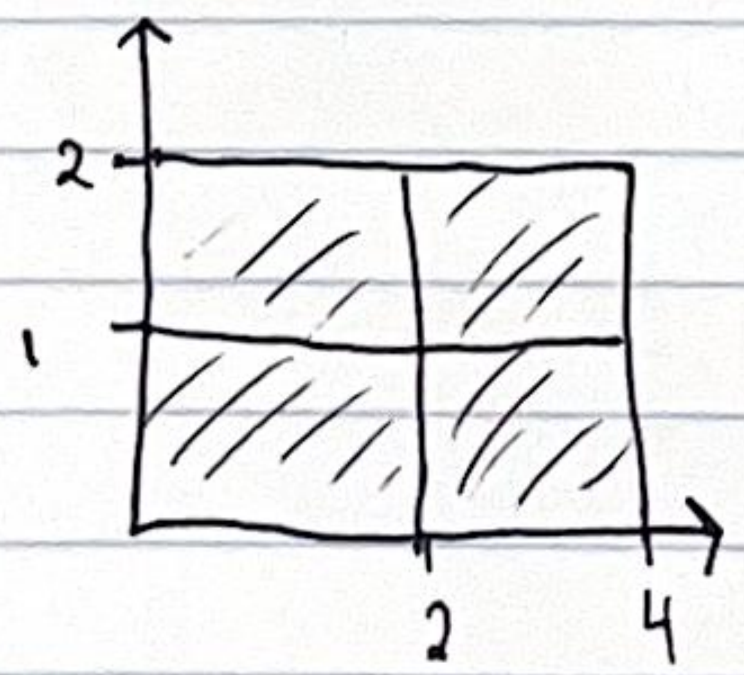
Riemann Sum:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

Ex Estimate volume of the solid that lies below $f(x,y) = x^2y + x$ and above $R = [0,4] \times [0,2]$. Use $m=2$ and $n=2$

a)



$$\iint_R (x^2y+x)\,dx \approx f(2,0)\cdot\Delta A + f(4,0)\Delta A + f(2,1)\Delta A + f(4,1)\Delta A$$

$$\approx \Delta A\left(f(2,0) + f(4,0) + f(2,1) + f(4,1)\right)$$

$$\approx 2(2 + 4 + 6 + 20)$$

$$\approx 64$$

Fubini's Theorem

If f is continuous on $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$

then:

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

double integral

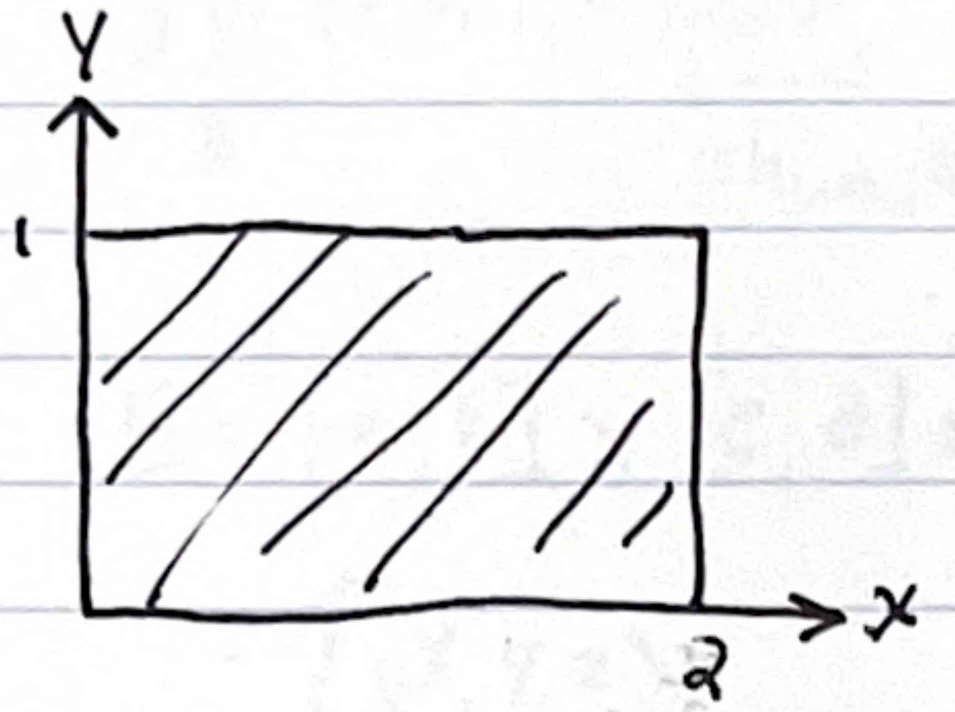
iterated single integral

Lecture

Mar 8, 2023

Applications of Double Integrals

Mass: If $P(x,y)$ is the (Area) density of a
then the total mass = $\iint P(x,y)\,dx\,dy$



$$\int_0^1\int_0^2 (4-x-y)\,dx\,dy =$$

$$\int_0^1 \left[4x - \frac{x^2}{2} - xy\right]_0^2 dy =$$

$$\int_0^1 [8-2-2y]\,dy =$$

$$\left[8y-2y-y^2\right]_0^1 =$$

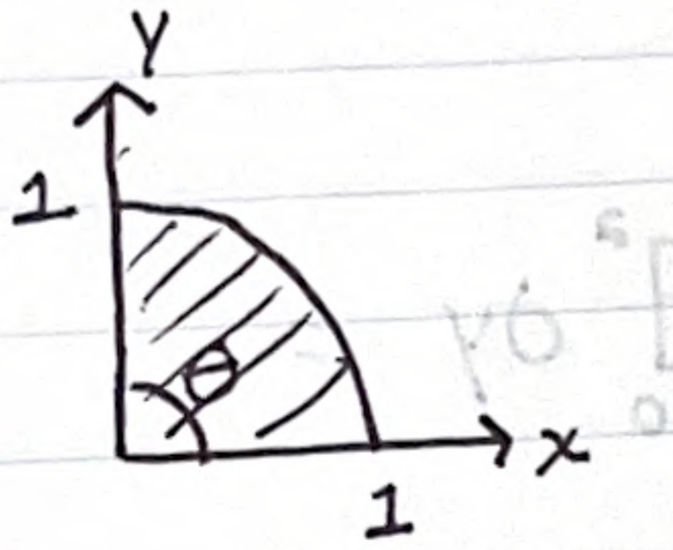
$$8-2-1=\boxed{5}$$

Moment of Inertia around x-axis: $I_x = \iint y^2 \cdot \rho \, dA$

around y-axis: $I_y = \iint x^2 \rho \, dA$

$r^2 = x^2 + y^2$

$I = I_x + I_y$



$I_x = \iint_D y^2 \rho \, dA = \int_0^{\frac{\pi}{2}}\int_0^1 r^2 \sin^2\theta \,(1-r^2)\, r \, dr \, d\theta$

$dA = r\,dr\,d\theta$

~~Avg area (of a rectang)~~

Avg value of a function

$f_{avg} = \frac{1}{A(R)} \iint_R f(x,y) \, dA$

Lecture

Tripple Integrals:

$$f: \mathbb{R}^3 \to \mathbb{R}$$

$$\iiint_B f(x,y,z)\,\underbrace{dv}_{\text{chundk of space}}$$

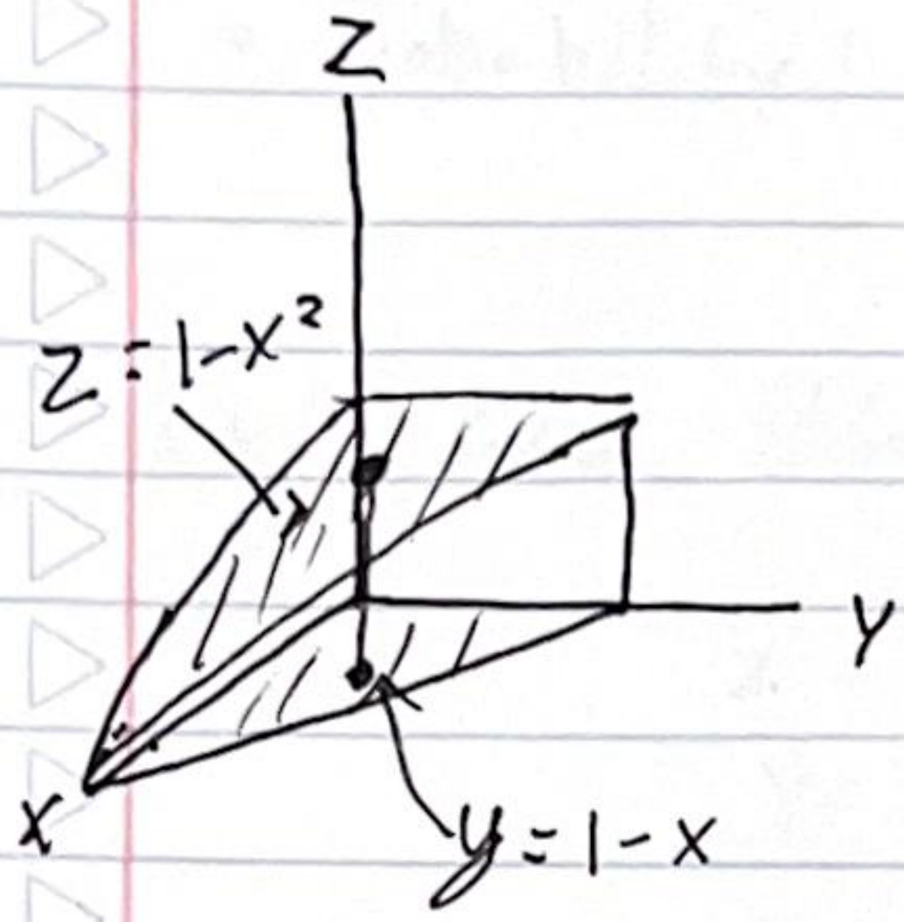
$$B = [a,b] \times [c,d] \times [r,s]$$
$$= \{(x,y,z)$$

If your function f is continuous, you are guarenteed a tripple integral.

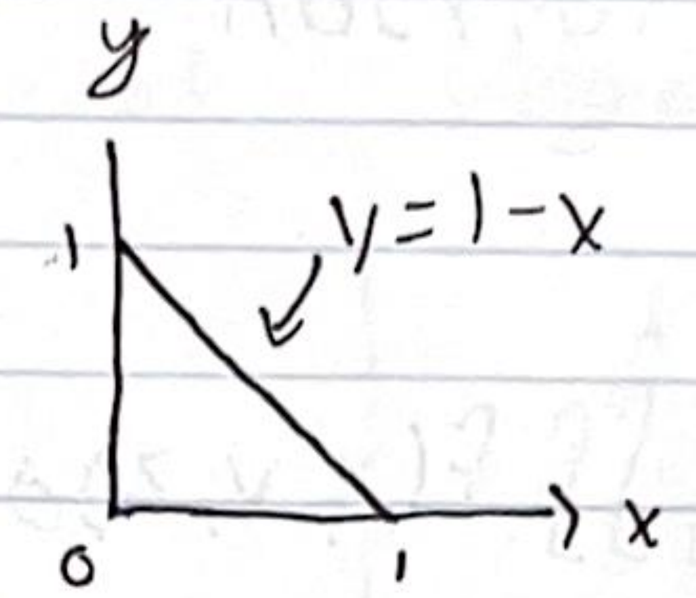
$$\int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Ex: $\int_0^2 \int_0^3 \int_0^1 (xy+z)\,dz\,dx\,dy$ (slicing by y, then ~~x~~, then z)

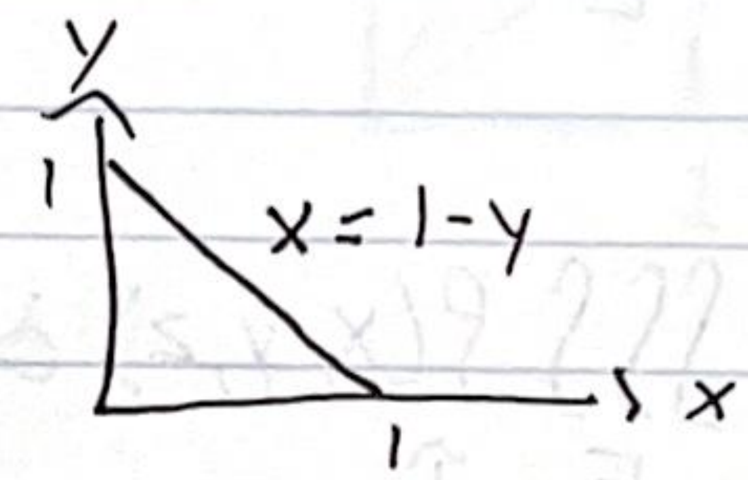
$$= \int_0^2 \int_0^3 \left[xyz + \frac{z^2}{2}\right]_{z=0}^{z=1} dx\,dy = \int_0^2 \int_0^3 xy + 1$$



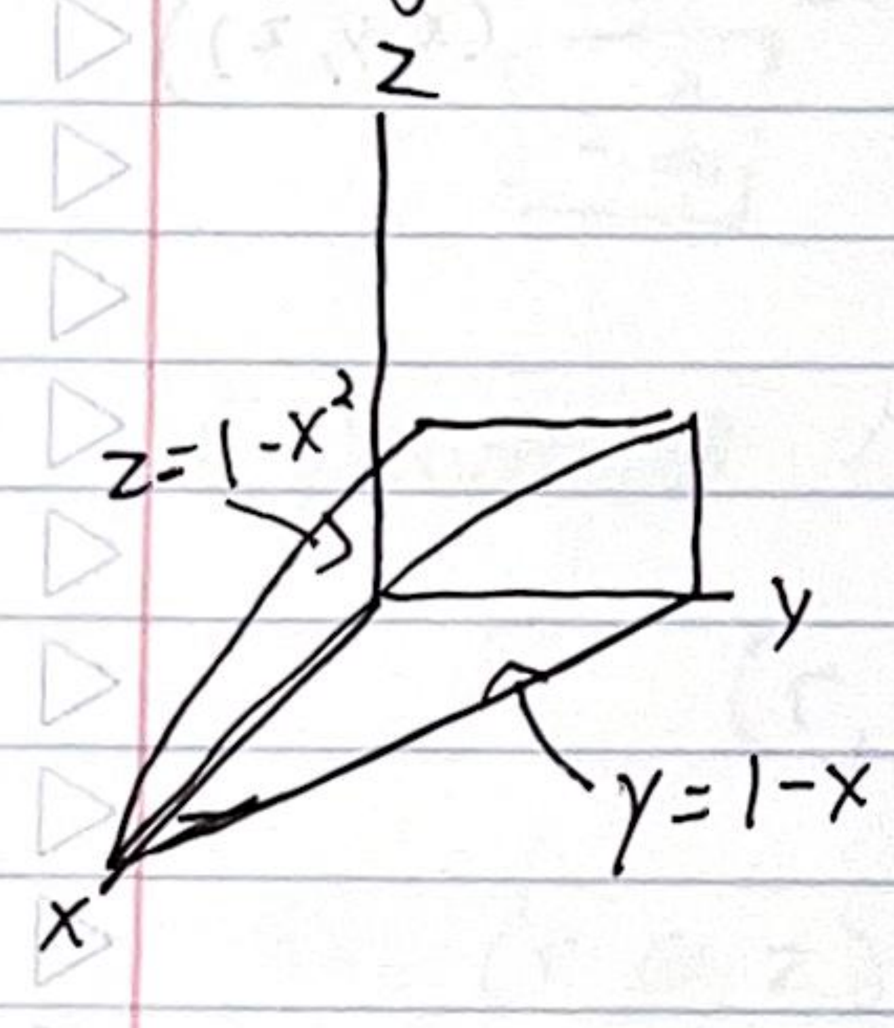
Project onto xy plane

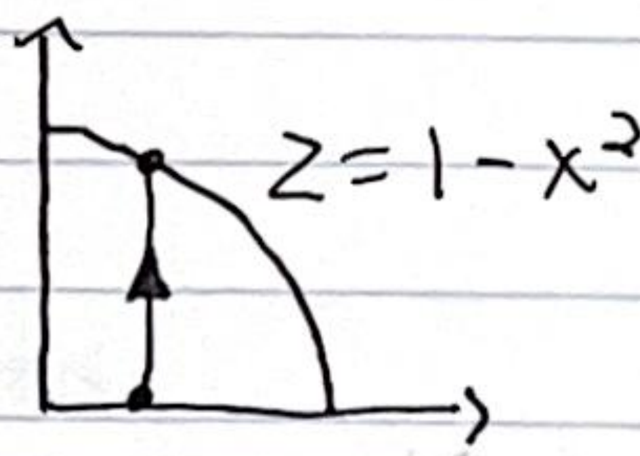


$$\int_0^1\int_0^{1-x}\int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dz$$



Project onto xz-plane





- Applications

  - Volume: $V(E)=\iiint_E 1\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} 1\,dz\right] dA$

given this is a type 1

• Average Value:

~~Doubl.~~ Two variables: $\frac{1}{A(D)} \iint_D f(x,y)\, dA$

Three Variables: $\frac{1}{V(E)} \iiint_E f(x,y,z)\, dV$

• mass: $m = \iiint_E \rho(x,y,z)\, dV$

$\uparrow$ density function

where $\rho(x,y,z)$ = density function ($\frac{\text{mass}}{\text{volume}}$ at each point (x,y,z))

• moments:

$$M_{yz} = \iiint_E x \cdot \rho(x,y,z)\, dV$$

$$M_{xz} = \iiint_E y \cdot \rho(x,y,z)\, dV$$

$$M_{xy} = \iiint z \cdot \rho(x,y,z)\, dV$$

• Center of Mass: $\bar{x} = \frac{M_{yz}}{\text{mass}}$, $\bar{y} = \frac{M_{xz}}{\text{mass}}$, $\bar{z} = \frac{M_{xy}}{\text{mass}}$

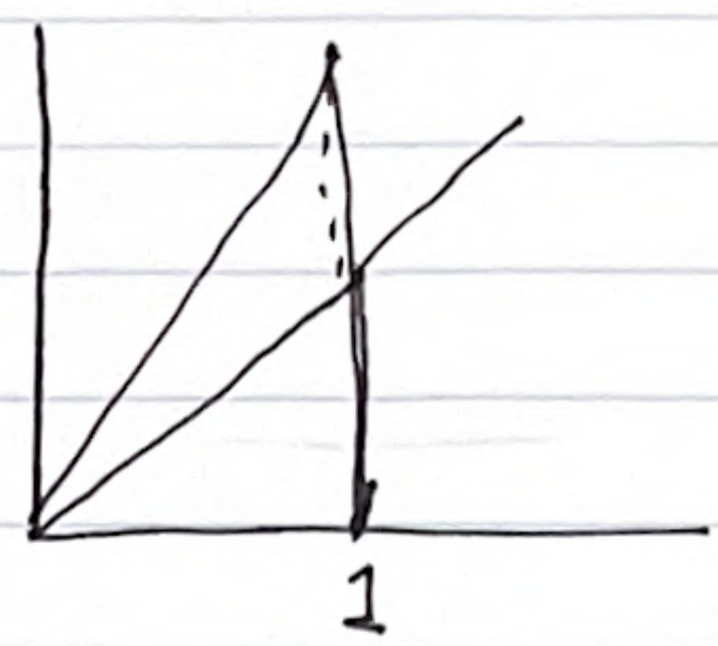
• Probability: X, Y, Z continuous random variables

Ex: Find total mass of wedge given:

$z = x + y$

$y = 2x$

$x = 1$



$$\text{mass} = \iiint_E \rho(x,y,z)\, dV$$

$$= \int_{x=0}^{1} \int_{y=0}^{2x} \int_{z=0}^{x+y} x \, dz \, dy \, dx$$

$$\boxed{=1}$$

Cylindrical Cordinates:

(r, θ, z)

Ex: $(r, \theta, z) = (4, \frac{-\pi}{4}, 8)$

$x = 4\cos(\frac{-\pi}{4})$

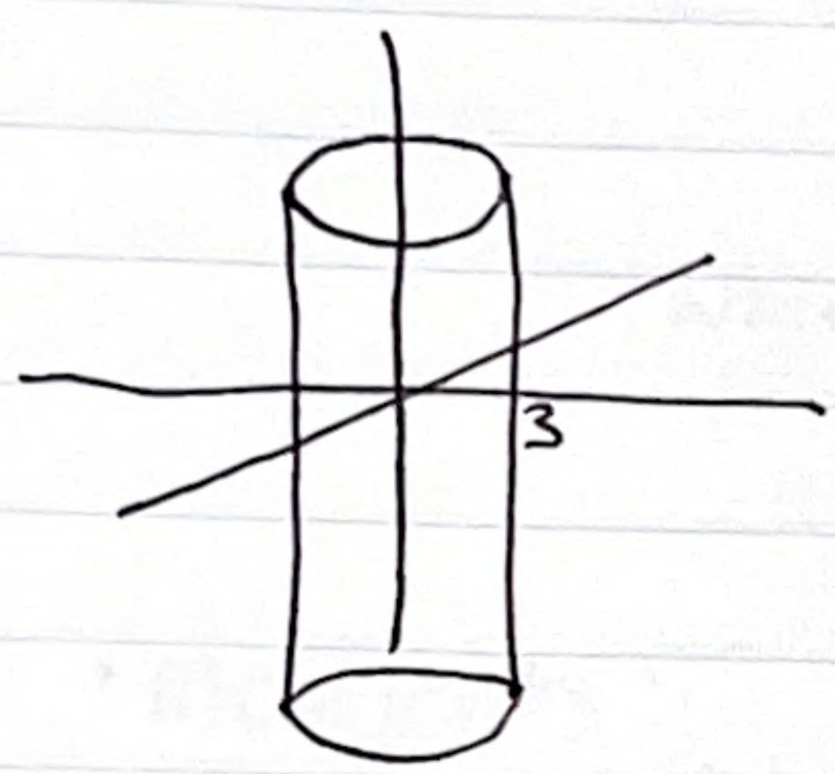
$y = 4\sin(\frac{-\pi}{4})$ $\Rightarrow (x,y,z) = (2\sqrt{2}, -2\sqrt{2}, 8)$

$z = 8$

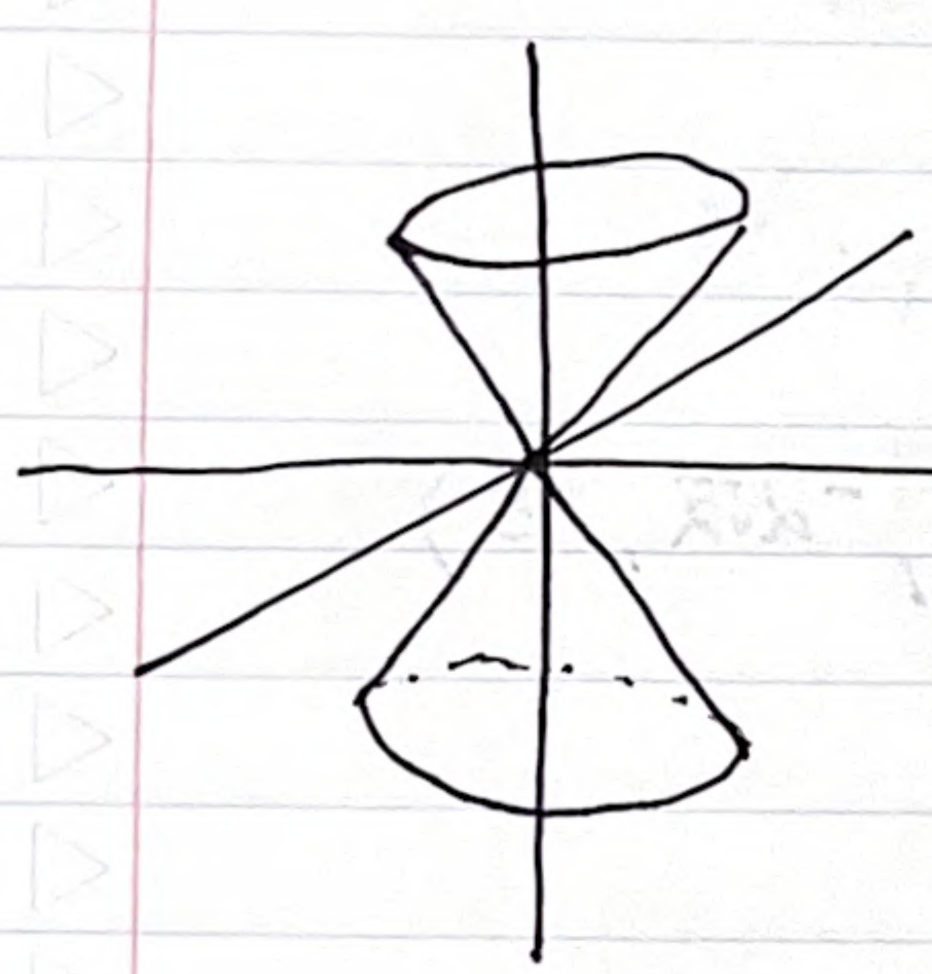
Ex: $(x, y, z) = (-2, 2\sqrt{3}, 3)$

$r^2 = (-2)^2 =$

Sketch: $r = 3$



$z = 2r$

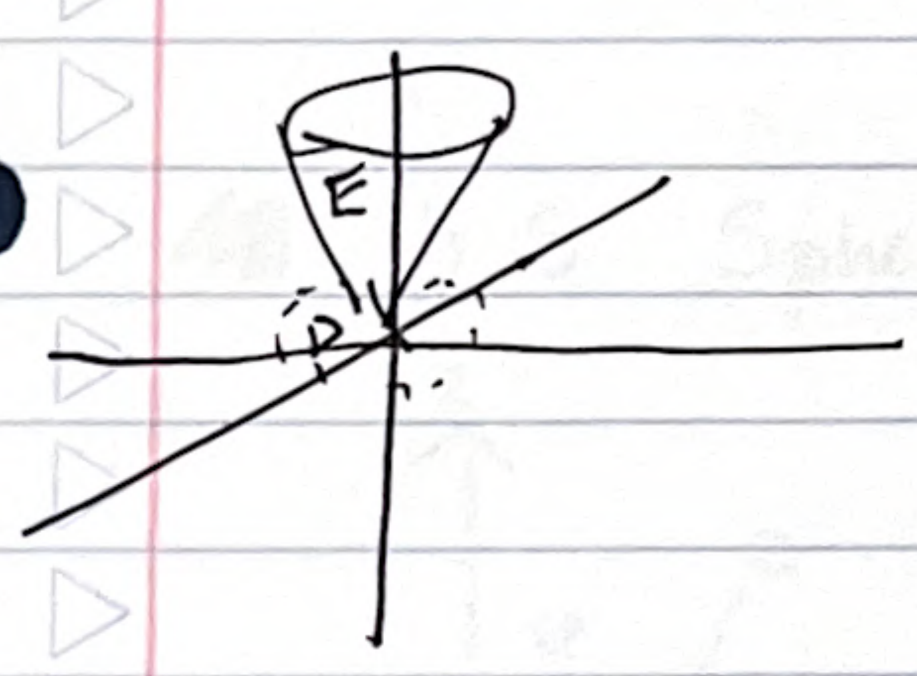


Triple Integrals in cylindrical coordinates

$$E = \{(x,y,z) \mid (x,y) \in D,\ z_1(x,y) \le z \le z_2(x,y)\}$$

$D = \{$

Example: Find mass of cone $x^2+y^2 = \frac{z^2}{4}$ between $z=0$ and $z=8$ w/ density $\rho(x,y,z) = 10-z$



$$r^2 = \frac{z^2}{4} \Rightarrow r = \frac{z}{2}$$

$$z = 2r$$

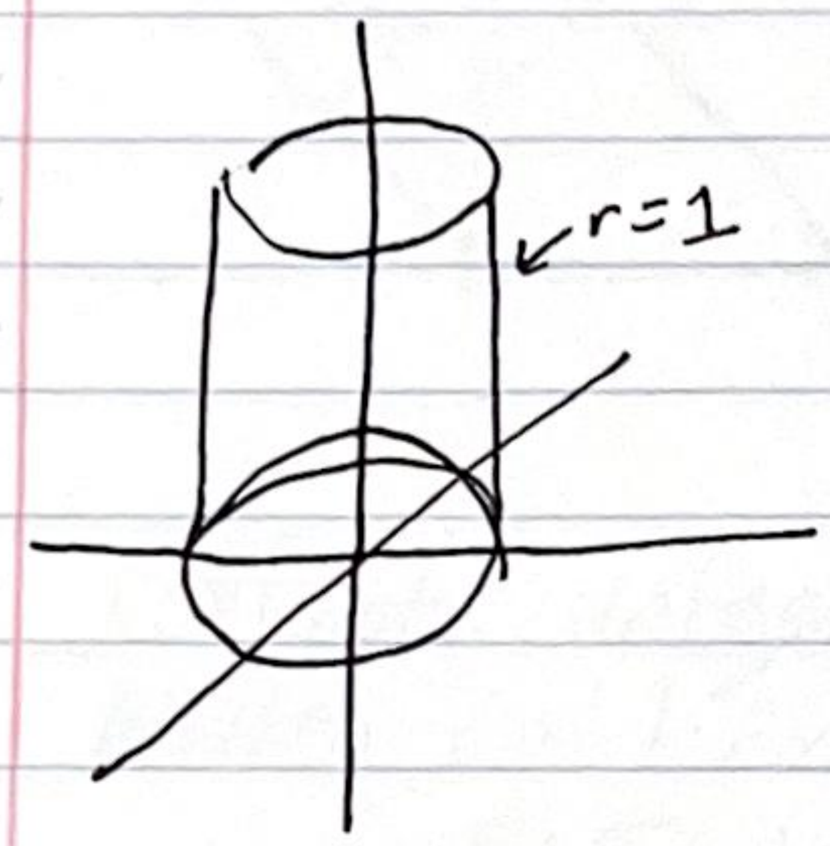
$$\text{mass} = \iiint_E \rho(x,y,z)\,dV$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{4} \int_{z=2r}^{8} (10-z)\,r\,dz\,dr\,d\theta$$

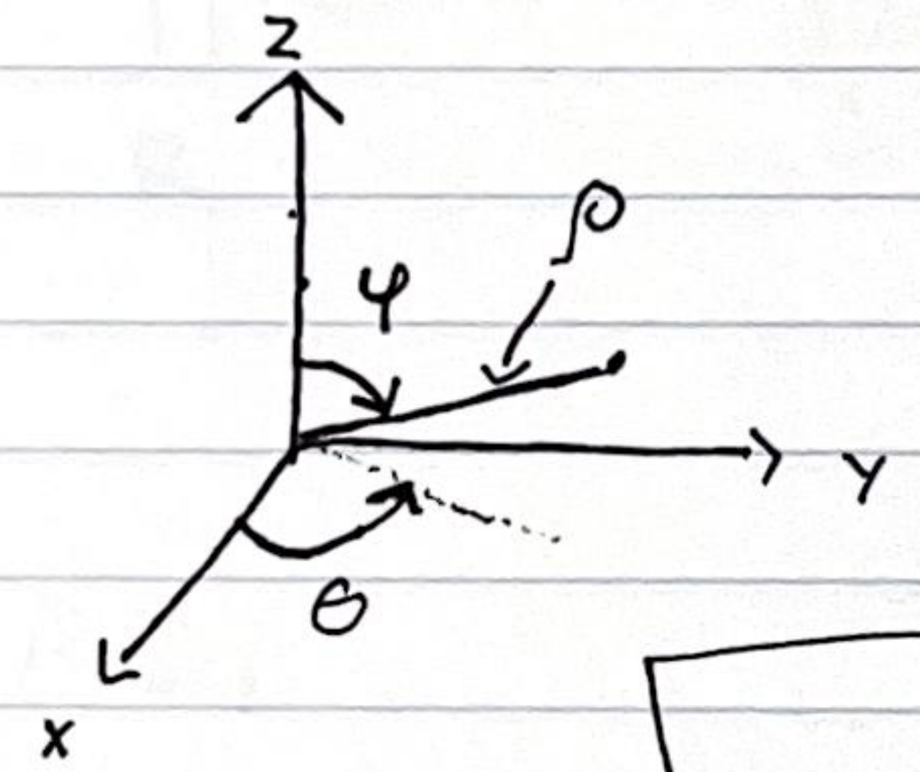
Lecture

Ex: Let E be bounded by $x^2+y^2=z$, $z=4$, $z=0$, and $z=1-x^2-y^2$

Two parts: bot and top



15.8 Spherical coordinates (ρ, θ, φ)



ρ rho	θ theta	φ phi
length of segment	angle between x-axis and proj of segment on x-y	angle between z-axis and the segment

Spherical $\to$ rectangular
$x=(\rho\sin\varphi)\cos\theta$
$y=(\rho\sin\varphi)\sin\theta$
$z=\rho\cos\varphi$
Rectangular $\to$ spherical
$\rho^2=x^2+y^2+z^2$
~~$\sin\theta=$~~ $\tan\theta=\frac{y}{x}$
$\cos\varphi=\frac{z}{\rho}$

$dV=\rho^2 \cancel{\sin\theta} \sin\varphi \, d\rho \, d\varphi \, d\theta$

$\rho\geq 0$

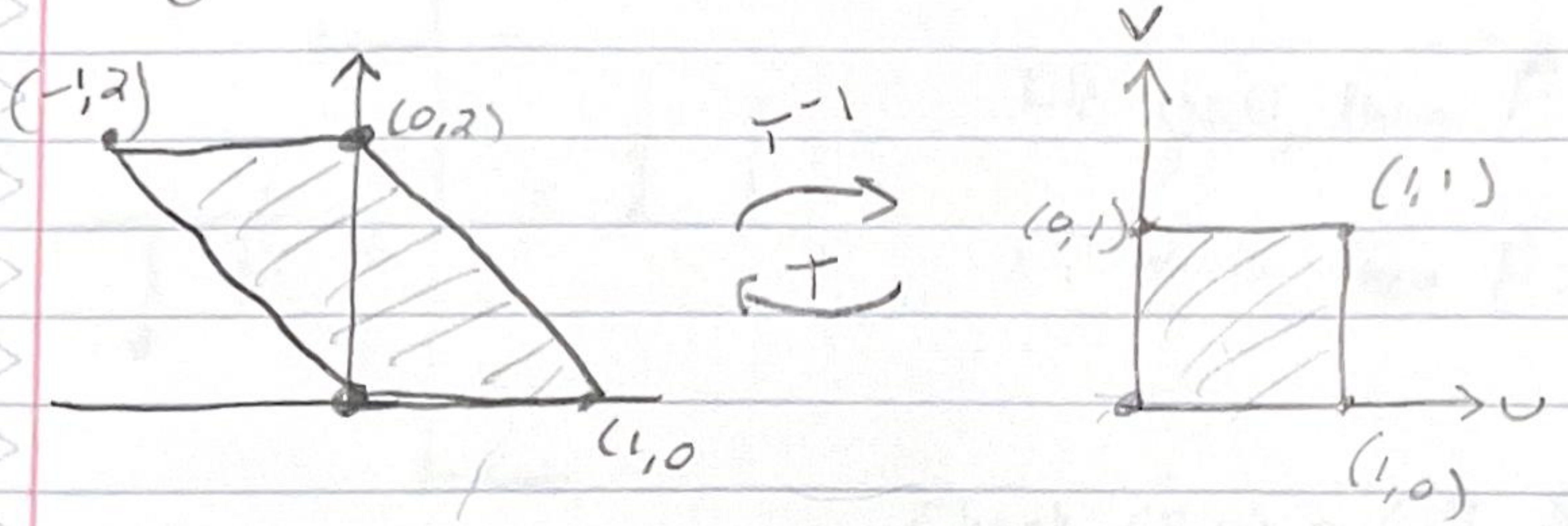
$0\leq\theta\leq 2\pi$

$0\leq\varphi\leq\pi$

Lecture

15.9 Change of variables in multiple integrals

Big Picture:



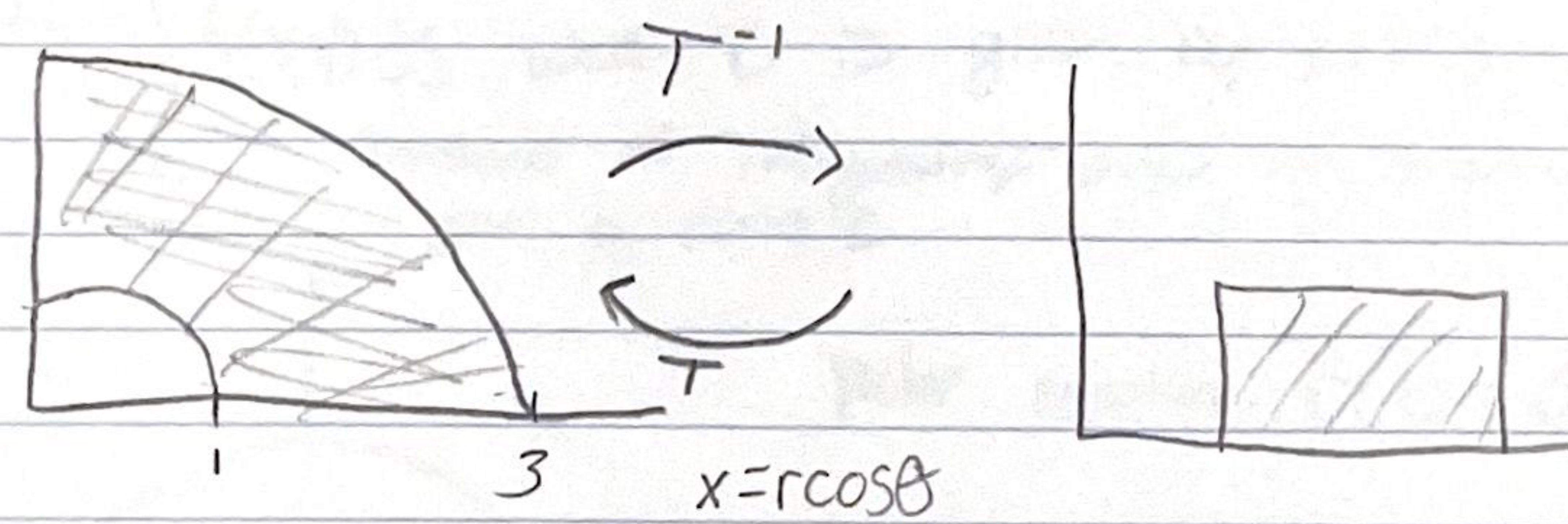
compute integral over nicer region S of fn written in terms of u and v

BUT: transformation distorted area

$$\iint_R f(x,y)\,dA = \iint_S f(u-v,\,2v)\cdot|J(u,v)|\,du\,dv$$

$$\text{Jacobian} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Ex: Polar



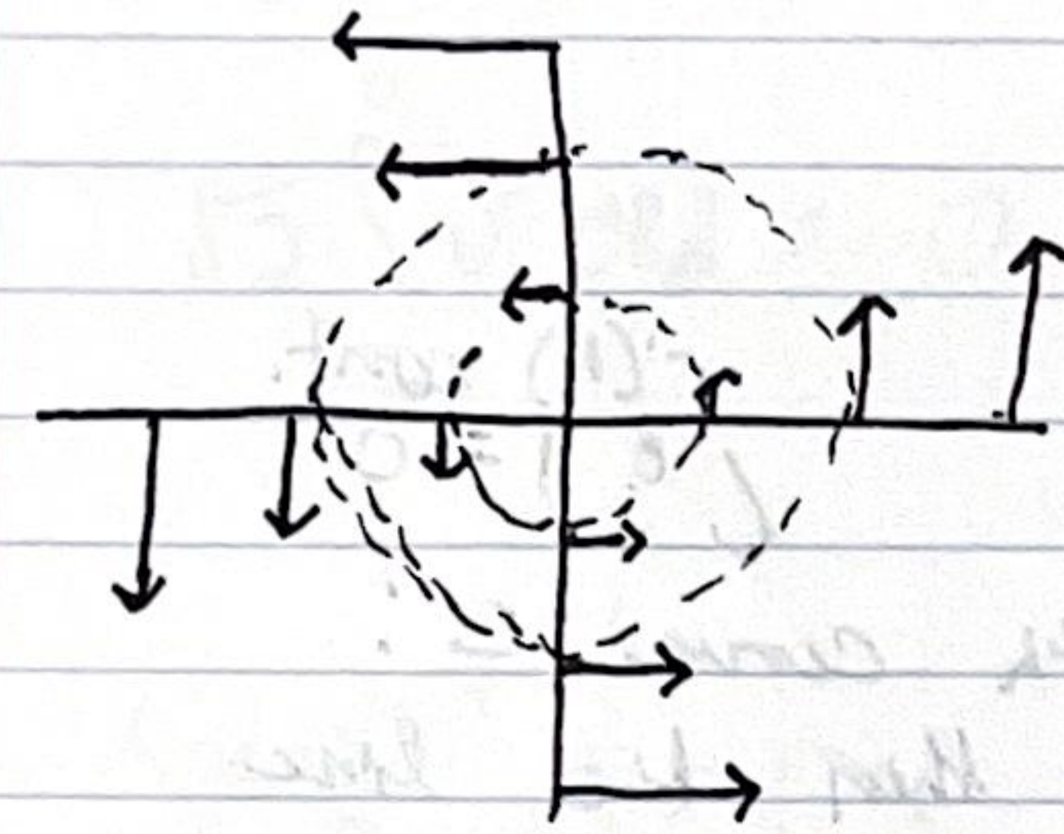
$x = r\cos\theta$

$z = r\sin\theta$

$$\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta,\,r\sin\theta)\,r\,dr\,d\theta$$

Lecture

Is $\vec{F}(x,y) = \langle \frac{-y}{2}, \frac{x}{2} \rangle$ conservative? (Non-conservative = cannot find potential function)



If $y=0$, then $\vec{F} = \frac{x}{2}\hat{j}$

If $x=0$, then $\vec{F} = \frac{-y}{2}\hat{i}$

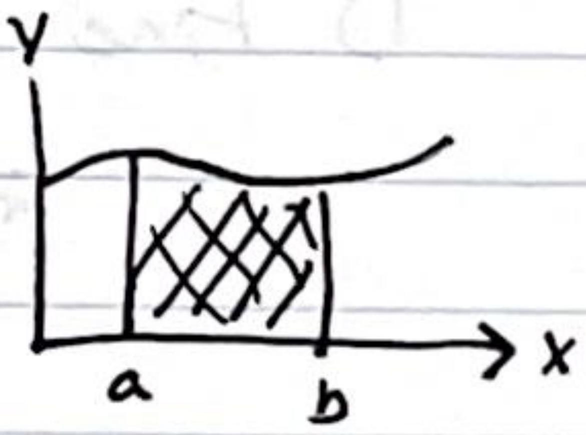
Check $\langle x, y \rangle \cdot \langle \frac{-y}{2}, \frac{x}{2} \rangle = \frac{-xy}{2} + \frac{xy}{2} = 0$

Suppose that $\vec{F} = \vec{\nabla} f$.

Since $\vec{\nabla} f$ points in the direction of greatest increase

16.2 Line Integrals

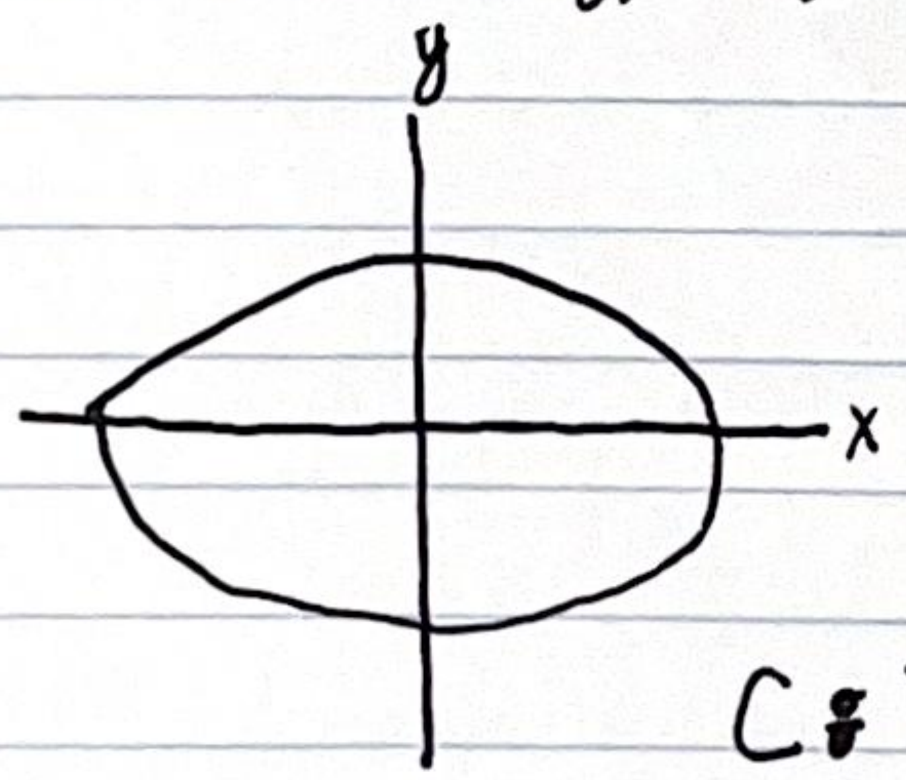
Calc I: $\int_a^b f(x)\,dx$ integrate over an integral $[a,b]$



Now: $\int_C f(x,y)\,ds$ where C is given by $x = x(t)$, $y = y(t)$

Instead of integrating over an interval $[a,b]$, integrate over a curve C.

Ex:



Vector equation: $\vec{r}(t) = \langle x(t), y(t) \rangle$ $a \le t \le b$

C: $x = 3\cos(t)$
$y = 2\sin(t)$
$0 \le t \le 2\pi$

split the ~~integral~~ interval into n equal sub intervals

$\Rightarrow$ this splits c into n arcs

$r'(t)$ cont.

& $!= 0$

Def: If f is defined on a smooth curve C:

$\vec{r}(t) = \langle x(t), y(t) \rangle, \ a \le t \le b$, then the line integral of f along C is:

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i) \cdot \Delta s_i$$

provided lim exists

<u>How to integrate f over C</u>

1) Find a smooth parametrization of C

$$\vec{r}(t) = \langle x(t), y(t) \rangle \quad a \le t \le b$$

2) Evaluate the integral

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\boxed{ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}$$

$$\int_C f\,ds = \int_a^b f(\vec{r}(t)) \cdot |\vec{r}\,'(t)|\,dt$$

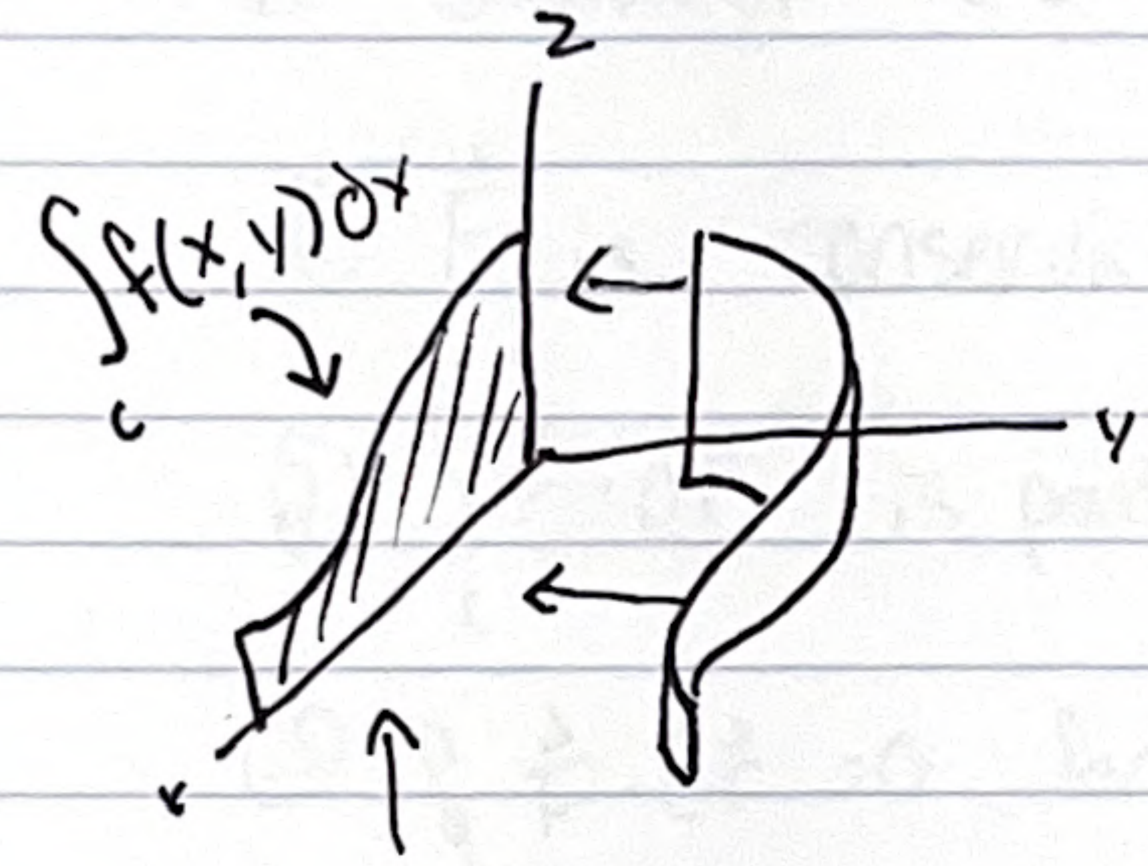
$$\int_0^{5\pi} (\cos(t))(\sin(t))^2 \cdot \sqrt{2}\, dt$$

let $u = \overset{\sin(t)}{\cancel{\cos(t)}}$

$du = \cos(t)\, dt$

$u = \sin(5\pi)$

$$\sqrt{2}\int_0^0 u^2\, du = 0$$



$$\int_C f\,dx = \int_a^b f(x(t), y(t)) \cdot x'(t)\, dt$$

dot product

projection of $\int_C f(x,y)dx$ ~~ondan~~ onto xz-plane

Leeture

Theorem:

Suppose $\vec{F}$ is continuous on an open connected region D.

The following are equivalent:

① $\vec{F}$ is conservative

② $\int_C \vec{F} \cdot d\vec{r}$ is path independent in D "$\vec{F}$ has path independence"

③ $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve in D "$\vec{F}$ has 0 circulation"

How to determine if $\vec{F}$ is conservative:

Suppose $\vec{F} = P\hat{i} + Q\hat{j} = \langle P, Q \rangle$ is conservative

then $\exists f(x,y)$ s.t. $\vec{\nabla} f = \vec{F}$ and so

$$\frac{\partial f}{\partial x} = P \quad \text{and} \quad \frac{\partial f}{\partial y} = Q$$

If $\vec{F}$ is conservative: $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Ex: $\vec{F} = xy\hat{i} + x^2\hat{j}$

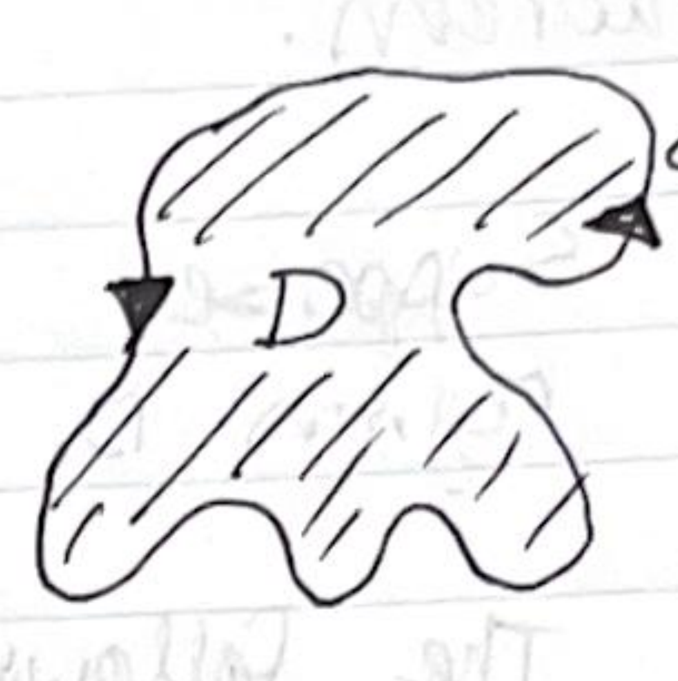
$\nabla f = \langle y, 2x \rangle$

$y \neq 2x$

not conservative

16.4 Greene's Theorem:

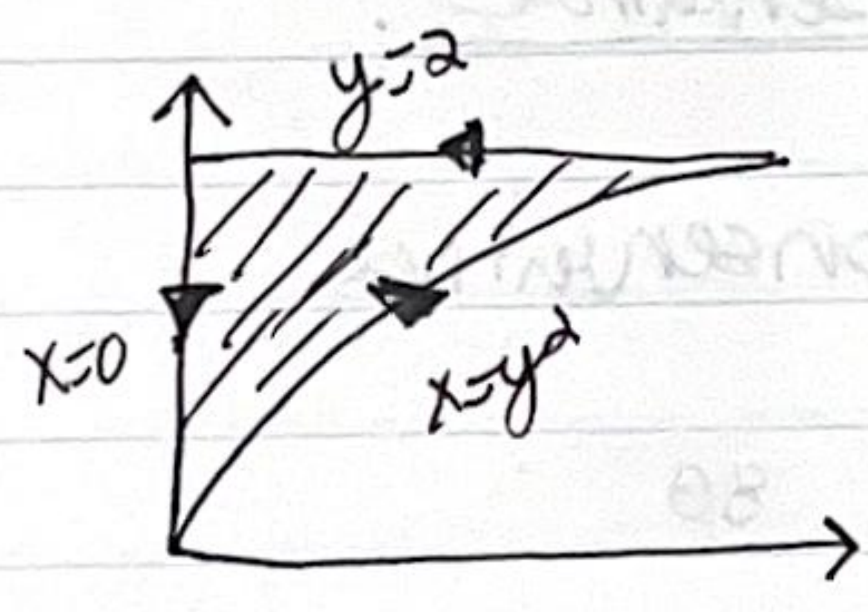
- C .... positively oriented, piecewise-smooth, simple, closed
- D... region bounded by C (includes c)
- P & Q have cont. partial derivatives on D



$$\Rightarrow \underbrace{\oint P\,dx + Q\,dy}_{} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$\vec{F} = \langle P, Q \rangle \Rightarrow \oint_C \vec{F} \cdot d\vec{r}$

Ex:



$$\oint 4xy\,\partial x + 3xy^2\,\partial y \overset{\text{greene}}{=} \int_0^2 \int_0^{y^2} 3y^2 - 4x \;\partial x\,\partial y =$$

$$\int_0^2 \left[3xy^2 - 2x^2\right]_0^{y^2} dy = \frac{y^5}{5}\Bigg|_0^2 = \boxed{\frac{32}{5}}$$

$f(x) = \cancel{x^2+y^2} \; -y$

$g(x) = x^2 + y^2$

$f'(x) = 0$

$g'(x) = 2x \cancel{+2y}$

$$\boxed{\cancel{\frac{2x(-y)}{(x^2+y^2)^2}}}$$

$-y$

Lecture 16.5

Curl and Divergence (3D)

Given $\vec{F} = \langle P, Q, R \rangle$

Define curl $\vec{F} = \vec{\nabla} \times F$

$$\nabla \times F = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

~~Ex~~ Theorem: If $f(x,y,z)$ has continuous second order partial derivatives, then

$$\text{curl}(\vec{\nabla} f) = \vec{0}$$

Verify

$$\text{curl } \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \langle f_{zy} - f_{yz}, -f_{zx} + f_{xz}, f_{yx} - f_{xy} \rangle$$

$\langle 0, 0, 0 \rangle$ by Clairant's Theorem

If curl $\vec{F} = \vec{0} \Rightarrow \vec{F}$ is conservative

Ex: Is $\vec{F} = y^2\hat{i} + (2xy+z)\hat{j} + y\hat{k}$ conservative?

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy+z & y \end{vmatrix} = \left\langle \frac{\partial y}{\partial y} - \frac{\partial(2xy+z)}{\partial z}, -\frac{\partial y}{\partial x} + \frac{\partial y^2}{\partial z}, \frac{\partial(2xy+z)}{\partial x} - \frac{\partial y^2}{\partial y} \right\rangle$$

Cannot take divergence of a divergent

But we can take the curl of a curl

Divergence of the gradient of $f(x,y,z)$

$$\text{div}(\vec{\nabla} f) = \vec{\nabla} \cdot (\vec{\nabla} f)$$

Laplace Operator

$$\nabla^2 = \nabla \cdot \nabla$$

16.8 Lecture

If $\vec{F} = \langle P, Q, R \rangle$ and P and Q only depend on x & y,
then:

$$\text{curl}(\vec{F}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ X & Y & 0 \end{vmatrix}$$

Stoke's Theorem:

$$\iint_D (\text{curl}(\vec{F})) \cdot \vec{n} \, dS$$

16.9 Divergence Theorem:

Given: Solid E & its boundary surface S with Outward orientation
vector field $\vec{F}$ (whose component have continuous
partial derivatives)

then: $$\underbrace{\oiint (\vec{F} \cdot \vec{n}) \, dS}_{\frac{\text{Flux}}{\Phi}} = \iiint_E \underbrace{(\text{div}(\vec{F}))}_{\vec{\nabla} \cdot \vec{F}} \, dV$$