Lecture 4 · Existence-Uniqueness Theorem

Last Time: Slope Fields
— behavior of solution curves

This time: When can we guarentee that a 1^{st} order initial value problem has a solution? Only one solution

Theorem: (Existance & Uniqueness)

Consider $\begin{cases} \frac{dy}{dx} = f(y,t) \\ y(t_0) = y_0 \end{cases}$

i) If $f(t,y)$ is continuous in neighborhood (t_0, y_0) then a solution exists is some rectangle $|t-t_0| < \delta$, $|y-y_0| < \varepsilon$

ii) If, in addition, $\frac{\partial f}{\partial y}(y,t)$ is continuous in a neighborhood of the point (t_0, y_0) then the solution is unique

Example: Recall that $\begin{cases} dy/dx = y^{1/3} \\ y(0) = 0 \end{cases}$ has two solutions $\Rightarrow y_1(t) = 0$ & $y_2(t) = \left(\frac{2}{3}t\right)^{3/2}$

Notice that we have $f(y,t) = y^{1/3}$ is continuous in a neighborhood of the point $(0,0)$ but $\frac{\partial f}{\partial y} = \frac{1}{3}y^{-2/3}$ is not continuous in a neighborhood of $(0,0)$

Example: Consider $\begin{cases} \partial y/\partial t = (y-5)\ln(|y-5|)+t \\ y(0)=5 \end{cases}$

Here $f(y,t) = (y-5)\ln(|y-5|)+t$ is continuous in a neighborhood of $(0,5)$

- so the theorem guarentees the existence of a solution for $t \simeq 0$

However $\frac{\partial f}{\partial y} = 1+\ln(|y-5|)$ is not continuous in a neighborhood of $(0,5)$ so the theorem does not guarentee uniqueness

Notice that if the initial condition were different (e.g. $y(0)=1$) then both parts of the theorem would apply.

- For some easy equations, we can carry out the iterations explicitly:

Example: Consider $\partial y/\partial t = 2t(1+y)$, $y(0)=0$

Here $y_0(t)=0$, $y_1(t) = 0 + \int_0^t 2s(1+0)\,ds = t^2$

$$y_2(t) = 0+\int_0^t 2s(1+s^2)\,ds = t^2+\frac{1}{2}t^4$$

$$y_3(t) = 0+\int_0^t 2s(1+s^2+\frac{1}{2}s^4)\,ds = t^2+\frac{1}{2}t^4+\frac{1}{6}t^6$$

$$\ldots \quad y_n(t) = 0+\int_0^t 2s(1+y_{n-1}(s))\,ds = t^2+\frac{1}{2}t^4+\ldots\frac{1}{n!}t^{2n}$$

Recall that $e^t = 1+t+\frac{1}{2}t^2+\ldots+\frac{t^n}{n!}+\ldots$

We recognize that $y_n(t) \to e^{t^2}-1$

Notice that if $y(t)=e^{t^2}-1$, then $\frac{\partial y}{\partial t} = 2te^{t^2} = 2t(y+1)$

So this is indeed a (the) solution

Example: Let's consider $\begin{cases} \frac{dy}{dt} = t^2 + y^2 \\ y(0) = 1 \end{cases}$

Here $y_0(t) = 1$ and for $n > 0$, $y_n(t) = 1 + \int_0^t s^2 + (y_{n-1}(s))^2 \, ds$

so $y_1(t) = 1 + t + \frac{1}{3}t^3$

$y_2(t) = 1 + t + t^2 + \frac{2}{3}t^3 + \frac{1}{6}t^4 + \frac{2}{15}t^5 + \frac{1}{63}t^7$

$y_3(t) = 1 + t + t^2 + \frac{4}{3}t^3 + \frac{5}{6}t^4 + \frac{8}{15}t^5 + \frac{29}{60}t^6 + \frac{47}{315} + \dots + \frac{1}{59535}t^{15}$

The pattern for the coefficients is not clear, so it would be hard to guess a solution

How does one show the sequence converges (in general)?

Consider the map T: functions $\to$ functions
that takes a function y to the function $T(y)(t) = y_0 + \int_{y_0}^t f(y(s), s) \, ds$

The Picard sequence is $y_0, y_1 = T(y_0), y_2 = T(y_1) = T(T(y_0)) = T^2(y_0)$
$y_3 = T(y_2) = T^3(y_0) \dots$
$y_n = T^n(0)$

What we show is that there is some appropriate notion of size of a function

$\|f\|$ and that T reduces distances

$$\|T(f) - T(g)\| \leq \rho\|f - g\|, \ \rho < 1$$

T is called a contraction
This is enough to show that $y_n = T^n(y_0)$ converges
If we call the limit y_∞, then notice that
$y_\infty = \lim_{n\to\infty} T^n(y_0) = \lim_{n\to\infty} (T(T^{n-1}(y_0))) = T(y_\infty)$

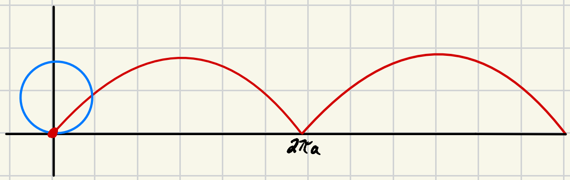
so y_∞ is a "fixed point" of T but then

$$y_\infty(t) = T(y_\infty)(t) = y_0 + \int_{t_0}^{t} f(y_\infty(s), s)\,ds$$

This is just another way of writing

$$\begin{cases} \frac{\partial y}{\partial t} = f(y_\infty, t) \\ y_\infty(t_0) = y_0 \end{cases}$$

Historical Context: The Cycloid



$$x(\theta) = a(\theta - \sin\theta)$$
$$y(\theta) = a(1 - \cos(\theta))$$

Lecture 6

Substitutions

$$\frac{dy}{dt} = F\left(\frac{y}{t}\right) \qquad y' = \sin\left(\frac{y}{t}\right) + \left(\frac{t}{y}\right)^2 + 3 + e^{y/t}$$

use sub: $v(t) = \frac{y}{t}$

ex:

$$\frac{dy}{dt} = \frac{t}{y} + \frac{y}{t} \qquad (y \neq 0)$$

use $v = \frac{y}{t} \Rightarrow y = vt$ $\rightarrow$ $\frac{dy}{dt} = \frac{d}{dt}(vt) = v't + v = F(v)$

substitute

$$v't + v = \frac{1}{v} + v$$

$$t\left(\frac{dv}{dt}\right) = \frac{1}{v} \qquad \text{(seperable equation)}$$

$$\int v\,dv = \int \frac{1}{t}\,dt \qquad F(v)$$

$$\frac{v^2}{2} = \ln(t) + C$$

Exact Equations

$$\psi(x,y)=c$$

$$\partial\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = 0$$

$$\frac{\partial\psi}{\partial y}\frac{dy}{dx} + \frac{\partial\psi}{\partial x} = 0$$

$$y = y(x)$$

$$N(x,y)\frac{dy}{dx} + M(x,y) = 0$$

$$M\,dx + N\,dy = 0$$

What if $\exists$ A ψ

$$M = \frac{\partial\psi}{\partial x} \quad \text{and} \quad N = \frac{\partial\psi}{\partial y}$$

if $M = \frac{\partial\psi}{\partial x}$ and $N = \frac{\partial\psi}{\partial y}$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2\psi}{\partial y\partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2\psi}{\partial x\partial y}$$

$$\Leftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Δ the solution to the eq is

$$\psi(x,y)=c$$

Example

$$(x^2+3y^2)\frac{dy}{dx} + 2xy = 0$$

$$\underbrace{2xy\,dx}_{M} + \underbrace{(x^2+3y^2)\,dy}_{N} = 0$$

$$\boxed{\begin{array}{c}\text{to check if it is an exact equation} \\ \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}\end{array}}$$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x \implies \frac{\partial \psi}{\partial x} = M = 2xy$$

$$\frac{\partial \psi}{\partial y} = N = x^2+3y^2$$

$$\int \frac{\partial \psi}{\partial x}\,dx = \int 2xy\,dx$$

$$\psi(x,y) = x^2y + f(y) \rightarrow \frac{\partial \psi}{\partial y} = \cancel{x^2} + f'(y) = \cancel{x^2} + 3y^2$$

$$f = y^3 + A \implies \psi(x,y) = x^2y + y^3 + A$$

since $\psi(x,y) = C \implies$

$$\psi(x,y) = x^2y + y^3 = C$$

Another example

$$2x\frac{\partial y}{\partial x}+y^2=0 \Rightarrow \underbrace{y^2}_{M}dx+\underbrace{2x}_{N}dy=0$$

$$\frac{\partial M}{\partial y}=2y$$

$$\frac{\partial N}{\partial x}=2$$

$\neq$ not exact solution

Example 3

$$\frac{\partial y}{\partial x}=\frac{2xy^2}{\sin(y)-2x^2y} \Rightarrow (\sin(y)-2x^2y)\,dy=2xy^2\,dx$$

$$\underbrace{2xy^2}_{M}dx+\underbrace{(2x^2y-\sin(y))}_{N}dy=0$$

$$\frac{\partial M}{\partial y}=4xy$$

$$\frac{\partial N}{\partial x}=4xy$$

cont.

Lecture 7

Brenoulli Equation

$$y' + P(t)y = Q(t)y^{\alpha} \quad , \quad \alpha \neq 0,1$$

use $v(t) = y^{1-\alpha}$

Example:

$$y' = \frac{y^2}{t^2} + ty$$

$$y' - ty = \frac{1}{t^2}y^2$$

$$\boxed{\begin{array}{l} \text{use} \quad v = y^{1-2} = y^{-1} = \frac{1}{y} \\ \\ y = \frac{1}{v} \\ \\ y' = \frac{-1}{v^2}v' \end{array}}$$

$$\frac{-1}{v^2}v' - t\frac{1}{v} = \frac{1}{t^2} \cdot \frac{1}{v^2}$$

$$v' + tv = -\frac{1}{t^2}$$

1st order equation for v

$$\rightarrow \mu(t) = e^{\frac{t^2}{2}}$$

$$\left(ve^{\frac{t^2}{2}}\right)' = -e^{t^2/2}\left(\frac{-1}{t^2}\right) = \frac{e^{t^2/2}}{t^2}$$

Suppose $F(t, \cancel{y}, y', y'') = 0$ but the equation does not explicitly have y

Use $v(t) = y'$

$y'' = v'$

$F(t, v, v')$

reducable equation

2nd $\rightarrow$ 1st order

<u>Example:</u> $y'' + \frac{2}{t} y' = t^2$

$v(t) = y'$

$v' = y''$

$\rightarrow$

$$v' + \frac{2}{t} v = t^2$$

$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln(t)} = t^2$$

$$t^2 v' + 2tv = t^4$$

$$(vt^2)' = t^4$$

$$vt^2 = \frac{t^5}{5} + A$$

$$y' = v = \frac{t^3}{5} + \frac{A}{t^2}$$

$$\int y' = \int \left(\frac{t^3}{5} + \frac{A}{t^2}\right) dt$$

$$\boxed{y = \frac{t^4}{20} - \frac{A}{t} + B}$$

$F(\cancel{t}, y, y', y'') = 0$, use $v(y) = y'$

$$y'' = \frac{d}{dt} y' = \frac{d}{dt} v(y(t)) = \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial t} = \frac{\partial v}{\partial y} \cdot v$$

Example: $y'' + 2yy' = 0$, use $v(y) = y'$

$$y'' = v\frac{\partial v}{\partial y}$$

$$v\frac{\partial v}{\partial y} + 2yv = 0$$

$v = 0$ | $v \neq 0$

$y' = 0 \rightarrow y = A$

$$\frac{\partial v}{\partial y} + 2y = 0 \rightarrow \frac{\partial v}{\partial y} = -2y \rightarrow v = A - y^2$$

$$\frac{\partial y}{\partial t} = A - y^2 \rightarrow \frac{\partial y}{A - y^2} = \partial t$$

Slope Fields

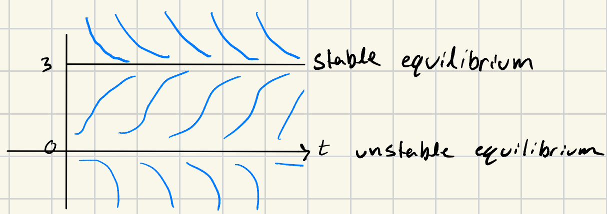
Autonomous EQs

Population Models

Ex: $P(t)$

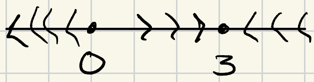
$$\frac{\partial P}{\partial t} = 3P - P^2 = P(3-P)$$

$\left.\begin{array}{l} P=0 \\ P=3 \end{array}\right\}$ critical points = Equilibrium Solutions



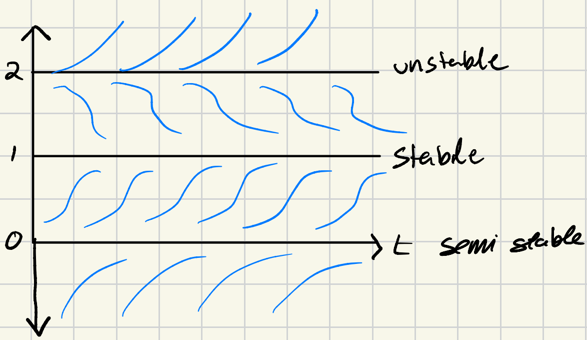
To visualize equilibrium

phase line



Ex: $\frac{dP}{dt} = P^2(P-1)(P-2)$

$$\left.\begin{array}{l} P=0 \\ P=1 \\ P=2 \end{array}\right\} \text{Equilibrium states}$$



Higher Order Linear ODEs

N^{th} Order Lin:

$$y^{n} + a_{n-1}(t)\,y^{n-1} + \ldots + a_1(t)\,y' + a_0 y = f(t)$$

$$y(t_0) = y_0$$

$$y'(t_0) = y_1$$

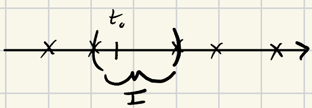
$$\vdots$$

$$y^{n-1}(t_0) \cdot y_{n-1}$$

Theorem of Existence and Uniqueness for a linear IVP

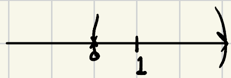
If $\exists$ an interval I around t_0 where a_{n-1}, a_{n-2}, a_0, f are continuous

$\Rightarrow$ $\exists$ one and only one solution to the IVP over all of I



<u>Example:</u>

$$\begin{cases} y'' + \frac{1}{t}y' - \frac{9}{t^2}y = 0 \\ y(1) = 0 \\ y'(1) = 1 \end{cases}$$



$I = (0, +\infty)$

Principle of Superposition For Linear Homogeneous Equations

If $y_1, y_2, \dots, y_n$ are solutions to a <u>lin</u> <u>hom</u> Eq

$\Rightarrow$ any linear combination $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_{n-1}$
is also a solution to the equation

<u>Example:</u>

$$y'' - 3y' + 2y = 0$$

$y_1 = e^t$
$y_2 = e^{2t}$

and $y = c_1 e^t + c_2 e^{2t}$ is also a solution

<u>Ex:</u>

$$y'' - y = 4 \quad \leftarrow \text{linear non homogenous}$$

$\left.\begin{array}{l} y_1 = 4 \\ y_2 = e^t - 4 \end{array}\right\}$ are solutions

But

$y = y_1 + y_2$ is not

<u>Ex:</u>

$$y'' + 2yy' = 0 \quad \leftarrow \text{not linear}$$

$\left.\begin{array}{l} y_1 = 1 \\ y_2 = \frac{1}{t} \end{array}\right\}$ Are solutions

But $y = 1 + \frac{1}{t}$ is not

<u>Ex:</u>

Equidimensional

$$\begin{cases} y'' + \frac{1}{t} y' - \frac{9}{t^2} = 0 \\ y(1) = 0 \\ y'(1) = 1 \end{cases}$$

Try $y = t^\alpha$

$y' = \alpha t^{\alpha - 1}$

$y'' = \alpha(\alpha - 1)(t^{\alpha - 2})$

plug in

$\alpha(\alpha-1)\cancel{t^{\alpha-2}} + \cancel{\frac{1}{t}}\alpha \cancel{t^{\alpha-1}} - \frac{9}{\cancel{t^2}}\cancel{t^\alpha} = 0$

$\alpha(\alpha - 1) + \alpha - 9 = 0$

$\alpha^2 - 9 = 0$

$\alpha = \pm 3$

$y_1 = t^3$

$y_2 = \frac{1}{t^3}$

$y = c_1 t^3 + \frac{c_2}{t^3}$

$y' = 3c_1 t^2 - \frac{3c_2}{t^4}$

Apply initial conditions

$y(1) = c_1 + c_2 = a$

$y'(1) = 3c_1 - 3c_2 = \ell$

$c_1 = \frac{1}{6}, \; c_2 = \frac{-1}{6}$

$$\underbrace{y = \frac{1}{6}t^3 - \frac{1}{6} \cdot \frac{1}{t^3}}_{\text{the solution to the IVP}}$$

To solve an N^{th} order lin-homogeneous IVP

1) Find N solutions to the equation: $y_1, y_2, \ldots, y_n$

2) Combine linearly: $y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$

3) Apply ICs to find the c's

Ex:

$y'' + 4y = 0$

$y_1 = \sin(2t)$

$y_2 = \sin(t)\cos(t)$

IC's: $y(0) = 1$
$y'(0) =$

$y = c_1 \sin(2t) + c_2 \sin(t)\cos(t)$

$y(0) = 0 + 0 \neq 1$

does not work because $\sin(2t) = 2\sin(t)\cos(t)$

Remember:

$\cos(2t) = 2\cos^2(t) - 1 = \cos^2(t) - \sin^2(t) =$

$1 - \sin^2(t)$

Linear Independence

Exam Review: Everything up until this past Friday

Find n solutions to the equation: $y_1, y_2, \dots y_n(t)$

Write $\alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n(t) = 0$

- If $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ is the only choice $\rightarrow$ $y_1, \dots y_n$ are linearly independent

- If $\exists$ other choices of $\alpha \rightarrow y_1, \dots, y_n$ are linearly dependent

Example:

$1, e^t, e^{-t}$

$\alpha_1 1 + \alpha_2 e^t + \alpha_3 e^{-t} = 0$

$\underbrace{\alpha_1 = \alpha_2 = \alpha_3 = 0}_{\text{linearly independent}}$

$\cos(t), e^{it}, e^{-it}$

$\alpha_1 \cos(t) + \alpha_2 e^{it} + \alpha_3 e^{-it} = 0$

$\underbrace{\alpha_1 = 1 \quad \alpha_2 = \frac{-1}{2} \quad \alpha_3 = \frac{-1}{2}}_{\text{linearly dependent}}$

Note: $\cos(t) = \frac{e^{it} + e^{-it}}{2}$

$\sin(t) = \frac{e^{it} - e^{-it}}{2i}$

$e^{it} = \cos(t) + i\sin(t)$

$e^{-it} = \cos(t) - i\sin(t)$

$$\alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n(t) = 0$$

$$\Downarrow$$

$$\alpha_1 y_1' + \alpha_2 y_2' + \dots + \alpha_n y_n'(t) = 0$$

$$\vdots$$

$$\Downarrow$$

$$\alpha_1 y_1^{(n-1)} + \alpha_2 y_2^{(n-1)} + \dots + \alpha_n y_n^{(n-1)}(t) = 0$$

linearly independent: $\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y^{(n-1)} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$$\det \begin{pmatrix} y_1 & \cdots & y_n \\ & \vdots & \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \neq 0$$

Wronskian iff $\exists$ only one solution $(\alpha_1 = \alpha_2 = \dots = \alpha_n = 0)$

$y \cdot \alpha = 0$ $\qquad W(y_1 \dots y_n) \neq 0 \rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0 \rightarrow y_1 \cdots y_n$ L.I.

$W = 0 \longrightarrow y_1 \cdots y_n$ L.D.

Why do we care about linear independence:

N^{th} ord. Lin hom IVP

1) Find N solutions to φ: $y_1, y_2, \dots, y_n$

2) $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$

3) apply initial conditions

$$y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) + \dots + c_n y_n(t_0) = a_0$$
$$y'(t_0) = c_1 y_1'(t_0) + \dots + c_n y_n'(t_0) = a_1$$
$$\dots$$
$$y^{n-1}(t_0) = c_1 y_1^{n-1}(t_0) + \dots + c_n y_n^{n-1}(t_0) = a_{n-1}$$

initial conditions

$$\begin{pmatrix} y_1(t_0) & \cdots & y_n \\ \vdots & \ddots & \\ y_1^{n-1} & \cdots & y_n^{n-1} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix}$$

Example:

$$y'' + \frac{1}{t}y' - \frac{9}{t^2}y = 0$$

$$y_1 = t^3 \quad y_2 = \frac{1}{t^3}$$

$$W \begin{vmatrix} t^3 & t^{-3} \\ 3t^2 & -3t^{-4} \end{vmatrix} = \frac{-3}{t} - \frac{3}{t} = \frac{-6}{t} \neq 0$$

$$y^n + a_{n-1}(t)y^{n-1} + \dots + a_0(t)y_0 = 0$$

n solutions $y_1, y_2, \dots y_n$

Calculate $W(y_1, \dots, y_n)$

Abel's Theorem:

W satisfies

$$W' + a_{n-1}W = 0$$

$$\to W(t) = W(t_0)e^{\int_{t_0}^{t} a_{n-1}(s)ds}$$

$$\to W(t) = 0$$

Method of Undetermined Coefficients

- If the LHS is constant coefficients and linear and the RHS is an exp, cos, sin, polynomial, or a product of these, try a y_p of the same form as the RHS plus all possible derivatives, each with a coefficient, multiply by t until you have no duplication with y_c, and find the coeff.

Ex: $y'' + y = e^t$

$$y_p = Ae^t$$

$$y_p'' = Ae^t$$

$$A\cancel{e^t} + A\cancel{e^t} = \cancel{e^t}$$

$$A + A = 1$$

$$A = \frac{1}{2}$$

Ex: $y' + y = t^2$

$$y_p = At^2 + Bt + C$$

$$y_p' = 2At + B$$

$$2At + B + At^2 + Bt + C = t^2$$

$$A = 1$$

$$2A + B = 0 \rightarrow B = -2$$

$$B + C = 0 \rightarrow C = 2$$

Ex $y'' + y' + y = \cos(2t)$

$$y_p = A\cos(2t) + B\sin(2t)$$

$$y_p' = -2A\sin(2t) + 2B\cos(2t)$$

$$y_p'' = -4A\cos(2t) - 4B\sin(2t)$$

$$-4Ac - 4Bs - 2As + 2Bc + Ac + Bs = C$$

$\cos(2t)$: $-4A + 2B + A = 1$ $\qquad -3A + 2B = 1$

$\sin(2t)$: $-4B - 2A + B = 0$ $\qquad -2A - 3B = 0$

$$A = \frac{-3}{13},\ B = \frac{2}{13}$$

$$y_p = \frac{-3}{13}\cos(2t) + \frac{2}{13}\sin(2t)$$

Ex: $y'' - y = e^t$

$y_p = Ae^t$

$y_p'' = Ae^t$

$Ae^t - Ae^t = e^t$

$\underbrace{0 = e^t}_{\text{problem}}$

$y_c = y'' - y = 0$

$y_c = e^{rt}$

$r^2 - 1 = 0$

$r = \pm 1$

$y_c = c_1 e^t + c_2 e^{-t}$

$y_p = Ae^t t$

$y_p' = Ae^t + Ate^t$

$y_p'' = 2Ae^t + Ate^t$

$2Ae^t + \cancel{Ate^t} - \cancel{Ae^t t} = e^t$

$2A\cancel{e^t} = \cancel{e^t}$

$2A = 1$

$A = \frac{1}{2}$

Ex:

$y'' - 2y' + y = e^t$

Let's find y_c first:

$y'' - 2y' + y = 0$

$e^{rt} \to r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$ twice

$y_c = C_1 e^t + C_2 te^t$

$y_p = Ae^t$ duplicates

$y_p = Ae^t t$ duplicates

no duplication

$y_p = Ae^t t^2$

Example: $y''' - y' = \cos(2t) + t^2$

y_c : $y''' - y' = 0$

$e^{rt} \to r^3 - r = 0$

$r(r^2 - 1) = 0$

$r = 0, 1, -1$

$y_c = C_1 + C_1 e^t + C_2 e^{-t}$

similar

must multiply by t

$y_p = \left(A\cos(2t) + B\sin(2t)\right) + (Ct^2 + Dt + E)t$

Example: $\ldots = t^2\cos(3t)$

$$y_p = At^2\cos(3t) + Bt^2\sin(3t) + Ct\cos(3t) + Dt\sin(3t) + E\cos(3t) + F\sin(3t)$$

then add all possible derivatives

Ex: $\ldots = \ln(t)$

$$y_p = A\ln(t) + \frac{B}{t} + \frac{C}{t^2} + \frac{D}{t^3} + \ldots$$

this method does not work for ln.

Annihilator Method

- LHS is const coeff
- RHS is Exp, sin, cos, polynomial, or product of these.

$$L y = f(t)$$

1. Find y_c

$$L y = 0$$
$$y_c = e^{rt} \to r = r_1, \ldots r_n$$
$$y_c = c_1 e^{r_1 t} + \ldots + c_n e^{r_n t}$$

2. Find an annihilator for $f(t)$ s.t.

$$\tilde{L} f = 0$$

3. Apply $\tilde{L}$ to the original eq.

$$\tilde{L} L y = \tilde{L} f = 0$$

$$\underbrace{\tilde{L} L}_{\text{const coeffs}} y = 0$$

4. Use $y = e^{rt} \to r = r_1, \ldots, r_n, r_{n+1}, \ldots$

$e^{r_1 t}, \ldots e^{r_n t}, e^{r_{n+1} t}, \ldots$

get rid of the ones already in y_c

$$y_p = A e^{r_{n+1} t} + \ldots + K e^{r_k t}$$

and find $A, \ldots, K$

Example: $y'' - 4y = 4e^{2t}$

y_c: $y'' - 4y = 0$

$y = e^{rt} \rightarrow r^2 - 4 = 0$

$r = \pm 2$

$y_c = c_1 e^{2t} + c_2 e^{-2t}$

y_p: $\tilde{L} e^{2t} = 0$

$(D-2)e^{2t} = 2e^{2t} - 2e^{2t} = 0$

derivative operator

$\tilde{L} = (D-2)$

plug into original eq:

$y'' - 4y = 4e^{2t}$
$(D^2-4)y = 4e^{2t}$ } rewrote

$\underset{\tilde{L}}{(D-2)}(D^2-4)y = (D-2)4e^{2t} = 0$

$(D-2)(D^2-4)y = 0$

$y = e^{rt}$

$(r-2)(r^2-4)\cancel{e^{rt}} = 0$

$r = 2, \pm 2$

$\cancel{e^{2t}}, te^{2t}, \cancel{e^{-2t}}$

already have in y_c

$y_p = Ate^{2t}$ → plug into original equation

$A = 1 \rightarrow y_p = te^{2t}$

$y = y_c + y_p$

$y = C_1 e^{2t} + C_2 e^{-2t} + te^{2t}$

Example: $y'''' + y'' = 3te^t + 2t^2$

$y_c = y'''' + y'' = 0$

$y = e^{rt} \rightarrow r^4 + r^2 = 0$

$r^2(r^2+1) = 0$

$r = 0, 0, \pm i$

$y = C_1 + C_2 t + C_3 \cos(t) + C_4 \sin(t)$

$y_p: \; 3te^t$

$\tilde{1}$

$(D-1)^2 te^t = 0$

$(D^4 + D^2)y = 3te^t$

$(D-1)^2(D^4+D^2)y = (D-1)^2 3te^t = 0$

$y = e^{rt} \rightarrow (r-1)^2(r^4+r^2) = 0$

$(r-1)^2 r^2 (r^2+1) = 0$

$r = 1, 1, 0, 0, \pm i$

$e^t, te^t, \cancel{1}, \cancel{t}, \cancel{\cos(t)}, \cancel{\sin(t)}$

$y_{p1} = A + Bte^t$

$y_{p2}: \; 2t^2$

$D^3(2t^2) = 0$

$(D^3)(D^4+D^2)y = (D^3)2t^2 = 0$

$y = e^{rt} \rightarrow r^3(r^4+r^2) = 0$

$r^5(r^2+1)$

$r = 0, 0, 0, 0, 0, \pm i$

$\cancel{1}, \cancel{t}, t^2, t^3, t^4, \cancel{\cos(t)}, \cancel{\sin(t)}$

already in y_c

$y_{p2} = Ct^2 + Dt^3 + Et^4$

$y_p = y_{p1} + y_{p2} \rightarrow$ find coeff

$y = y_p + y_c$

Variation of Parameters

Ex: $y'' + y = \ln(t)$

y_c: $y'' + y = 0$

$y_c = c_1\cos(t) + c_2\sin(t)$

$y_p = \mu_1(t)\cos(t) + \mu_2(t)\sin(t)$

$y_p' = \mu_1'(t)\cos - \mu_1(t)\sin + \mu_2'(t)\sin + \mu_2\cos$

$\mu_1' c + \mu_2' s = 0$ ← add this constraint (don't want anything higher than μ')

$y_p'' = -\mu_1' s - \mu_1\cos + \mu_2'\cos - \mu_2\sin$

plug into original equation →

$-\mu_1'\sin - \cancel{\mu_1\cos} + \mu_2'\cos - \cancel{\mu_2\sin} + \cancel{\mu_1\cos} + \cancel{\mu_2\sin} = \ln(t)$

$-\mu_1'\sin + \mu_2'\cos = \ln(t)$ → put next to constraint:

$$\begin{cases} \mu_1'\cos + \mu_2'\sin = 0 \\ -\mu_1'\sin + \mu_2'\cos = \ln(t) \end{cases}$$

$\mu_2' = \frac{-\cos}{\sin}\mu_1'$

sub ↳

$-\mu_1'\sin - \frac{\cos^2}{\sin}\mu_1' = \ln(t)$

$$\begin{cases} \mu_1' = -\sin\ln(t) \\ \mu_2' = \cos\ln(t) \end{cases}$$

$\mu_1 = \int -\sin\ln(t)\,dt$

$\mu_2 = \int \cos\ln(t)\,dt$

plug back into y_p ↳

$y_p = -\cos(t)\int^t \sin(z)\ln(z)\,dz + \sin(t)\int^t \cos(z)\ln(z)\,dz$

$y = y_c + y_p$

$$\mu_1 = -\int \frac{y_2 \cdot f}{W(y_1, y_2)}\,dt$$

$$\mu_2 = \int \frac{y_1 f}{W(y_1, y_2)}\,dt$$

Laplace Transform

(Note: is a linear function because $L(f) + L(g) = L(f+g)$)

Given $f(t)$,

$$L(f) = \int_0^\infty e^{-st} f(t)\,dt$$

Ex:

$$L(1): \int_0^\infty e^{-st}\,dt = \frac{-1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s}$$

$s>0$

$$L(t): \int_0^\infty e^{-st}\, t\,dt = \frac{-1}{s} e^{-st}\, t \Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, 1\,dt = \frac{-1}{s^2} e^{-st}\Big|_0^\infty = \frac{1}{s^2}$$

$$L(e^{at}): \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{-1}{s-a}$$

$s>a$

$$L(e^{ivt}) = \frac{1}{s-iv} = \frac{s+iv}{s^2+v^2} = \frac{s}{s^2+v^2} + i\frac{v}{s^2+v^2}$$

$\shortparallel$

$$L(\cos(vt) + i\sin(vt)) = L(\cos(vt)) + iL(\sin(vt))$$

$$L(f') = \int_0^\infty e^{-st} f'(t)\,dt =$$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-s) e^{-st} f(t)\,dt$$

$$= -f(0) + sL(f)$$

$$L(f'') = sL(f') - f'(0)$$

$$= s^2 L(f) - sf(0) - f'(0)$$

I.V.P. $\quad Y = L(y)$

$$y' + 3y = 0$$
$$y(0) = 2$$

$$L(y') + 3L(y) = 0$$
$$sY - y(0) + 3Y = 0$$
$$(s+3)Y - 2 = 0$$
$$Y = \frac{2}{s+3}$$
$$y = 2e^{-3t}$$

$$\begin{cases} y' + 3y = t \\ y(0) = 2 \end{cases}$$

$$sY - y(0) + 3Y = \frac{1}{s^2}$$
$$(s+3)Y - 2 = \frac{1}{s^2}$$
$$(s+3)Y = \frac{1}{s^2} + 2$$
$$Y = \frac{1}{s^2(s+3)} + \frac{2}{s+3}$$

$$\frac{1}{s^2(s+3)} = \frac{A}{s+3} + \frac{B}{s} + \frac{C}{s^2} = \frac{As^2 + B(s^2+3s) + C(s+3)}{(s+3)s^2}$$
$$= \frac{3C + (3B+C)s + (A+B)s^2}{(s+3)s^2}$$

$$3C = 1 \qquad C = \tfrac{1}{3}$$
$$3B + C = 0 \qquad B = -1/9$$
$$A + B = 0 \qquad A = 1/9$$

$$Y = \frac{1}{9}\cdot\frac{1}{s+3} - \frac{1}{9}\cdot\frac{1}{s} + \frac{1}{3}\cdot\frac{1}{s^2} + \frac{2}{s+3}$$
$$= \frac{19}{9}\cdot\frac{1}{s+3} - \frac{1}{9}\frac{1}{s} + \frac{1}{3}\cdot\frac{1}{s^2}$$
$$y = \frac{19}{9}e^{-3t} - \frac{1}{9} + \frac{1}{3}t$$

$$y'' - y' - 2y = 2e^{t}$$

$y(0) = 3$
$y'(0) = 4$

$$s^2 Y - sy(0) - y'(0) - [sY - y(0)] - 2Y = \frac{2}{s-1}$$

$$(s^2 - s - 2)Y - s3 - 4 + 3 = \frac{2}{s-1}$$

$$(s-2)(s+1)Y = \frac{2}{s-1} + 3s + 1$$

$$Y = \frac{2}{(s-1)(s-2)(s+1)} + \frac{3s}{(s-2)(s+1)} + \frac{1}{(s-2)(s+1)}$$

$$Y = \frac{-1}{s-1} + \frac{3}{s-2} + \frac{1}{s+1}$$

$$y = -e^{t} + 3e^{2t} + e^{-t}$$

Definitions

$$H(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

Heaviside

$$L(H(t-a)) = \frac{1}{s}e^{-as}$$

$$L(f(t)H(t-a)) = \frac{1}{s}e^{-as} f(t+a)$$

Watch Monday Lecture

Free Damped Oscillator

$(F_{external} = 0)$ $(\gamma > 0)$

$Mx'' + \gamma x' + Kx = 0$

$x = e^{rt}$

$Mr^2 + \gamma r + k$

$r = \frac{-\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mK}}{2m}$

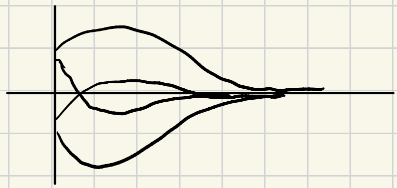
$M, K, \gamma > 0$

If $\gamma^2 > 4MK$ (over damped)

$R_1 \neq R_2 < 0$

$x = Ae^{\left(\frac{-\gamma}{2m} + \frac{\sqrt{\gamma^2 - 4MK}}{2m}\right)} + Be^{\frac{-\gamma}{2m} - \frac{\sqrt{\gamma^2 - 4MK}}{2m}}$

$= e^{\frac{-\gamma t}{2m}}\left(Ae^{\frac{\sqrt{\gamma^2-4MK}}{2m}t} + Be^{-\frac{\sqrt{\gamma^2-4MK}}{2m}t}\right.$

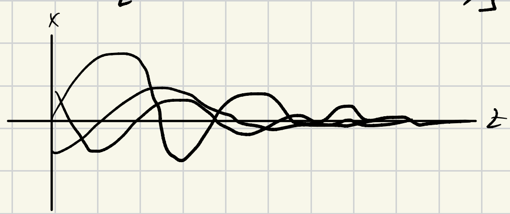


If $\gamma^2 < 4MK$ (underdamped)

$r = \frac{-\gamma}{2M} \pm \frac{i\sqrt{4MK - \gamma^2}}{2m}$

Natural Frequency: $\sqrt{\frac{K}{M}}$

$x(t) = e^{\frac{-\gamma}{2m}t}\left[A\cos\left(\frac{\sqrt{4MK-\gamma^2}}{2m}t\right) + B\sin\left(\frac{\sqrt{4MK-\gamma^2}}{2m}t\right)\right]$

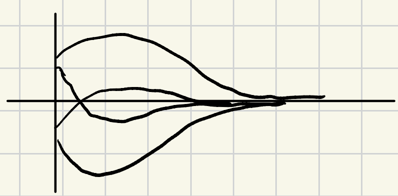


Critically Damped
and overdamped
look similar to the eye,
but have different numbers.

If $\gamma^2 = 4MK$ (critically damped)

$R_1 = R_2$

$x(t) = Ae^{\frac{-\gamma}{2M}t} + Bte^{\frac{-\gamma}{2m}t}$



Forced Damped Oscillator.

sinusoidal force

$$Mx'' + \gamma x' + Kx = F\cos(\omega t)$$

X_c: $Mx'' + \gamma x' + Kx = 0$

$$X_c = e^{\frac{-\gamma}{2M}t}\left[Ae^{+\sqrt{\ }\,t} + Be^{-\sqrt{\ }\,t}\right]$$

$X_p = C\cos(\omega t) + D\sin(\omega t)$

$X_p' = \dots$

$X_p'' = \dots$

Plug into original eq

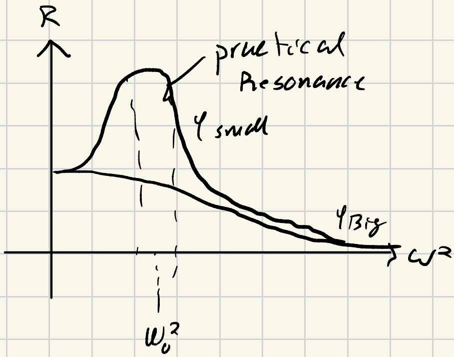
$= R\cos(\omega t - \delta)$; $R = \dfrac{\frac{F}{M}}{\sqrt{(\omega_0^2-\omega^2)^2 + \frac{\gamma^2\omega^2}{M^2}}}$

$\delta = \tan^{-1}\left(\frac{D}{C}\right)$

$$C = \frac{\frac{F}{M}(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2 + \frac{\gamma^2\omega^2}{M^2}} \; ; \; D = \frac{\frac{\gamma}{M}\omega\frac{F}{M}}{\dots}$$

$$X = X_c + X_p = \underbrace{e^{\frac{-\gamma}{2M}t}\left[A\cdots + B\cdots\right]}_{\text{Transient}} + \underbrace{R\cos(\omega t - \delta)}_{\text{steady state}}$$

X_{Tr} $\qquad X_{ss}$



Complexification

$$Mx'' + Kx = F\cos(\omega t) \rightarrow Mx'' + Kx = Re(Fe^{i\omega t}) \rightarrow Re(M\tilde{x}'') + Re(K\tilde{x}) = Re(Fe^{i\omega t})$$

$\tilde{x}$ complex

$x = Re(\tilde{x})$

$\tilde{x} = Ae^{i\omega t}$; $\tilde{x}'' = -\omega^2 Ae^{i\omega t}$

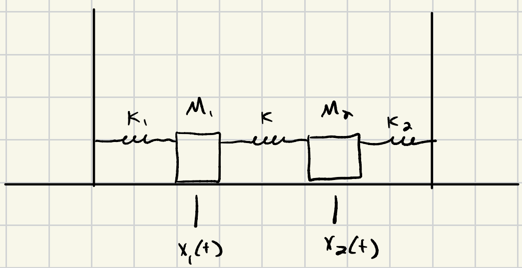
Matrix Multiplication

$$\underset{M\times N}{A}\,\underset{N\times P}{B} = \underset{M\times P}{C}$$

Row-by-column

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 13 & 18 \\ 17 & 24 \\ 21 & 30 \end{bmatrix}$$

Examples



$$\begin{cases} M_1 x_1'' = -K_1 x_1 - K x_1 + K x_2 + g_1(t) \\ M_2 x_2'' = K x_1 - K x_2 - K_2 x_2 + g_2(t) \end{cases}$$

let $x_3 = x_1'$ $\quad x_3' = x_1''$

$x_4 = x_2'$ $\quad x_4' = x_2''$

$$x_1' = x_3$$

$$x_2' = x_4$$

$$x_3' = -\frac{(K+K_1)x_1}{M_1} + \frac{Kx_2}{M_1} + \frac{g_1(t)}{M_1}$$

$$x_4' = \frac{K}{M_2}x_1 - \frac{(K+K_2)x_2}{M_2} + \frac{g_2(t)}{M_2}$$

$$\underline{x}(t) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{(K+K_1)}{M_1} & \frac{K}{M_1} & 0 & 0 \\ \frac{K}{M_2} & -\frac{(K+K_2)}{M_2} & 0 & 0 \end{pmatrix} \qquad g = \begin{pmatrix} 0 \\ 0 \\ g_1 \\ g_2 \end{pmatrix}$$

$$\underline{x}' = A(t)\underline{x} + g$$

Ex)

$$y''' + t^2 y'' + y y' = \cos(t)$$

$y_1 = y$ $\quad\quad$ $y_1' = y_2$

$y_2 = y' = y_1'$ $\quad \rightarrow \quad$ $y_2' = y_3$

$y_3 = y'' = y_2'$ $\quad\quad$ $y_3' = -t^2 y_3 - y_1 y_2 + \cos(t)$

$$\underline{y}(t) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \qquad \underline{F} = \begin{pmatrix} y_2 \\ y_3 \\ -t^2 y_3 - y_1 y_2 + \cos(t) \end{pmatrix} \qquad \underline{y}' = \underline{F}(t, \underline{y})$$

Theorem of Existance + Uniqueness

For $\begin{cases} \underline{y}' = \underline{F}(t, \underline{y}) \\ \underline{y}(t_0) = y_0 \end{cases}$

- If $\exists$ a region around (t_0, y_0) $\forall$ $\underline{F}(t, \underline{y})$ is continuous $\rightarrow$ a solution exists around (t_0, y_0)

- And if

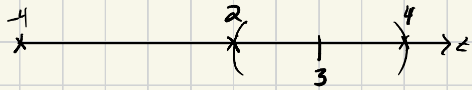
$\frac{dF_1}{dy_1}, \frac{dF_1}{dy_2}, \ldots \frac{dF_1}{dy_n}, \frac{dF_2}{dy_1}, \ldots \frac{dF_n}{dy_n}$ are continuous $\rightarrow$ sol is unique

Ex

$$y_1' = t^2 y_1 + \frac{1}{t-2} y_2 + \cos(t)$$

$$y_2' = \frac{t}{t+4} y_1 + \sin(t) y_2 + \frac{1}{t-4}$$

$$\begin{cases} y_1(3) = 0 \\ y_2(3) = 1 \end{cases}$$



$$I = (2, 4)$$

Lin Hom Syst

$$y' = A(t) y$$

suppose we have n-sols $v_1, v_2, \dots v_N$

$$\underline{v}_1 = \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} \qquad \underline{v}_2 = \begin{pmatrix} 1 \\ t^2 - 1 \\ t^2 + 1 \end{pmatrix}$$

$$\text{Det} \begin{pmatrix} v_1 & v_2 & \dots & v_N \end{pmatrix} = \underbrace{W(t)}_{\text{Wronskian}}$$

if $W(t) \neq 0$, $v_1, v_2, \dots v_N$ are lin ind

if $W(t) = 0$, $v_1, v_2, \dots v_N$ are lin dep

Abel's Theorem

$$\frac{dW}{dt} = T_r(A) \cdot W(t)$$

$\rightarrow$ $W(t) = W(t_0) e^{\int T_r(A)\, dt}$ $\rightarrow$ $v_1, v_2, \dots, v_N$ lin ind

$\rightarrow$ $W = 0$ $\rightarrow$ $v_1, v_2, \dots, v_N$ lin dep

$$y' = Ay + f$$
$$y(t_0) = y_0$$

$M(t)$
$$\begin{cases} \phi'(t) = A\phi \\ \phi(0) = I \end{cases}$$

$$\begin{cases} y' = ay \\ y(0) = 1 \end{cases} \rightarrow y = e^{at}$$

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^2}{3!} + \dots$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \dots = I + \sum_{K=1}^{\infty} \frac{A^K t^K}{K!}$$

$$\phi(t) = e^{At} = I + \sum_{K=1}^{\infty} \frac{A^K t^K}{K!}$$

$$e^{At}\Big|_{t=0} = I; \quad \left(e^{At}\right)' = 0 + \sum_{K=1}^{\infty} \frac{A^K K t^{K-1}}{K!} =$$

$$A + \sum_{K=2}^{\infty} \frac{A^K t^{K-1}}{(K-1)!} = A + A\sum_{K=2}^{\infty} \frac{A^{K-1} t^{K-1}}{(K-1)!} =$$

$$A\left(I + \sum_{K=1}^{\infty} \frac{A^K t^K}{K!}\right) = Ae^{At}$$

$A_{n\times n}$ is Diagonalizable

If it has N lin. ind. eigenvectors

$\varphi_1, \varphi_2, \varphi_3, \dots \varphi_n \rightarrow$ eigenvectors $\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n \rightarrow$ eigenvalues

$$(\varphi_1, \varphi_2, \dots \varphi_n) = U \qquad \Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \\ \vdots & & \ddots \; \lambda_n \end{pmatrix}$$

$$AU = U\Lambda$$

$$AUU^{-1} = U\Lambda U^{-1}$$

$$\star \begin{cases} A = U\Lambda U^{-1} \rightarrow \text{given } a \text{ is diagonalizable} \\ U^{-1}AU = \Lambda \end{cases}$$

$$A^2 = AA = U\Lambda \cancel{U^{-1}U}\Lambda U^{-1} = U\Lambda^2 U^{-1}$$

$$\phi(t) = e^{At} = I + \sum_{k=1}^{\infty} \frac{A^k t^k}{k!} = I + \sum_{k=1}^{\infty} U\Lambda^k U \frac{t^k}{k!} = UIU^{-1} + U\left(\sum_{k=1}^{\infty} \frac{\Lambda^k t^k}{k!}\right)U^{-1} = U\left[I + \sum_{k=1}^{\infty}\frac{\Lambda^k t^k}{k!}\right]U^{-1} =$$

$$= U\begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots \\ 0 & e^{\lambda_2 t} & \\ \vdots & & \ddots & e^{\lambda_n t} \end{pmatrix}U^{-1}$$

$$= e^{At} = \phi$$

Ex)

$$y' = A_{n\times n} y \qquad A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 3 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad U^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\lambda_1 = -1 \qquad \lambda_2 = \lambda_3 = 3$$

$$\Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & e^{3t} & 0 \\ e^{-t} & e^{3t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e^{-t}+e^{3t} & -e^{-t}+e^{3t} & 0 \\ -e^{-t}+e^{3t} & e^{-t}+e^{3t} & 0 \\ 0 & 0 & 2e^{3t} \end{pmatrix} = \phi(t)$$

$$y(t) = \phi(t) y_0$$

Putzer's Algorithm

Another way of finding the exp. of a matrix

e^{At} is the fundemental solution matrix of $\frac{\partial \bar{v}}{\partial t} = A\bar{v}$ which $= I_{n \times n}$ at $t=0$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \dots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

If $\bar{v}_1, \dots, \bar{v}_n$ is a complete set of eigenvectors for A w/ $A\bar{v}_j = \lambda_j \bar{v}_j$

and $U = \begin{bmatrix} \bar{v}_1 & \dots & \bar{v}_n \end{bmatrix}$ then $A = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^{-1}$

$$e^{At} = U \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} U^{-1}$$

Cayley-Hamilton Theorem

If A is an $n \times n$ matrix
and $\lambda_1 - \lambda_n$ is its eigenvalues
and we define

$B_0 = I$
$B_1 = A - \lambda I$
$B_2 = (A - \lambda_2 I) B_1$
$\vdots$
$B_n = (A - \lambda_n I) B_{n-1}$

$$\frac{1}{2}\int_0^2 (2-x)\,dx = \frac{1}{2}\left[2x - \frac{x^2}{2}\right]_0^2 = \frac{1}{2}\left[4-2\right]$$

$$1$$

$$A_n = \frac{2}{L}\int_L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$A_n = \int_0^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

let $u = 2-x$ $\quad v = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$

$du = -dx$ $\quad dv = \cos\left(\frac{n\pi x}{2}\right)dx$

$$= (2-x)\left(\frac{2}{n\pi}\right)\sin\left(\frac{n\pi x}{2}\right) + \int \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)dx$$

$$\frac{2}{n\pi}\left[(2-x)\sin\left(\frac{n\pi x}{2}\right) - \left(\frac{2}{n\pi}\right)\cos\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$\frac{2}{n\pi}\left[0 - \underbrace{\frac{2}{n\pi}\cos(n\pi)} - 0 + \underbrace{\frac{2}{n\pi}}\right]$$

$$\frac{2}{n\pi}\left[\frac{2}{n\pi} - \frac{2}{n\pi}\cos(n\pi)\right]$$

$$\left(\frac{2}{n\pi}\right)^2\left[1-\cos(n\pi)\right]$$

$$A_n = \frac{4\left[1-\cos(n\pi)\right]}{(n\pi)^2}$$

$$\left|\begin{bmatrix} 4-\lambda & 3 \\ 2 & 3-\lambda \end{bmatrix}\right| = 0$$

$$(4-\lambda)(3-\lambda) - 6 = 0$$

12

$$r^2 + r - 6 = 0$$

$(r$

$$r = -3, 2$$

$$\cancel{y_h = C_1 e^{-3t} + C_2 e^{2t}}$$

$$y_p = At^2 + Bt + C$$

$$y_p' = 2At + B$$

$$y_p'' = 2A$$

$$y_p = -2t^2 + t + 1$$

$$2A + 2At + B - 6At^2 - 6Bt - 6C = 12t^2 - 10t - 9$$

$$-6A = 12 \qquad A = -2$$

$$B = 1$$

$$C = 1$$

$$y = y_p + y_h$$

$$y = C_1 e^{-3t} + C_2 e^{2t} - 2t^2 + t + 1$$

$$1 = C_1 + C_2 + 1$$

$$-1 \qquad\qquad -1$$

$$C_1 = -C_2$$

$$0 = C_1 e^{-3} - C_1 e^{2} \cancel{-2 + 1 + 1}$$

$$C_1 e^{-3} = C_1 e^{2}$$

$$C_1 = 0 \qquad C_2 = 0$$

$(D^2+\lambda)y$

$D = \pm\sqrt{\lambda}\, i$

$\underline{\lambda > 0}$

$y = C_1 \cos(\sqrt{\lambda}\, x)$

$y' = -C_1 \sqrt{\lambda} \sin(\sqrt{\lambda}\, x) + C_2 \sqrt{\lambda} \cos(\sqrt{\lambda}\, x)$

$r^2 + 1 = 0$

$r = \pm i$

$y_h = C_1 \cos(x) + C_2 \sin(x)$

$20 = C_1$

$a = -20$