KINK-EQUIVALENCE
OF MATRICES, SPANNING SURFACES 4-MANIFOLDS, & QUADRATIC FORMS

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BASED ON JOINT WORK WITH:

HUGH HOWARDS,

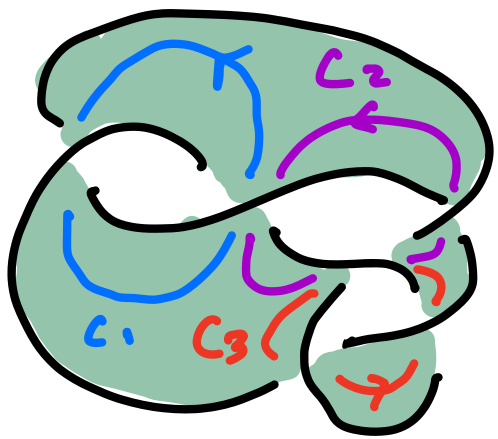
FRANK MOORE,

JOHN TOLBERT

A GOERITZ MATRIX G of a spanning surface F measures how much F twists:

EX:

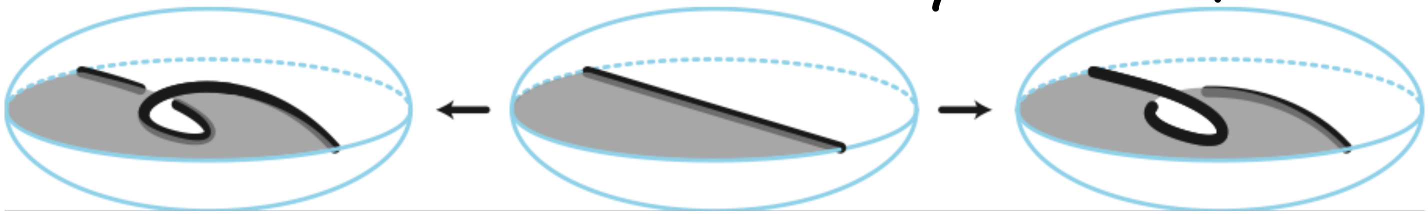
F:



$$G: \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

THM (GREENE): A KNOT IS ALTERNATING IFF IT HAS SPANNING SURFACES WHOSE GOERITZ MATRICES ARE POSITIVE- AND NEGATIVE-DEFINITE.

THEOREM (K): ALL CHECKERBOARD (CB) SURFACES (FOR ALL DIAGRAMS) OF A GIVEN KNOT IN S^3 ARE "KINK-EQUIVALENT", I.E. RELATED BY ISOTOPY & "KINK" MOVES:



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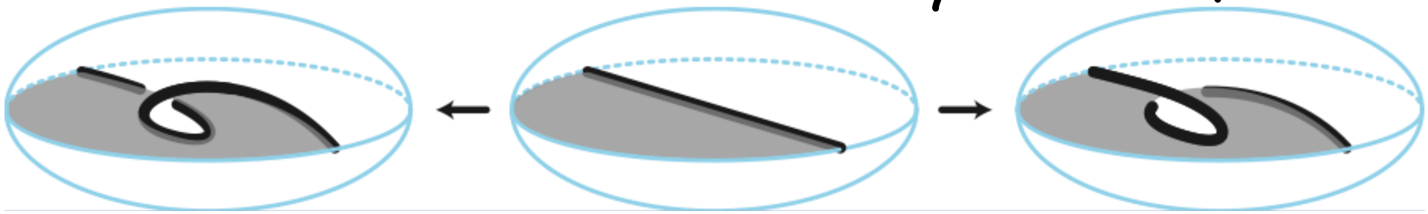
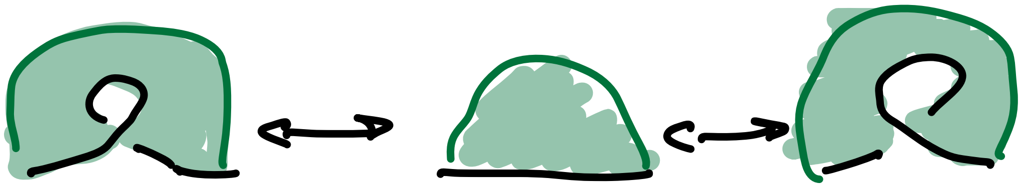
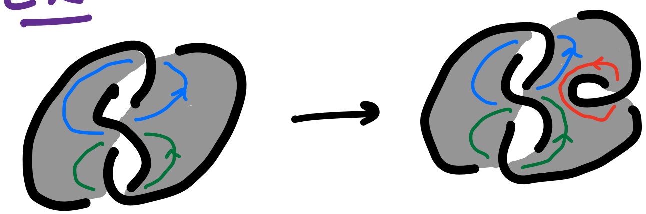


KINK MOVES CHANGE A GOERITZ MATRIX G LIKE THIS:



$$\begin{bmatrix} G & \vec{0} \\ \vec{0}^T & -1 \end{bmatrix} \longleftrightarrow G \longleftrightarrow \begin{bmatrix} G & \vec{0} \\ \vec{0}^T & 1 \end{bmatrix}$$

EX



$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

DEFN: Two symmetric integer matrices are **kink-equivalent** if they are related by these moves:

$$G \longleftrightarrow P^T G P \qquad G \longleftrightarrow \begin{bmatrix} G & 0 \\ 0 & \pm 1 \end{bmatrix}$$

(P unimodular)

Q: How much does kink-equivalence of matrices differ from kink-equivalence of surfaces?

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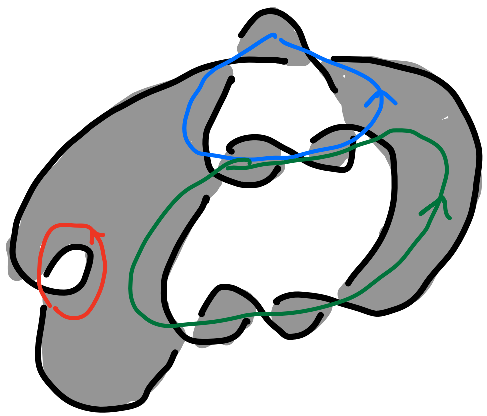
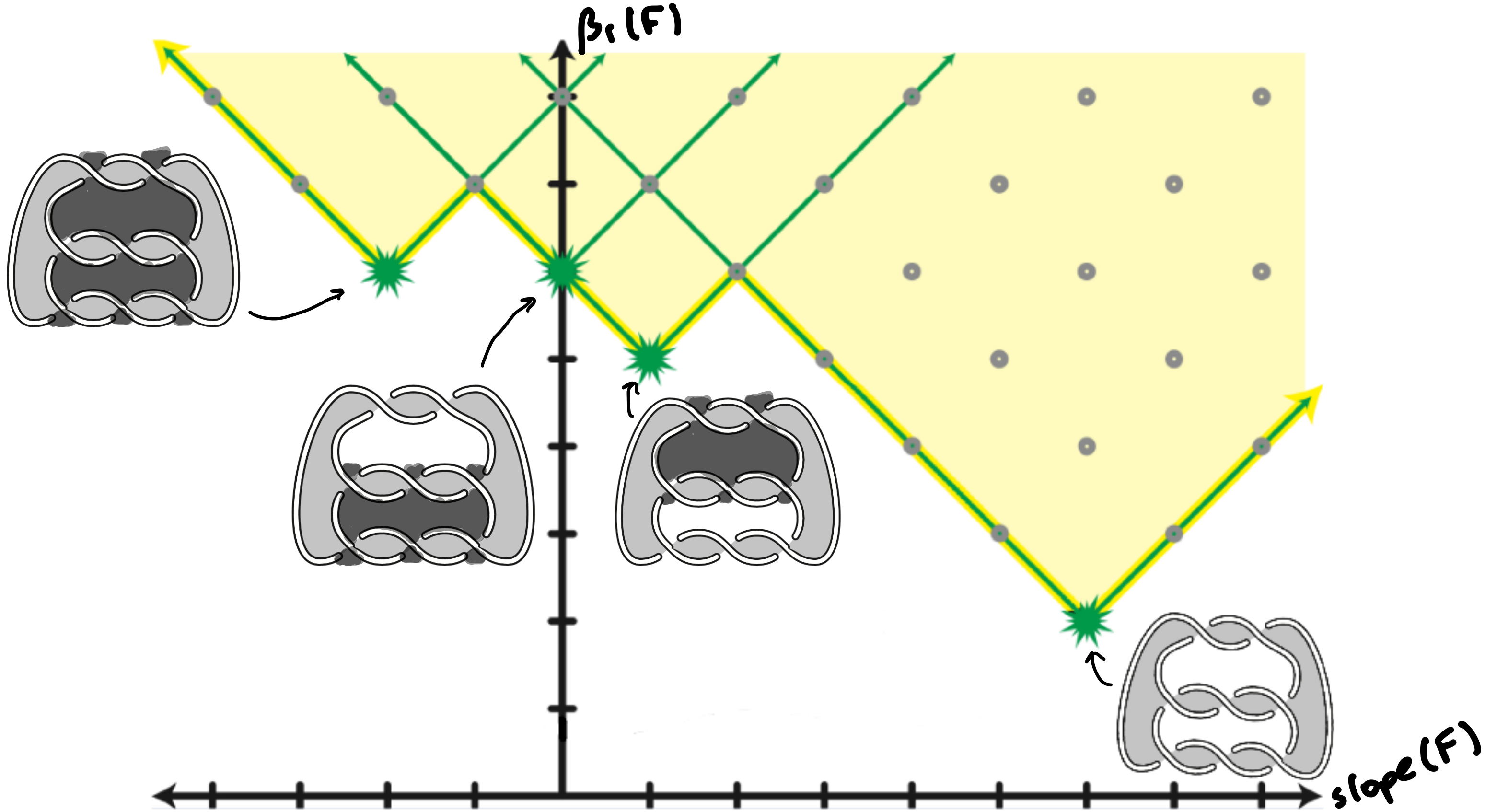
F = 

HAS GOERITZ MATRICES

$$G_1 = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{AND} \quad G_2 = \begin{bmatrix} -3 & 6 & 0 \\ 6 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

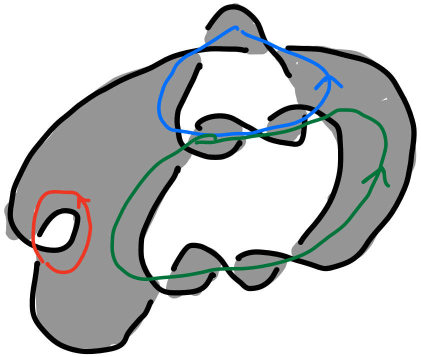
BUT F, UNLIKE G_2, ADMITS NO POSITIVE-UNKINKING MOVE (EVEN AFTER ISOTOPY) $\rightsquigarrow$ "fake unkinking move"

Geography Problem: Given a knot K, identify all pairs $(\beta_1(F), \text{slope}(F))$ realized by spanning surfaces F for K.



THEOREM (ADAMS-K): Given an alternating diagram D of a knot K the adequate state surfaces from D determine the geography of K as shown above.

EX:

F = 

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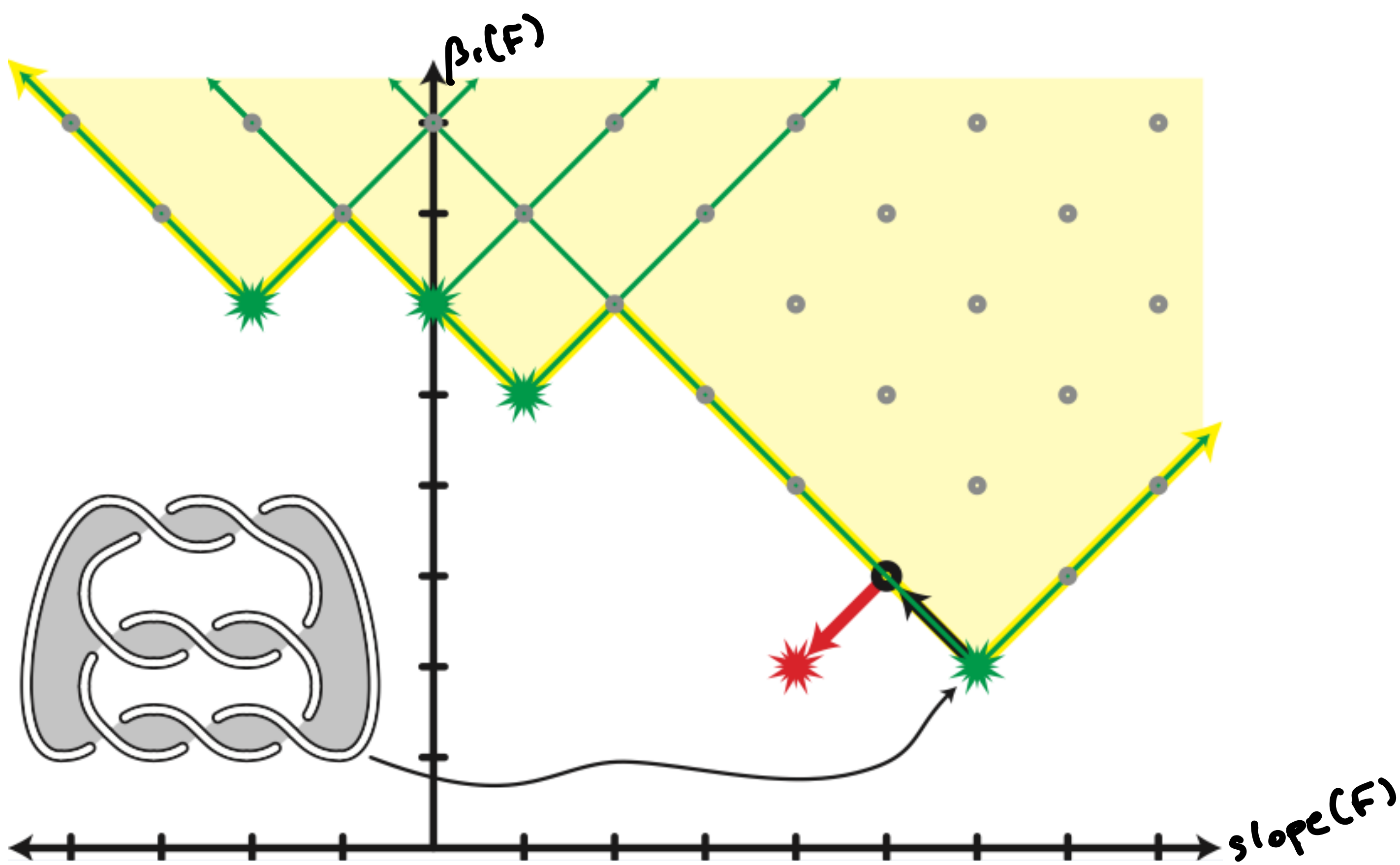

Q: Which positive-definite integer matrices are kink-equivalent ($\sim$) to negative-definite matrices?

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PROP (H-K-M-T): If G decomposes as $G = I + CC^T = [I \ C] \cdot [I \ C]^T$ then $G \sim -(I + C^T C)$, which is negative-definite.

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Q: For which positive-definite integer matrices G does there exist an integer matrix C such that $G = C^T C$?

A: Not all, e.g.:

$$G = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{bmatrix}$$

THM (H. HOWARDS, K., F. MOORE, J. TOLBERT): Given a symmetric integer matrix G, TFAE:

1. G is kink-equivalent to a positive-definite matrix.
2. G is kink-equivalent to a negative-definite matrix.
3. G is nonsingular.

MOREOVER... [$\underbrace{\text{LINEAR BOUND}}$ ON THE # OF MOVES NEEDED]

↳ COMES FROM LAGRANGE'S FOUR SQUARES THM

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COR: EVERY KNOT IS "ALTERNATING UP TO FAKE UNKINKING MOVES"

COR: GIVEN A SIMPLY-CONNECTED CLOSED TOPOLOGICAL 4-MANIFOLD WHOSE INTERSECTION PAIRING IS NONSINGULAR w/ $n_\pm$ $\pm$ eigenvalues, $\exists$ $\pm$definite 4-MANIFOLDS $M_\pm$ AND THESE HOMEOMORPHISMS:

$$M_+ \#_{i=1}^{n_-} \overline{\mathbb{CP}^2} \cong M \#_{i=1}^{4n_-} \mathbb{CP}^2 \quad \text{AND} \quad M_- \#_{i=1}^{n_+} \overline{\mathbb{CP}^2} \cong M \#_{i=1}^{4n_+} \mathbb{CP}^2$$

DEFN: Two symmetric RATIONAL MATRICES are KINK-EQUIVALENT if they are related by these moves:

$$G \longleftrightarrow P^T G P \quad (P \text{ unimodular}) \qquad\qquad G \longleftrightarrow \begin{bmatrix} G & 0 \\ 0 & \pm 1 \end{bmatrix}$$

THM (H. HOWARDS, K., F. MOORE, J. TOLBERT): Given a symmetric rational ~~integer~~ matrix G, TFAE:

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MOREOVER... [LINEAR BOUND ON THE # OF MOVES NEEDED]

COR: LET $q: \mathbb{Q}^n \to \mathbb{Q}$ be a nonsingular quadratic form w/ $n_\pm$ $\pm$ eigenvalues. Writing $q_0 : x \mapsto x^2$, $\exists$ $\pm$definite quadratic forms that satisfy these unimodular congruences:

$$q + q_0^{\oplus n_-} \cong q_+ \oplus (-q_0)^{n_-} \quad \text{AND} \quad q + (-q_0)^{\oplus n_+} \cong q_- \oplus q_0^{n_+}$$

OPEN Q: ARE [3] AND [-3] KINK-EQUIVALENT?

OPEN Q:

$$\left\{\begin{matrix}\text{Spanning}\\ \text{surfaces}\\ \text{in } \mathbb{R}^3\end{matrix}\right\} \subsetneq \left\{\begin{matrix}\text{Free}\\ \text{Spanning}\\ \text{surfaces}\\ \text{in } \mathbb{R}^3\end{matrix}\right\} \underset{?}{\subseteq} \left\{\begin{matrix}\text{unkinked}\\ \text{Checkerboard}\\ \text{surfaces}\\ \text{in } \mathbb{R}^3\end{matrix}\right\} \subsetneq \left\{\begin{matrix}\text{Checkerboard}\\ \text{surfaces}\\ \text{in } \mathbb{R}^3\end{matrix}\right\}$$

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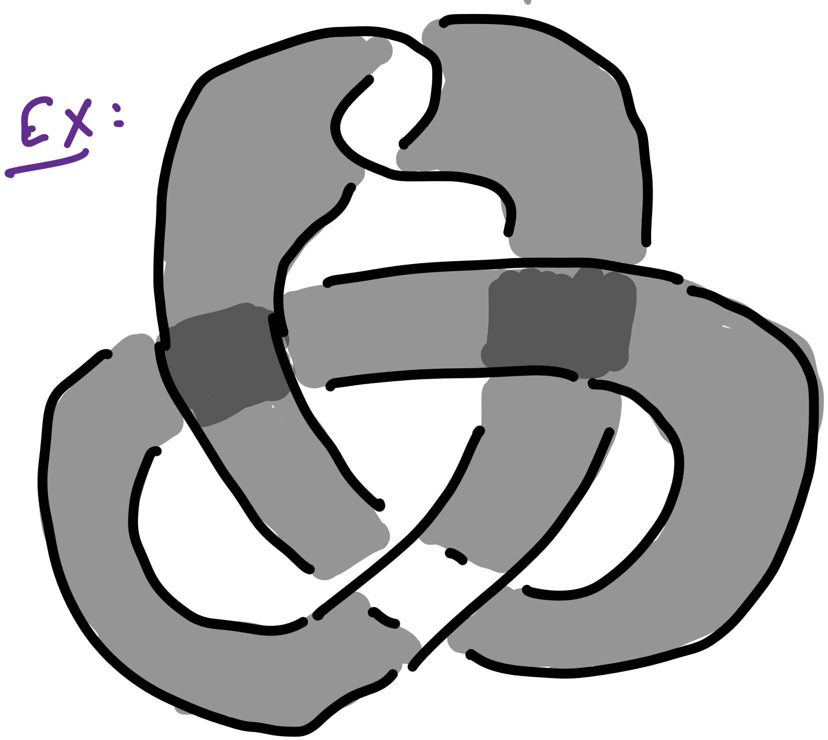
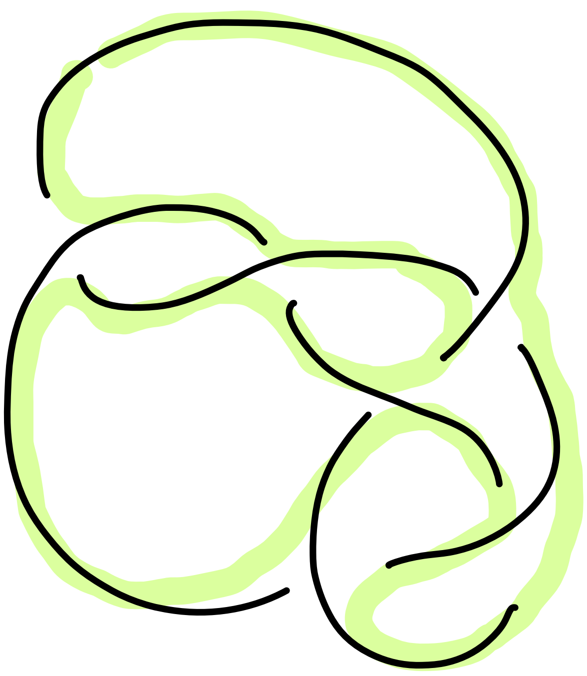
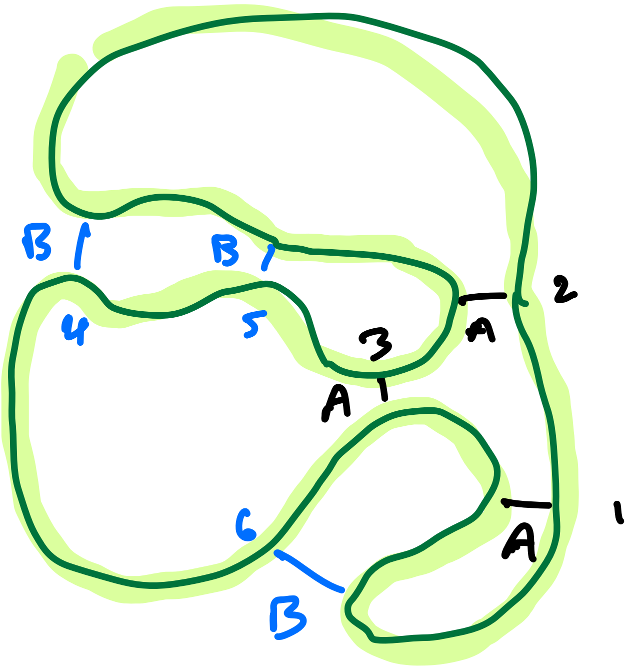


THANK YOU!





$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} I & 0 & \\ 0 & -I & A \\ \hline & & I \quad 0 \\ A^T & & 0 \quad -I \end{array}\right]$$