AVERAGE CROSSCAP NUMBER OF A 2-BRIDGE KNOT

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BASED ON JOINT WORK WITH

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PROLOGUE: EVERY RATIONAL NUMBER $\frac{p}{q}$ CAN BE EXPRESSED AS A CONTINUED FRACTION:

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}}$$

"depth"

where $a_0, \cdots, a_k \in \mathbb{Z}$.

QUESTION: Given $\frac{p}{q}$, what is the smallest depth among all of its continued fraction representations?

NOTATION:

ADDITIVE FORM:

$$\frac{p}{q} = [a_1, \dots, a_k] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}}$$

SUBTRACTIVE FORM:

$$\frac{p}{q} = [b_1, \dots, b_\ell]_- = \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_\ell}}}}$$

BUT WE CAN DISTRIBUTE $-$'S, SO WHO CARES?!

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BUT WE CAN DISTRIBUTE $-$'s, SO WHO CARES?!

FACT: Given $0 < \frac{p}{q} < 1$, $\exists!$ expressions $\frac{p}{q} = [a_1, \dots, a_k] = [b_1, \dots, b_\ell]_-$ IN WHICH ALL $a_i \geq 1$, $b_i \geq 2$, and k is odd.

MOREOVER, GIVEN these expressions, reversing them gives:

$$[a_k, \dots, a_1] = \frac{p'}{q} = [b_\ell, \dots, b_1]$$

WHERE $p' \cdot p \equiv 1 \pmod{q}$.

OUTLINE

① BACKGROUND:

- $\mathbb{Q} - \{0\} \longleftrightarrow$ 2-BRIDGE KNOTS & LINKS

  $\frac{p}{q} \longleftrightarrow K_{p/q}$

- UNORIENTED genus Γ & crosscap number γ

- Hatcher-Thurston surfaces as state surfaces

② UNORIENTED GENUS $\Gamma(K_{\frac{p}{q}})$ AND

AVERAGE UNORIENTED GENUS $\overline{\Gamma}(c)$

③ CROSSCAP NUMBER $\gamma(K_{\frac{p}{q}})$ AND

AVERAGE CROSSCAP NUMBER $\overline{\gamma}(c)$

$\mathbb{Q} - \{0\} \longrightarrow \{\text{2-BRIDGE LINKS}\}$

GIVEN $0 < \frac{p}{q} < 1$:

① Use the division algorithm and $[a_1, \dots, a_{k-1}, a_k, 1] = [a_1, \dots, a_{k-1}, a_k + 1]$ TO WRITE $\frac{p}{q} = [a_1, \dots, a_k]$ WHERE k is odd and all $a_i \geq 1$.

② THEN $K_{\frac{p}{q}}$ IS

THE "NESTED C'S"

PLAT CLOSURE

OF THE BRAID

$\sigma_1^{a_1} \sigma_2^{-a_2} \sigma_1^{a_3} \sigma_2^{-a_4} \cdots \sigma_1^{a_k}$

NOTE: THIS DIAGRAM IS ALTERNATING BECAUSE ALL $a_i \geq 1$.

$\{\text{2-BRIDGE LINKS}\} \longrightarrow \mathbb{Q} - \{0\}$

THE DOUBLE-BRANCHED COVER OF S^3 w/ branch set any 2-bridge link K is a lens space $L(p,q)$.

THE CLASSIFICATION OF LENS SPACES:

<u>FACT</u>: $L(p,q) \underset{\text{homeo}^+}{\cong} L(p',q') \iff \underbrace{q' = q \text{ AND } p' \equiv p^{\pm 1} \pmod{q}}_{*}$

TELLS US THAT THE MAPPING $K \longmapsto \frac{p}{q}$ IS WELL-DEFINED UP TO $*$

<u>THM (SCHUBERT '54)</u>: $K_{\frac{p}{q}}$ AND $K_{\frac{p'}{q'}}$ ARE ISOTOPIC $\iff *$

<u>COR</u>: $K_{\frac{p}{q}}$ IS ISOTOPIC TO ITS MIRROR IMAGE $\iff p^2 \equiv 1 \pmod{q}$

$\iff [a_1, \ldots, a_n]$ is a palindrome

DEFNS:

- A SPANNING SURFACE FOR A KNOT OR LINK $K \subset S^3$ IS A CONNECTED SURFACE $F \subset S^3$ WITH $\partial F = K$.
- ITS COMPLEXITY $\beta_1(F)$ = RANK$(H_1(F))$ = # of holes in F = # of cuts that reduce F to a disk.
- THE UNORIENTED GENUS $\Gamma(K)$ AND CROSSCAP NUMBER $\gamma(K)$ ARE:

$$\Gamma(K) = \text{MIN}\{\beta_1(F) \,/\, F \text{ IS A SPANNING SURFACE FOR } K\}$$

$$\gamma(K) = \text{MIN}\{\beta_1(F) \,/\, F \text{ is a 1-sided spanning surface for } K\}$$

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FACT: Every link satisfies $\gamma(L) = \{\Gamma(L), \Gamma(L)+1\}$

REASON: Even if every surface F realizing $\Gamma(L)$ is 2-sided, we can make any such F 1-sided by attaching a single .

EX:

$\Gamma = 2$  $\beta_1 = 2$ $\longrightarrow$ $\beta_1 = 3$  $\gamma = 3$

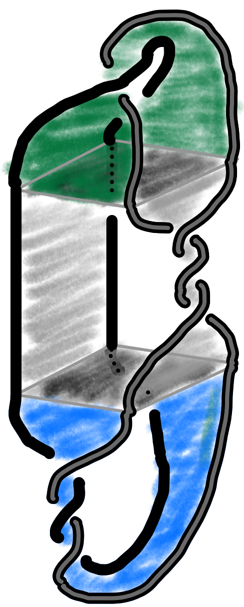
GIVEN $0 < \frac{p}{q} < 1$, HATCHER–THURSTON DESCRIBE A CORRESPONDENCE

SUBTRACTIVE CONTINUED FRACTION REPRESENTATIONS $\frac{p}{q} = [b_1, \ldots, b_\ell]_-$ or $\frac{p}{q} = 1 + [b_1, \ldots, b_\ell]_-$, each $|b_i| \geq 2$ $\longleftrightarrow$ ESSENTIAL SPANNING SURFACES F FOR $K_{p/q}$

UNDER WHICH

THE DEPTH ℓ OF $[b_1, \ldots, b_\ell]_-$ or $1 + [b_1, \ldots, b_\ell]_-$ equals $\beta_1(F)$.

EX: $\frac{11}{18} = [2, 3, 4]_- \rightsquigarrow$



UPSHOT: THE SMALLEST DEPTH AMONG ALL CONTINUED FRACTION REPRESENTATIONS OF p/q equals $\Gamma(K_{p/q})$.

Q: HOW TO FIND UNORIENTED GENUS $\Gamma(K_{p/q})$?

OPTION: Apply earlier results of Hatcher-Thurston, Hirasawa-Teragaito, or Hoste-Shanahan-Van Cott...

Q: WHAT IF WE ALSO WANT TO FIND THE AVERAGE UNORIENTED GENUS $\overline{\Gamma}(c)$ AMONG ALL 2-BRIDGE c-CROSSING KNOTS?

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Q: WHAT IF WE ALSO WANT TO FIND THE AVERAGE UNORIENTED GENUS $\overline{\Gamma}(c)$ AMONG ALL 2-BRIDGE c-CROSSING KNOTS?

IDEA: USE PROPERTIES OF ALTERNATING LINKS TO GET A NEW FORMULA FOR $\Gamma(K_{p/q})$, ONE WELL-SUITED FOR RECURSION...

EVENTUALLY THIS WILL LEAD TO:

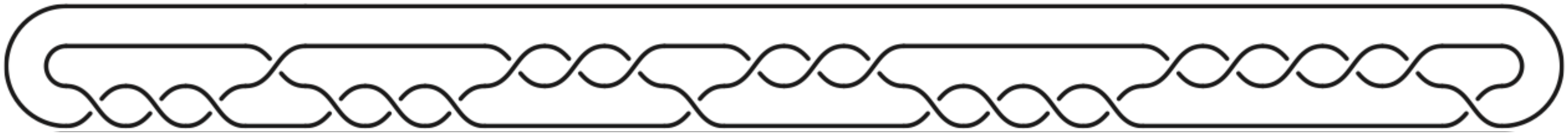
THEOREM: THE AVERAGE UNORIENTED GENUS $\overline{\Gamma}(c)$ AND AVERAGE CROSSCAP NUMBER $\overline{\gamma}(c)$ AMONG ALL c-CROSSING 2-BRIDGE KNOTS SATISFY

$$\overline{\Gamma}(c) = \frac{c}{3} + \frac{1}{9} + \varepsilon_1(c) \qquad \overline{\gamma}(c) = \overline{\Gamma}(c) + \varepsilon_2(c)$$

WHERE $\varepsilon_1(c) \to 0$ AND $\varepsilon_2 \to 0$ AS $c \to \infty$.

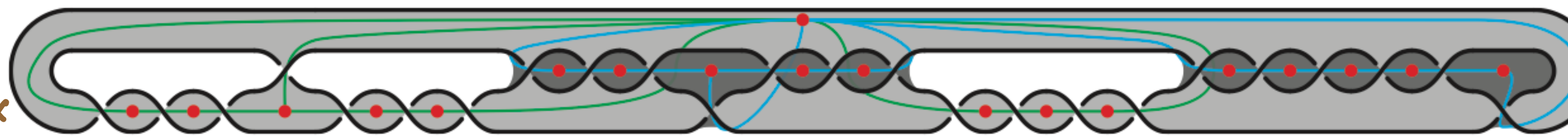
STATE SURFACES:

LINK DIAGRAM D



KAUFFMAN STATE x

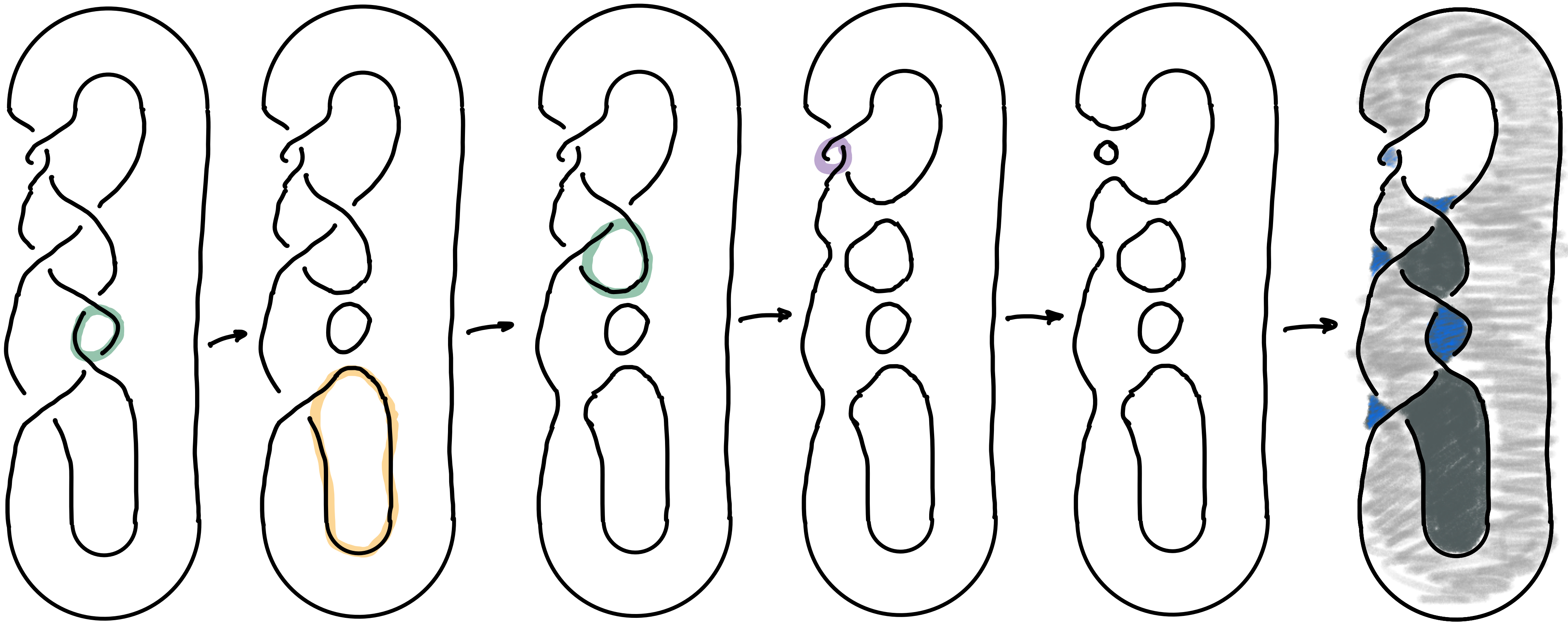
STATE SURFACE F_x (w/ state graph) G_x



FACT: If a diagram D has c crossings and a state x of D has $|x|$ state circles, then its state surface F_x satisfies

$$\beta_1(F_x) = 1 + c - |x|.$$

WE APPLY RESULTS OF ADAMS-K ABOUT ALTERNATING LINKS TO FIND (SURFACES REALIZING) THE UNORIENTED GENUS $\Gamma(K_{p/q})$:



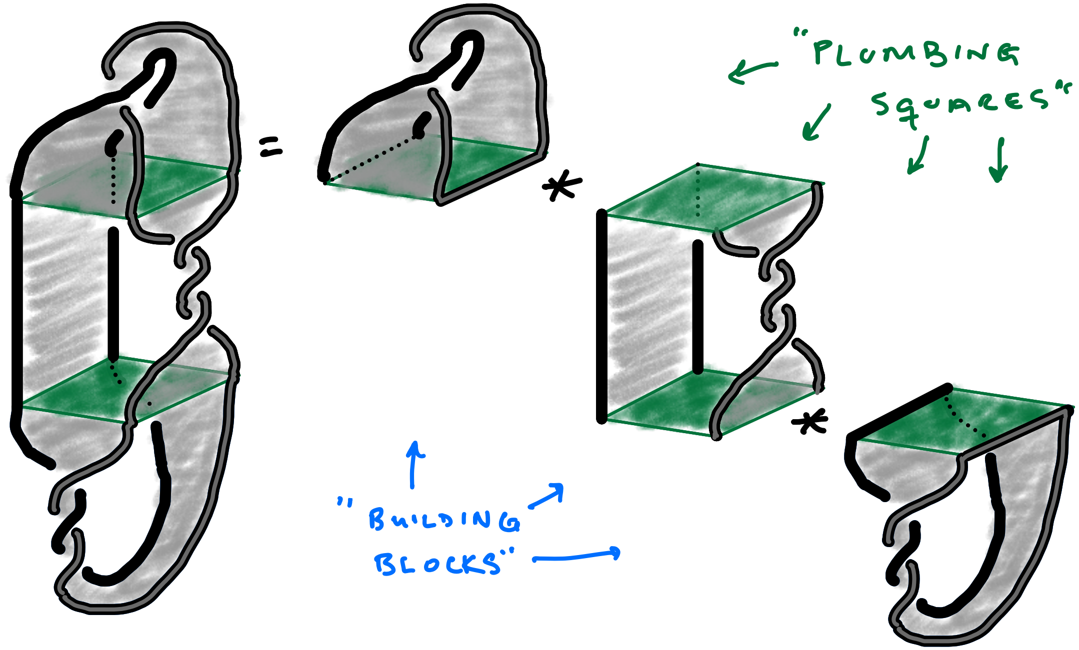
PROCEDURE (ADAMS-K):

(IN 2-BRIDGE SETTING)

- SMOOTH AROUND A 1-GON IF THERE IS ONE
- IF NOT, SMOOTH AROUND A BIGON
- REPEAT

THEOREM (ADAMS-K): The resulting surface F always realizes unoriented genus: $\beta_1(F) = \Gamma(K)$

"RECALL": HATCHER-THURSTON SURFACES ARE PLUMBINGS OF ANNULI & MÖBIUS BANDS:



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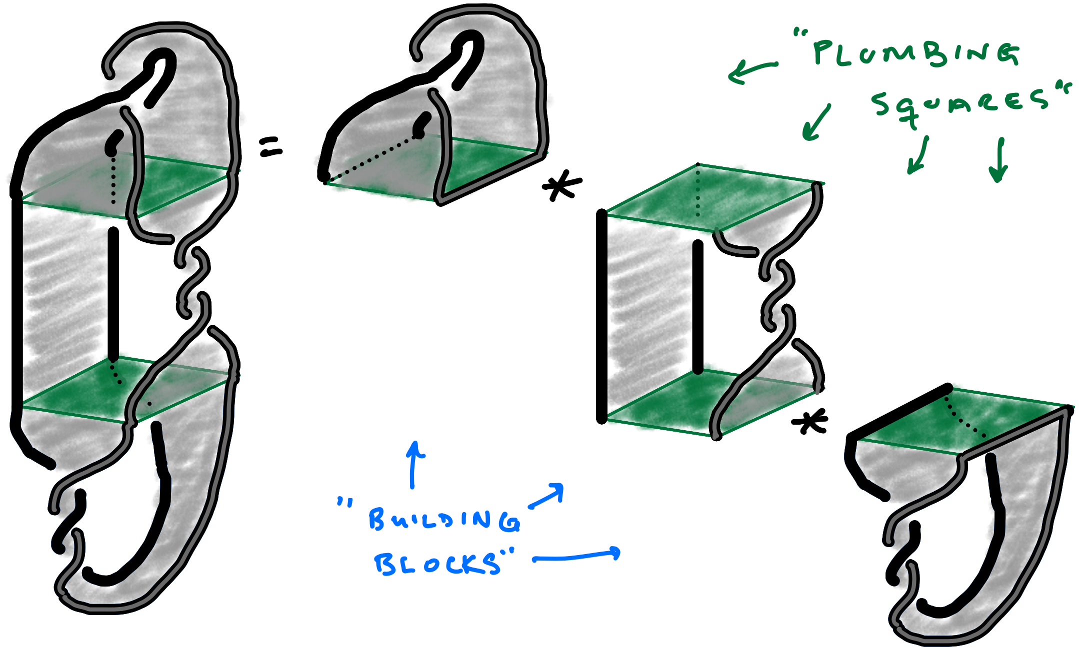
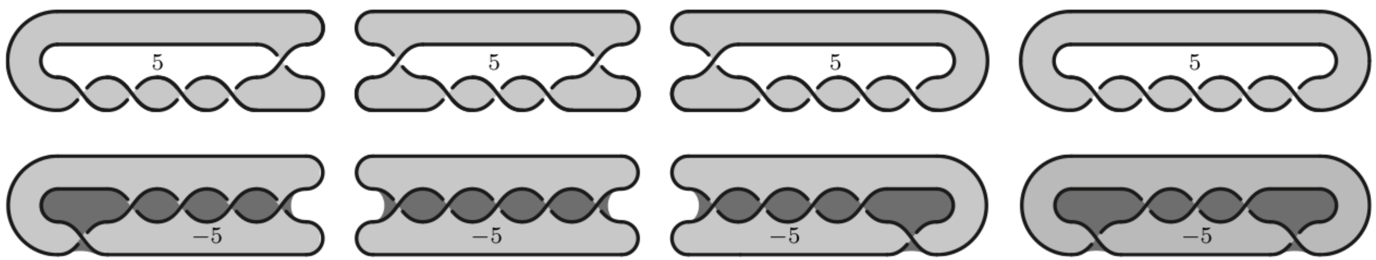
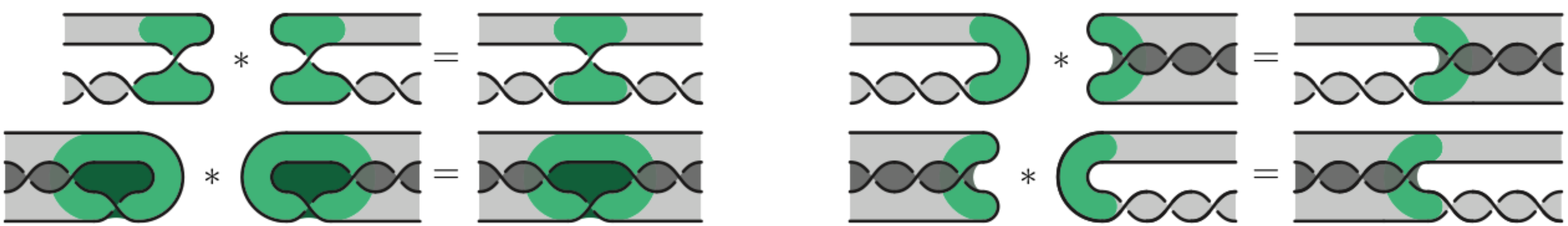


OBS: ALL HATCHER-THURSTON SURFACES OF $K_{p/q}$ CAN BE REALIZED AS STATE SURFACES OF A SINGLE ALTERNATING DIAGRAM:

TYPICAL BUILDING BLOCKS:



PLUMBING SQUARES:

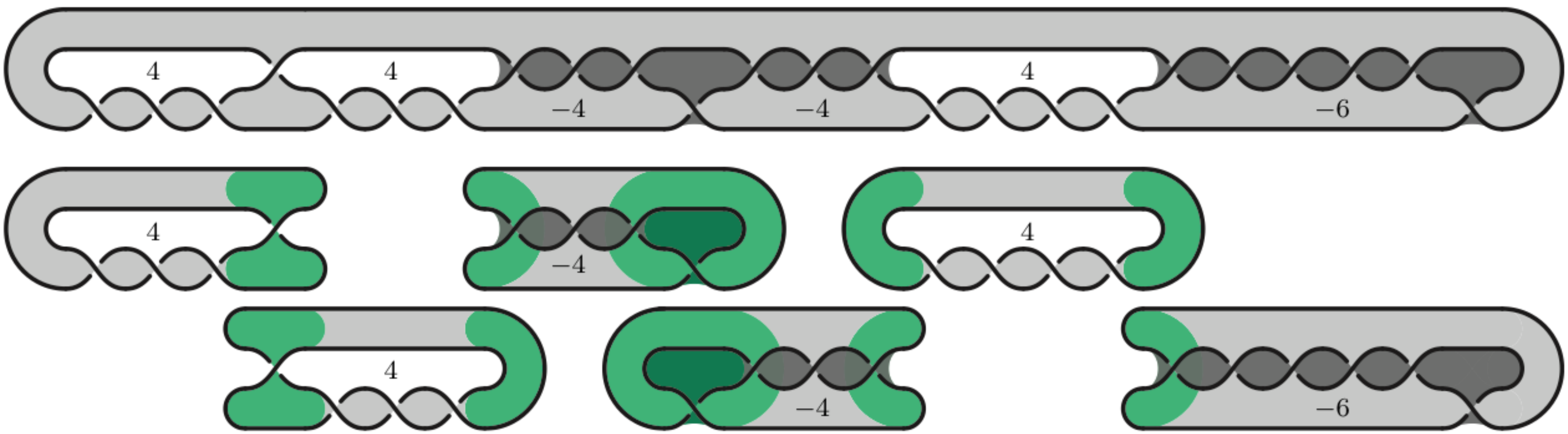


EXAMPLES OF HATCHER–THURSTON (H-T) SURFACES AS STATE SURFACES

all-even!

- THE H-T SURFACE [4, 4, −4, −4, 4, −6] IS AN ALGORITHMIC SEIFERT SURFACE

2-SIDED



EXAMPLES OF HATCHER–THURSTON (H-T) SURFACES AS STATE SURFACES

all-even!

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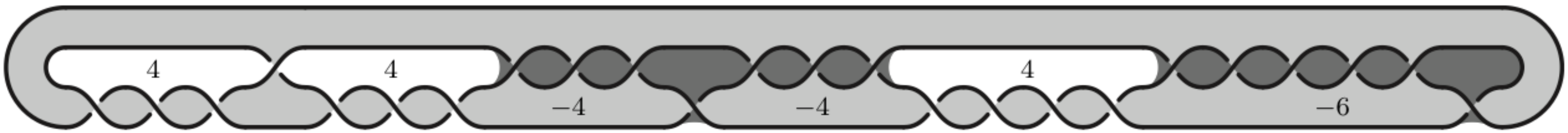
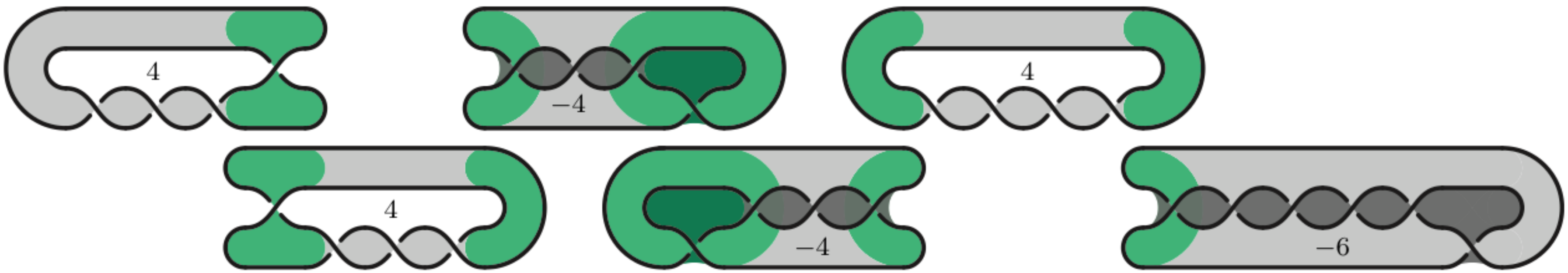
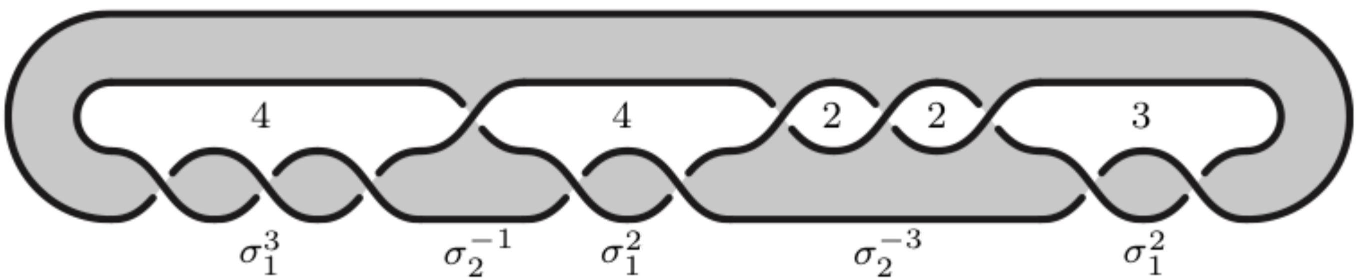




• THE H-T SURFACE FOR THE POSITIVE SUBTRACTIVE FORM $[4,4,2,2,3]_-$ IS A CHECKERBOARD SURFACE FOR A KNOT DIAGRAM D, :



WHEREAS THE ADDITIVE FORM $[3,1,2,3,2]$ DESCRIBES D VIA THE BRAID $\sigma_1^3 \sigma_2^{-1} \sigma_1^2 \sigma_2^{-3} \sigma_1^2$.

THEOREM: IF A DIAGRAM D OF A 2-BRIDGE KNOT OR LINK K CORRESPONDS TO THE POSITIVE SUBTRACTIVE CONTINUED FRACTION $[b_1, \dots, b_k]^-$, THEN $\Gamma(K) = w - z$, WHERE

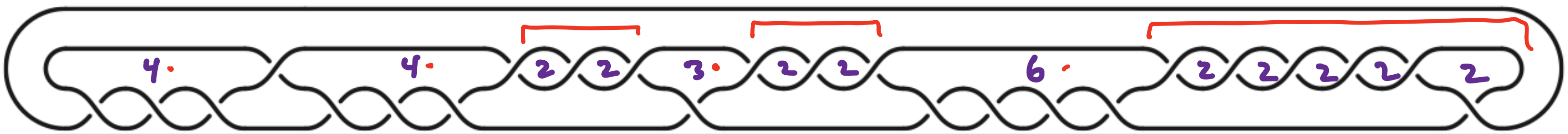
- $w = \#\{i : b_i \geq 3\} + \#$ of strings of 2's in $[b_1, \dots, b_k]^-$
- $z = \#$ of TIMES $2, \underbrace{3, 3, \dots, 3}_{\text{at least one } 3}, 2$ appears in $[b_1, \dots, b_k]^-$

EXAMPLE:

$z = 7$

$w = 1$

D=



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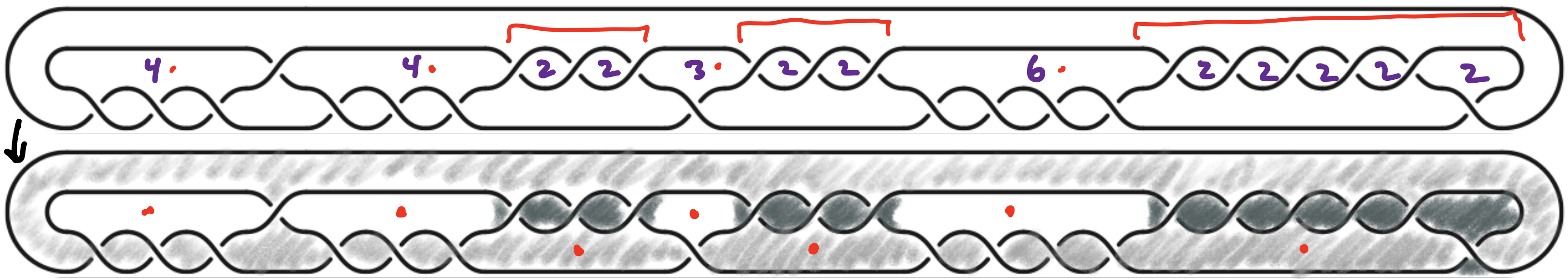

$\beta_1 = z = 7$

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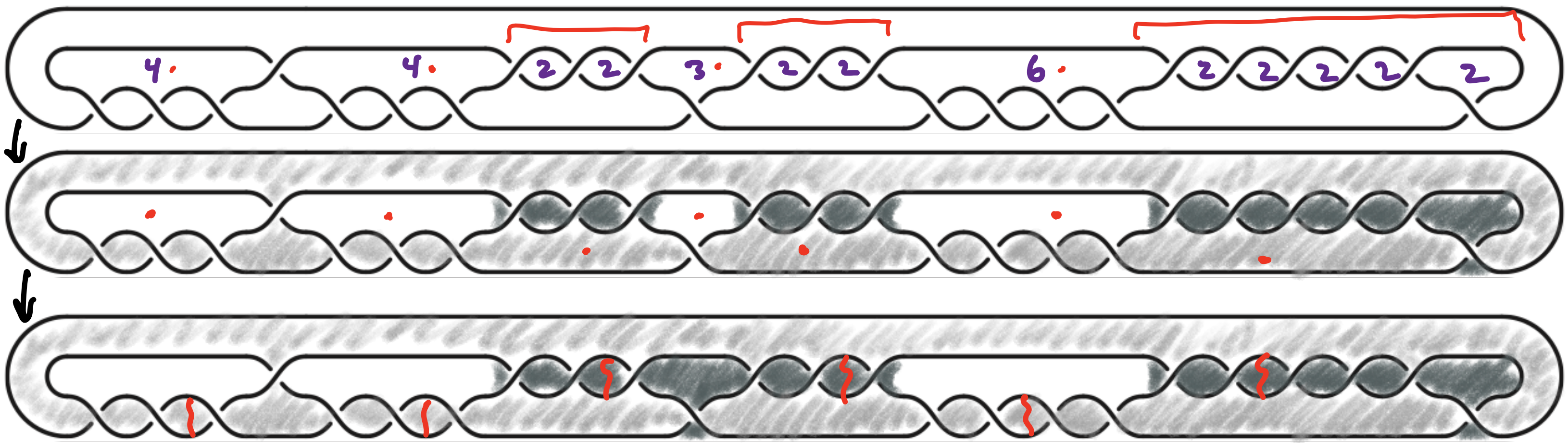
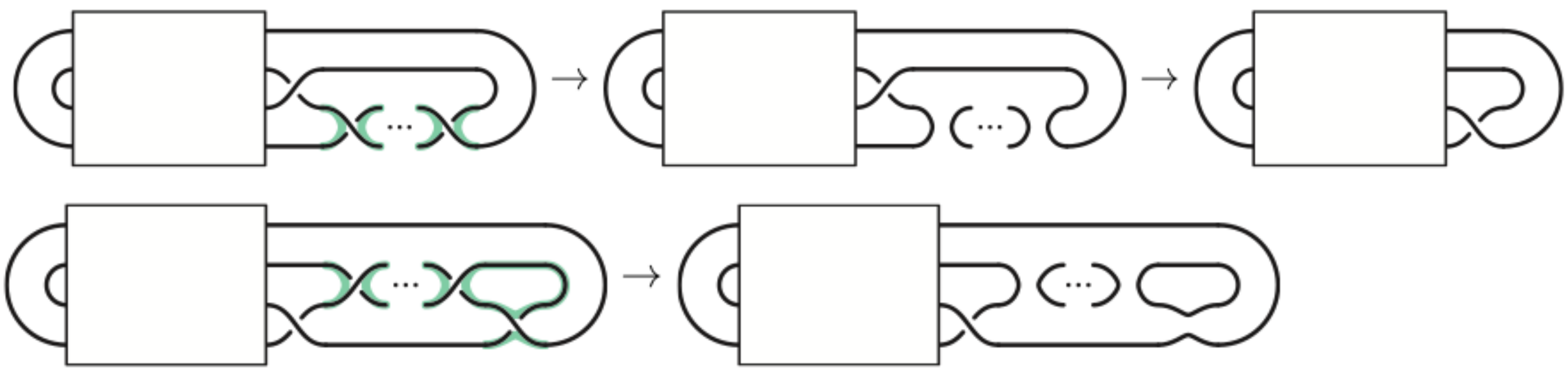


$\beta_1 = z = 7$

$\beta_1 = z - w = 6$

IDEA OF PROOF (by induction on k):



KNOTS

TO COMPUTE AVERAGE UNORIENTED GENUS, WE DENOTE:

$K_c = \{c\text{-crossing 2-bridge knots}\}$ where nonisotopic mirror images are considered distinct

TUPLES

$K(c) = \{ (b_1, \ldots, b_k) \in \mathbb{Z}^k, k \in \mathbb{Z}^+, \text{ all } b_i \geq 2, K_{[b_1, \ldots, b_k]_-} \in K_c \}$

$K^p(c) = \{ \text{palindromic } (b_1, \ldots, b_k) \in K(c) \}$

THEN FOR $c \geq 4$: $|K_c| = \frac{1}{2}\left(|K(c)| + |K^p(c)| \right) = \begin{cases} \frac{2^{c-2}-1}{3} & c \text{ even} \\ \frac{2^{c-2}+2^{(c-1)/2}}{3} & c \equiv 1 \pmod 4 \\ \frac{2^{c-2}+2^{(c-1)/2}+2}{3} & c \equiv 3 \pmod 4 \end{cases}$

ERNST-SUMNERS

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ERNST-SUMNERS

ALSO DENOTING:

$$W(c) = \sum_{\vec{b} \in K(c)} w(K_{[\vec{b}]_-}) \qquad Z(c) = \sum_{\vec{b} \in K(c)} z(K_{[\vec{b}]_-})$$

$$W^p(c) = \sum_{\vec{b} \in K^p(c)} w(K_{[\vec{b}]_-}) \qquad Z^p(c) = \sum_{\vec{b} \in K^p(c)} z(K_{[\vec{b}]_-})$$

WE HAVE:

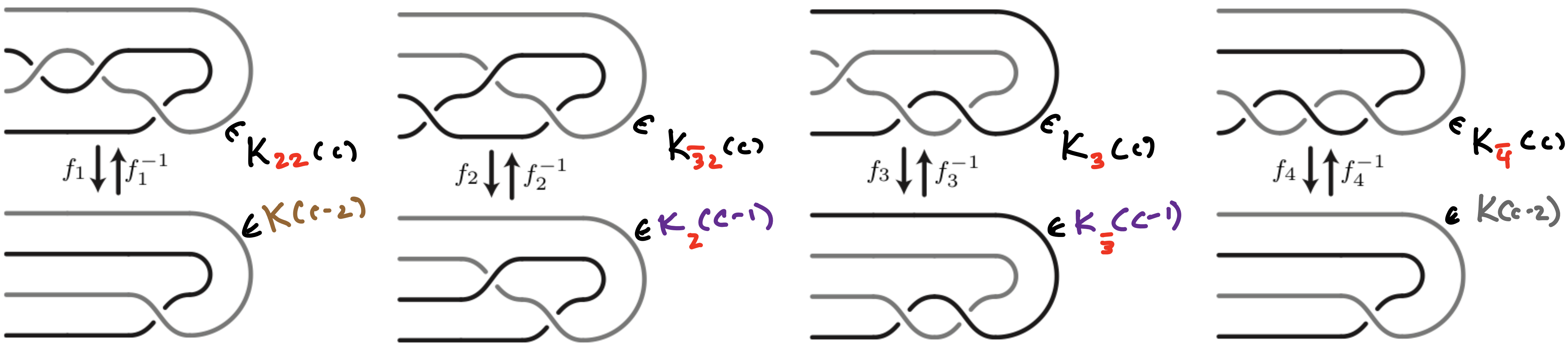
$$\overline{\Gamma}(c) = \frac{1}{|K_c|} \cdot \sum_{K \in K_c} \Gamma(K) = \frac{1}{|K_c|}\left(\sum_{K \in K_c} w(K) - z(K)\right) = \frac{1}{2|K_c|}\left(W(c) - Z(c) + W^p(c) - Z^p(c)\right)$$

Q: HOW TO COMPUTE $W(c)$, $Z(c)$, $W^P(c)$, $Z^P(c)$?

Sketch for $W(c)$ (others are similar):

① PARTITION $K(c) = K_{22}(c) \sqcup K_{\bar{3}2}(c) \sqcup K_3(c) \sqcup K_{\bar{4}}(c)$ and

establish a bijection $g = f_1 \cup f_2 \cup f_3 \cup f_4 : K(c) \to K(c-2) \sqcup \underbrace{K(c-1)}_{= K_2(c-1) \sqcup K_{\bar{3}}(c-1)} \sqcup K(c-2)$



② Use g to get a recursion of the form $W(c) = W(c-1) + 2W(c-2) + D_W$ where D_W comes from the ways that g changes w.

③ Solve the recursion using initial values.

THIS APPROACH REVEALS THAT $W^P(c) = 0 = Z^P(c)$ when c is even

and, writing $d = \frac{c-1}{2}$, YIELDS THE FOLLOWING FORMULAS:

$$W(c) = c \cdot 2^{c-4} \quad \text{for } c \geq 4$$

$$Z(c) = \frac{3c-8}{27}(2^{c-4}) + \frac{14}{27}(-1)^c - (-1)^c \cdot \frac{2}{3} \cdot \delta_{1,c \bmod 3} \quad \text{for } c \geq 6$$

$$W^P(c) = \frac{1+3d}{3}(2^{d-1}) - \frac{2}{3}(-1)^d \quad \text{for odd } c \geq 11$$

$$Z^P(c) = \frac{(3d+1)}{27}(2^{d-1}) - \frac{14}{27}(-1)^d + \frac{2}{3}(-1)^d(\delta_{1,d \bmod 3} + 3\delta_{2,d \bmod 3}). \quad \text{for odd } c \geq 11$$

THEOREM: LET $c \geq 11$ AND WRITE $d = \frac{c-1}{2}$. THEN THE AVERAGE UNORIENTED GENUS AMONG ALL c-CROSSING 2-BRIDGE KNOTS IS:

$$\overline{\Gamma}(c) = \frac{c}{3} + \frac{1}{9} + \varepsilon_1(c)$$

WHERE
$$\varepsilon_1(c) = \begin{cases} \dfrac{c-2+3\,\delta_{1,c \bmod 3}}{3(2^{c-2}-1)} & c \text{ even} \\[2ex] \dfrac{(6d+3)2^{d+1}-4-18\,\delta_{2,d \bmod 3}}{9(2^{c-2}-2^d)} & c \equiv 1 \pmod 4 \\[2ex] \dfrac{(6d+3)2^{d+1}-6c-2+18\,(\delta_{1,d \bmod 3}+2\,\delta_{2,d \bmod 3})}{9(2^{c-2}-2^d+2)} & c \equiv 3 \pmod 4 \end{cases}$$

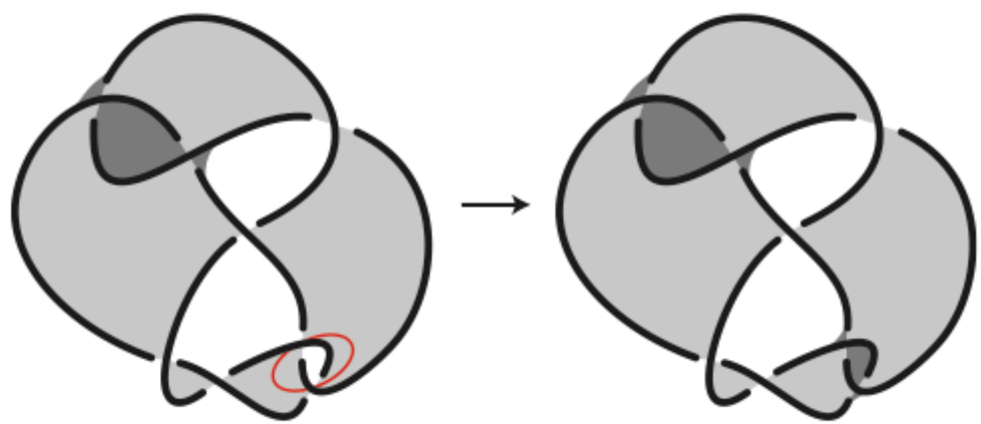
SINCE $\varepsilon_1(c) \longrightarrow 0$ AS $c \to \infty$, $\overline{\Gamma}(c) \longrightarrow \frac{c}{3} + \frac{1}{9}$ AS $c \to \infty$.

TO DETERMINE WHETHER THE CROSSCAP NUMBER $\gamma(K_{p/q})$ is $\Gamma(K_{p/q})$ or $\Gamma(K_{p/q})+1$, we prove:

THEOREM: SUPPOSE THAT A DIAGRAM D OF A 2-BRIDGE KNOT OR LINK K CORRESPONDS TO THE ALL-EVEN SUBTRACTIVE CONTINUED FRACTION $[e_1, \ldots, e_k]_-$, AND ASSUME THAT D HAS MORE THAN TWO CROSSINGS. THEN TFAE:

1. D HAS A UNIQUE MINIMAL COMPLEXITY STATE SURFACE F_x, AND F_x IS 2-SIDED.
2. $\gamma(L) = \Gamma(L)+1$.
3. Each $|e_i| \geq 4$.

WE ALSO PROVE A RELATED THEOREM ABOUT ALTERNATING LINKS IN GENERAL. IN ONE DIRECTION, THE IDEA IS $\longrightarrow$



TO COMPUTE AVERAGE CROSSCAP NUMBER, WE DENOTE:

$K_c = \{c\text{-crossing } 2\text{-bridge knots}\}$ $\quad L_c = \{c\text{-crossing } 2\text{-bridge } \overset{2\text{-component}}{\wedge} \text{ links}\}$

$$K^{\mathcal{E}}(c) \text{ resp. } L^{\mathcal{E}}(c) = \left\{ (e_1, \dots, e_k) \in (2\mathbb{Z})^k,\ k \in \mathbb{Z}^+,\ \text{all } |e_i| \geq 4,\ K_{[e_1, \dots, e_k]^-} \in K_c \text{ resp. } L_c \right\}$$

$E(c) = K^{\mathcal{E}}(c) \cup L^{\mathcal{E}}(c)$, $\Delta(c) = |K^{\mathcal{E}}(c)| - |L^{\mathcal{E}}(c)|$, so $|K^{\mathcal{E}}(c)| = \frac{1}{2}\left(|E(c)| + \Delta(c)\right)$

TO COMPUTE AVERAGE CROSSCAP NUMBER, WE DENOTE:

$K_c = \{c\text{-crossing 2-bridge knots}\}$ $\quad L_c = \{c\text{-crossing 2-bridge 2-component links}\}$

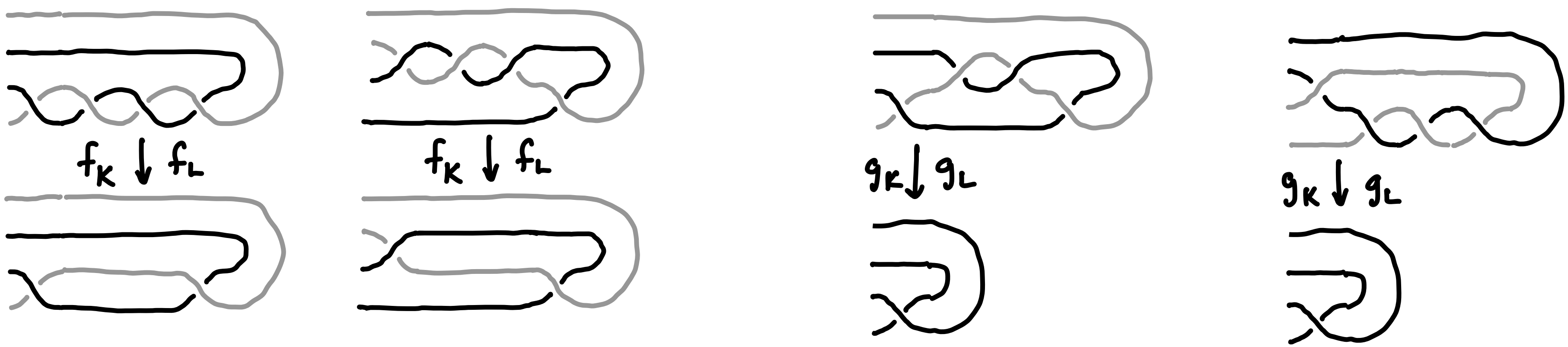
$K^{\varepsilon}(c)$ resp. $L^{\varepsilon}(c)$ $= \left\{ (e_1, \dots, e_k) \in (2\mathbb{Z})^k,\ k \in \mathbb{Z}^+,\ \text{all } |e_i| \geq 4,\ K_{[e_1, \dots, e_k]_-} \in K_c \text{ resp. } L_c \right\}$

$E(c) = K^{\varepsilon}(c) \cup L^{\varepsilon}(c)$, $\Delta(c) = |K^{\varepsilon}(c)| - |L^{\varepsilon}(c)|$, so $|K^{\varepsilon}(c)| = \frac{1}{2}\left(|E(c)| + \Delta(c)\right)$

PARTITION $K^{\varepsilon}(c) = K^{\varepsilon}_{4}(c) \sqcup K^{\varepsilon}_{\bar{6}}(c)$ AND $L^{\varepsilon}(c) = L^{\varepsilon}_{4}(c) \sqcup L^{\varepsilon}_{\bar{6}}(c)$ AND DEFINE

$f_K : K^{\varepsilon}_{\bar{6}}(c) \to K^{\varepsilon}(c-2)$ $\qquad g_K : K^{\varepsilon}_{4}(c) \to L^{\varepsilon}(c-3) \sqcup L^{\varepsilon}(c-4)$

$f_L : L^{\varepsilon}_{\bar{6}}(c) \to L^{\varepsilon}(c-2)$ $\qquad g_L : L^{\varepsilon}_{4}(c) \to K^{\varepsilon}(c-3) \sqcup K^{\varepsilon}(c-4)$



GET RECURSIONS $\quad |E(c)| = |E(c-2)| + |E(c-3)| + |E(c-4)|$

AND $\quad \Delta(c) = \Delta(c-2) - \Delta(c-3) - \Delta(c-4)$

THE RECURSIONS $|E(i)| = |E(i-2)| + |E(i-3)| + |E(i-4)|$ AND $\Delta(i) = \Delta(i-2) - \Delta(i-3) - \Delta(i-4)$

HAVE CHARACTERISTIC POLYNOMIALS $x^4 - x^2 - x - 1 = (x+1)(x^3 - x^2 - 1)$

AND $y^4 - y^2 + y + 1 = (y+1)(y^3 - y^2 + 1)$,

WHOSE ROOTS x_i AND y_i WE CAN WRITE IN TERMS OF

$\alpha = \sqrt[3]{\frac{1}{2}(29 + 3\sqrt{93})}$, $\omega = e^{\pi i/3}$, AND $\beta = -\sqrt[3]{\frac{1}{2}(25 + 3\sqrt{69})}$:

$x_1 = -1$, $x_2 = \frac{1}{3}(1 + \alpha + \alpha^{-1}) \approx 1.5$, $x_3 = \frac{1}{3}(1 - \alpha\omega - \alpha^{-1}\omega^{-1}) \approx -.2 - .7i$, $x_4 = \overline{x_3}$

$y_1 = -1$, $y_2 = \frac{1}{3}(1 + \beta + \beta^{-1}) \approx -.8$, $y_3 = \frac{1}{3}(1 - \beta\omega^{-1} - \beta^{-1}\omega) \approx .8 - .7i$, $y_4 = \overline{y_3}$

THE RECURSIONS $|E(c)| = |E(c-2)| + |E(c-3)| + |E(c-4)|$ AND $\Delta(c) = \Delta(c-2) - \Delta(c-3) - \Delta(c-4)$

HAVE CHARACTERISTIC POLYNOMIALS $x^4 - x^2 - x - 1 = (x+1)(x^3 - x^2 - 1)$

AND $y^4 - y^2 + y + 1 = (y+1)(y^3 - y^2 + 1)$,

WHOSE ROOTS x_i AND y_i WE CAN WRITE IN TERMS OF

$$\alpha = \sqrt[3]{\tfrac{1}{2}(29 + 3\sqrt{93})}, \qquad \omega = e^{\pi i/3}, \qquad \text{AND} \qquad \beta = -\sqrt[3]{\tfrac{1}{2}(25 + 3\sqrt{69})} :$$

$$x_1 = -1, \quad x_2 = \tfrac{1}{3}(1 + \alpha + \alpha^{-1}) \approx 1.5, \quad x_3 = \tfrac{1}{3}(1 - \alpha\omega - \alpha^{-1}\omega^{-1}) \approx -.2 - .7i, \quad x_4 = \overline{x_3}$$

$$y_1 = -1, \quad y_2 = \tfrac{1}{3}(1 + \beta + \beta^{-1}) \approx -.8, \quad y_3 = \tfrac{1}{3}(1 - \beta\omega^{-1} - \beta^{-1}\omega) \approx .8 - .7i, \quad y_4 = \overline{y_3}$$

WE CAN THEN WRITE

$$\boxed{\begin{aligned} |E(c)| &= u_1 x_1^{c-4} + u_2 x_2^{c-4} + u_3 x_3^{c-4} + u_4 x_4^{c-4} \\ \text{AND} \quad \Delta(c) &= v_1 y_1^{c-4} + v_2 y_2^{c-4} + v_3 y_3^{c-4} + v_4 y_4^{c-4} \end{aligned}}$$

INITIAL VALUES

WHERE, BY CRAMER'S RULE,

$$u_1 = \frac{\begin{vmatrix} 2 & 1 & 1 & 1 \\ 0 & x_2 & x_3 & x_4 \\ 2 & x_2^2 & x_3^2 & x_4^2 \\ 2 & x_2^3 & x_3^3 & x_4^3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{vmatrix}} = \frac{2}{3}, \qquad v_1 = \frac{\begin{vmatrix} -2 & 1 & 1 & 1 \\ 0 & y_2 & y_3 & y_4 \\ -2 & y_2^2 & y_3^2 & y_4^2 \\ 2 & y_2^3 & y_3^3 & y_4^3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 \\ y_1^3 & y_2^3 & y_3^3 & y_4^3 \end{vmatrix}} = -2$$

AND LIKEWISE $u_2 \approx .7$, $u_3 \approx .3 - .2i$, $u_4 = \overline{u_3}$, $v_2 \approx 1.1$, $v_3 \approx -.5 - .1i$, $v_4 = \overline{v_3}$

AFTER SIMILARLY ACCOUNTING FOR PALINDROMES, WE OBTAIN:

THEOREM: FOR $c \geq 7$, THE NUMBER OF c-CROSSING 2-BRIDGE KNOTS K WITH $\gamma(K) = \Gamma(K)+1$ (RATHER THAN $\gamma(K)=\Gamma(K)$) IS:

$$\frac{1}{2}\left(|K^E(c)| + |K^{EP}(c)|\right) = \frac{1}{4}\sum_{i=1}^{4}\left(u_i x_i^{\frac{c-7}{2}}\left(x_i^{\frac{c-1}{2}} + \delta_{0,c \ mod \ 2}\right) + v_i y_i^{c-4}\right),$$

WHERE THE u_i, v_i, x_i, AND y_i ARE FROM THE LAST SLIDE.

ERGO, WITH THESE SAME VALUES AND DENOTING $d = \frac{c-1}{2}$:

THEOREM: THE PORTION $\varepsilon_2(c)$ OF c-crossing 2-bridge knots with $\gamma(K) = \Gamma(K)+1$ IS:

$$\varepsilon_2(c) = \begin{cases} \displaystyle\sum_{i=1}^{4} \frac{3\left(u_i x_i^{d-3}\left(x_i{}^d + \delta_{0,c \ mod \ 2}\right) + v_i y_i^{c-4}\right)}{4(2^{c-2}-1)} & c \ even \\ \displaystyle\sum_{i=1}^{4} \frac{3\left(u_i x_i^{d-3}\left(x_i{}^d + \delta_{0,c \ mod \ 2}\right) + v_i y_i^{c-4}\right)}{4(2^{c-2}+2^d)} & c \equiv 1 \pmod 4 \\ \displaystyle\sum_{i=1}^{4} \frac{3\left(u_i x_i^{d-3}\left(x_i{}^d + \delta_{0,c \ mod \ 2}\right) + v_i y_i^{c-4}\right)}{4(2^{c-2}+2^d+2)} & c \equiv 3 \pmod 4 \end{cases}$$

IN PARTICULAR, $\varepsilon_2(c) \to 0$ AS $c \to \infty$.

IN SUMMARY:

THEOREM: THE AVERAGE UNORIENTED GENUS $\overline{\Gamma}(c)$ AND AVERAGE CROSSCAP NUMBER $\overline{\gamma}(c)$ AMONG ALL c-CROSSING 2-BRIDGE KNOTS SATISFY

$$\overline{\Gamma}(c) = \frac{c}{3} + \frac{1}{9} + \varepsilon_1(c) \qquad \overline{\gamma}(c) = \overline{\Gamma}(c) + \varepsilon_2(c)$$

WHERE $\varepsilon_1(c) \to 0$ AND $\varepsilon_2 \to 0$ AS $c \to \infty$.

THANKS!