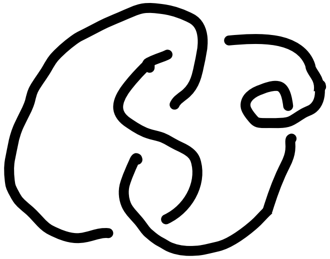
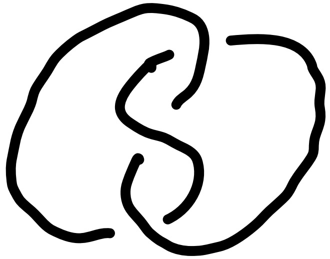
Downstream lessons from a geometric proof of Tait's flyping conjecture

Thomas Kindred

Wake Forest University



NON-REDUCED ALTERNATING



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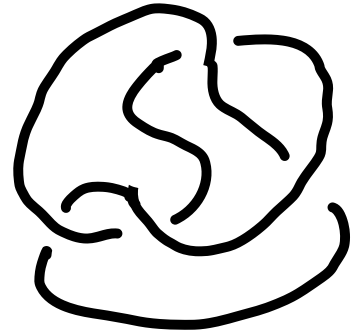


NON ALTERNATING

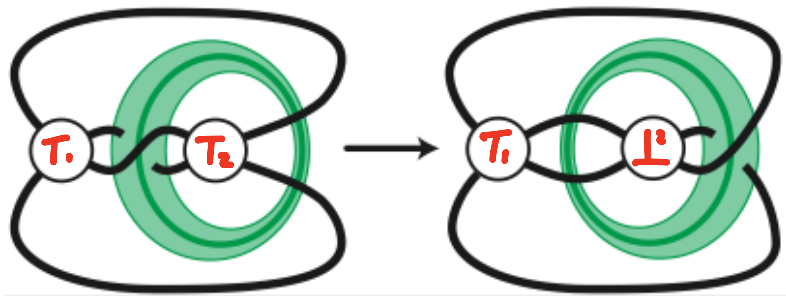
TAIT'S CONJECTURES (1898):

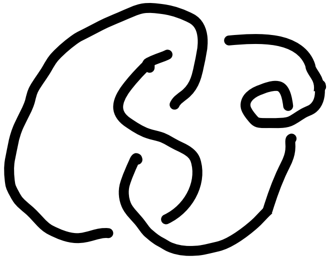
Let D & D' be reduced alternating diagrams of a knot K.

① CROSSINGS: $c(D) = c(D') = c(K)$

② WRITHE = $|\ \ | - |\ \ |$: $w(D) = w(D')$

③ IF K IS PRIME, THEN D & D' ARE RELATED BY A SEQUENCE OF FLYPE MOVES:





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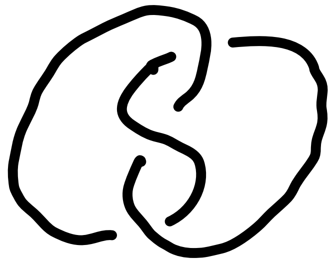

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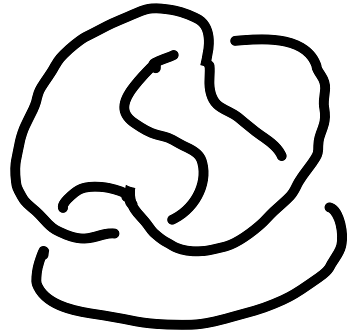

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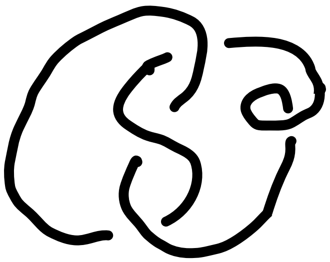
PROVEN IN 1987-93 by COMBOS KAUFFMAN, MENASCO, MURASUGI, AND THISTLETHWAITE.

PROOFS ALL USED THE JONES POLYNOMIAL (DISCOVERED IN 1985)



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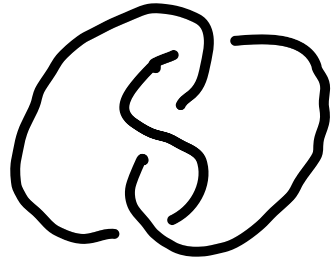

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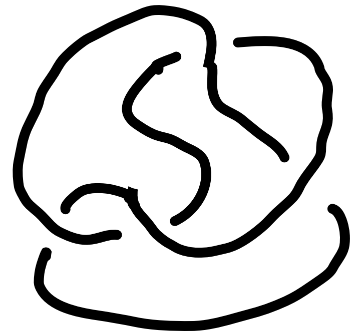

NON ALTERNATING

TAIT'S CONJECTURES (1898): Let D & D' be reduced alternating diagrams of a knot K.

① $\underbrace{c(D)=c(D')}=c(K)$

② reproved without the Jones polynomial by Greene in 2017

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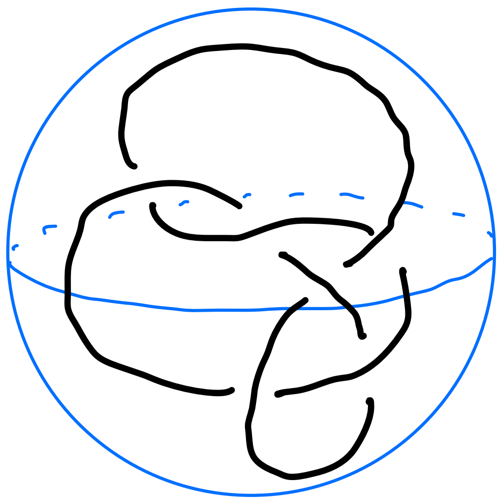
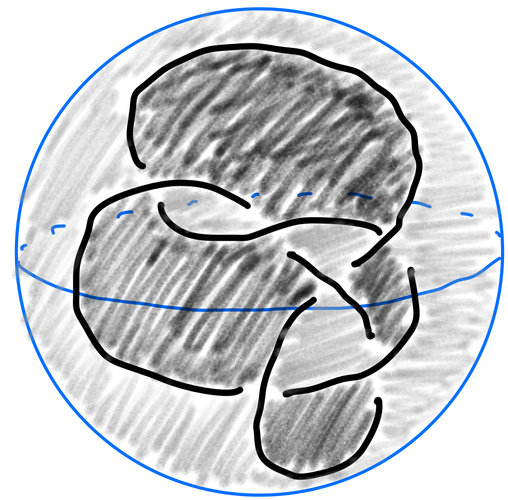
GEOMETRIC PROOF OF TAIT'S FLYPING CONJECTURE (K), MAIN IDEA:

$$\left\{ \begin{array}{c} \text{REDUCED} \\ \text{ALTERNATING} \\ \text{LINK DIAGRAMS} \end{array} \right\} \xleftrightarrow[\text{(GREENE, K)}]{} \left\{ \begin{array}{c} \text{PAIRS } (F_+, F_-) \text{ OF} \\ \text{ESSENTIAL } \pm\text{-DEFINITE} \\ \text{SURFACES WITH } \partial F_+ = \partial F_- \end{array} \right\}$$





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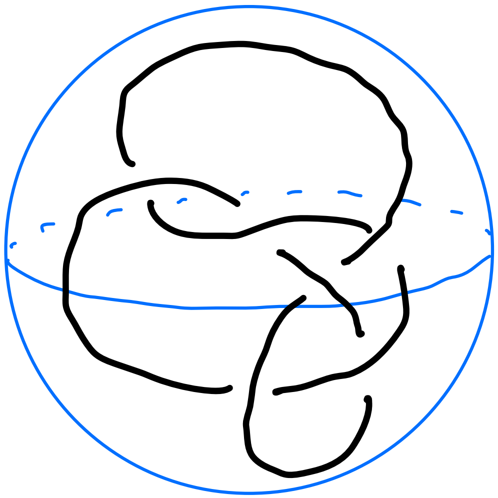
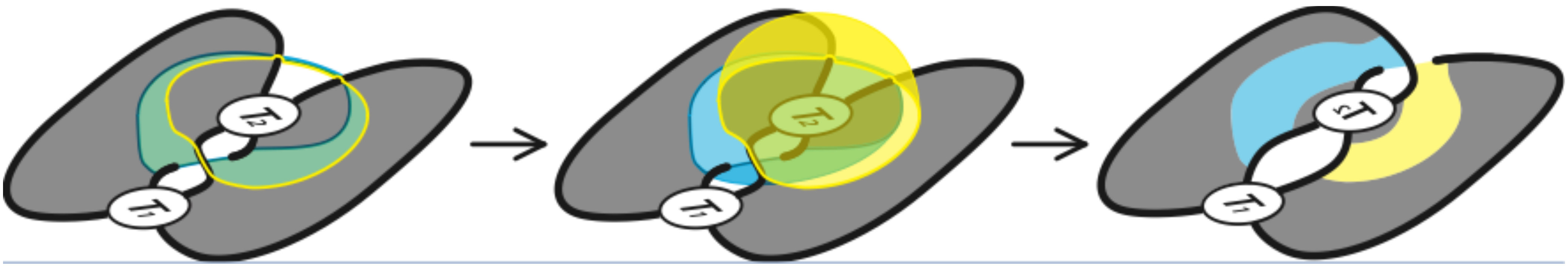
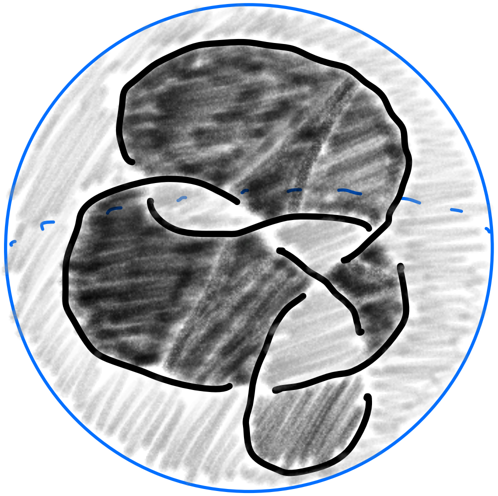
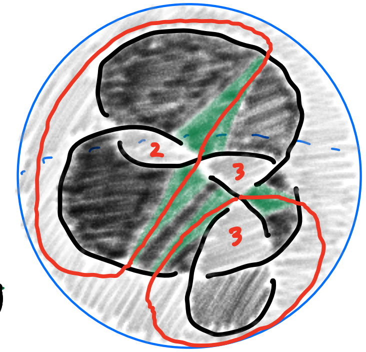


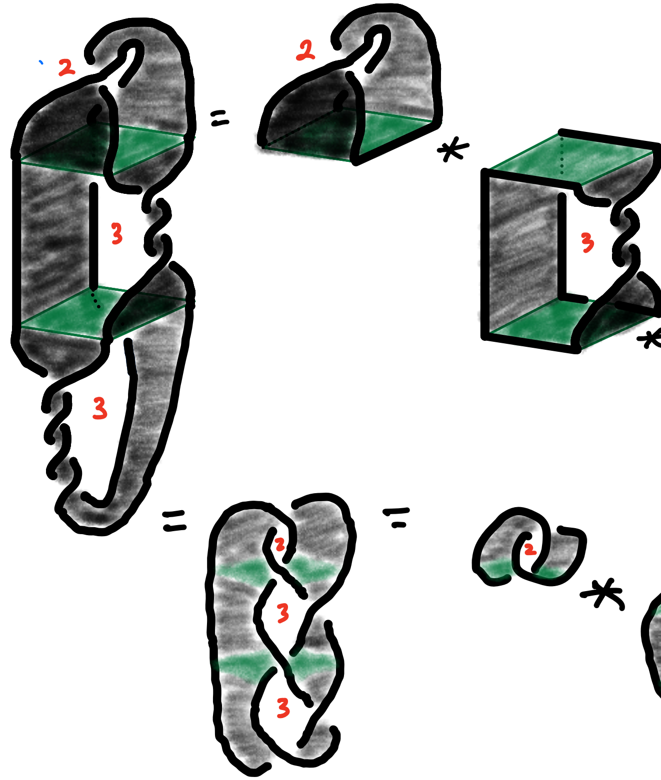
FLYPE MOVE $\longleftrightarrow$ CERTAIN "REPLUMBING" F_+ OR F_-

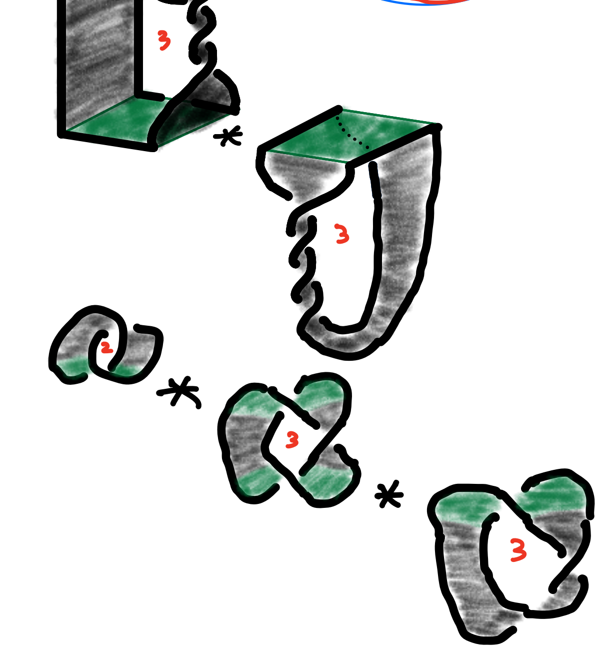


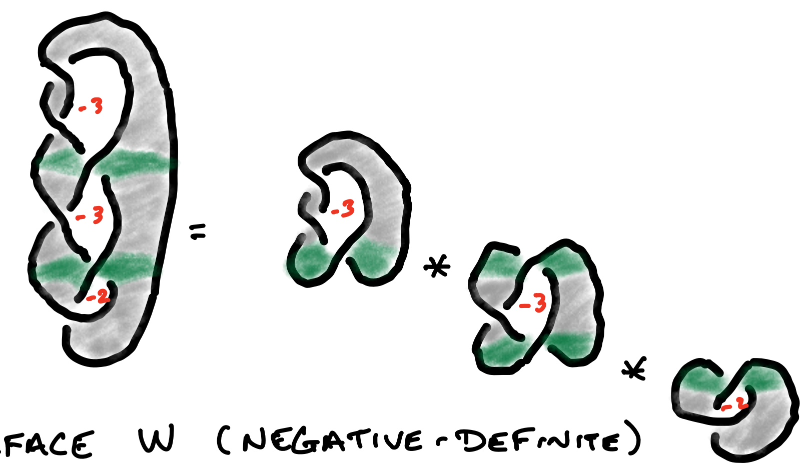
BLACK SURFACE B
(POSITIVE-DEFINITE)

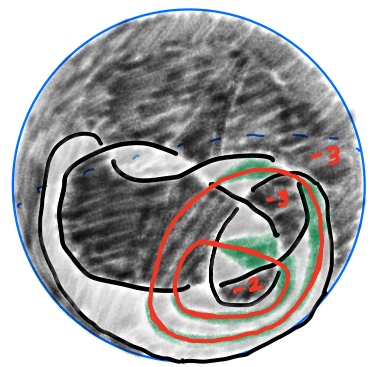




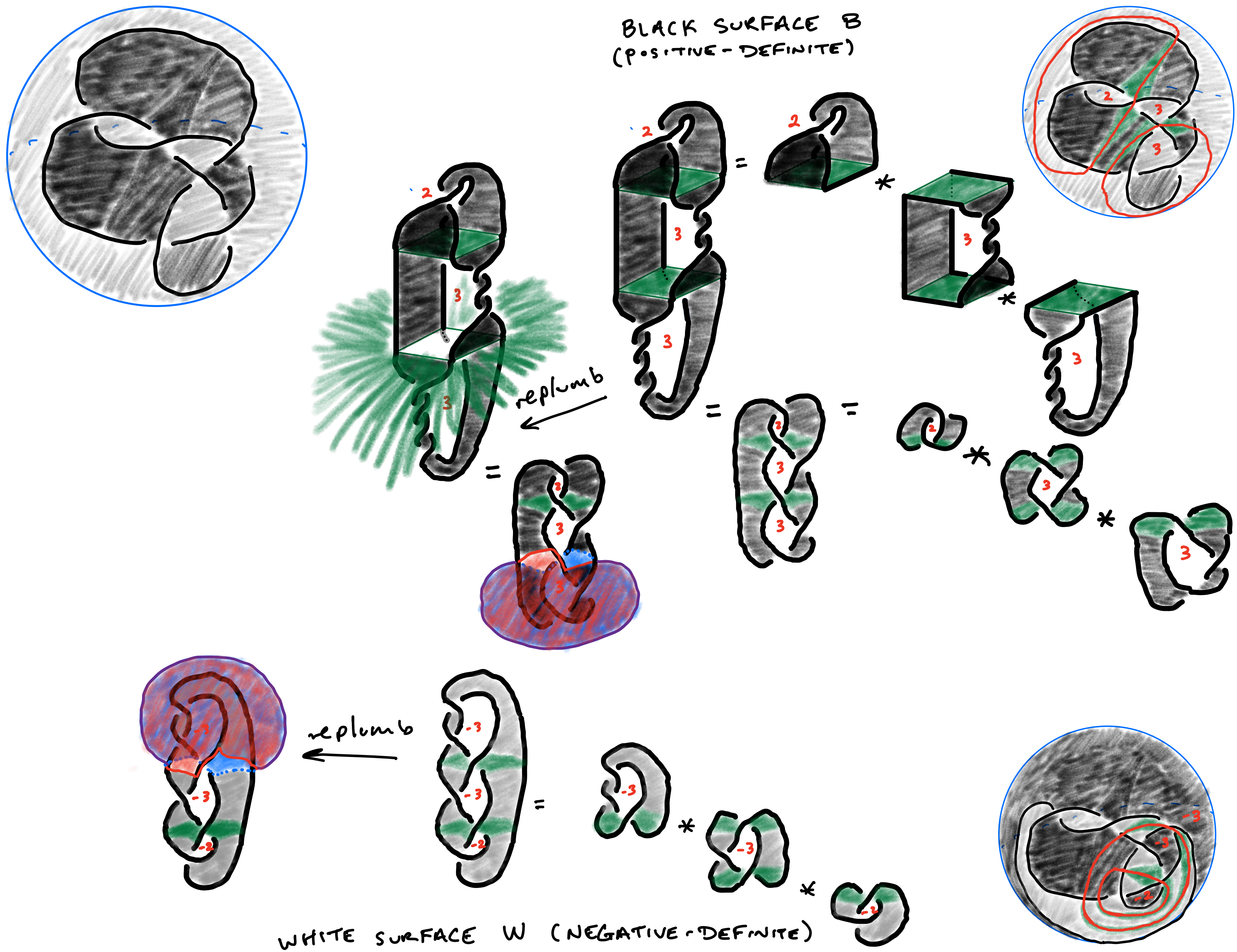








WHITE SURFACE W (NEGATIVE-DEFINITE)


BLACK SURFACE B
(POSITIVE-DEFINITE)
replumb
replumb
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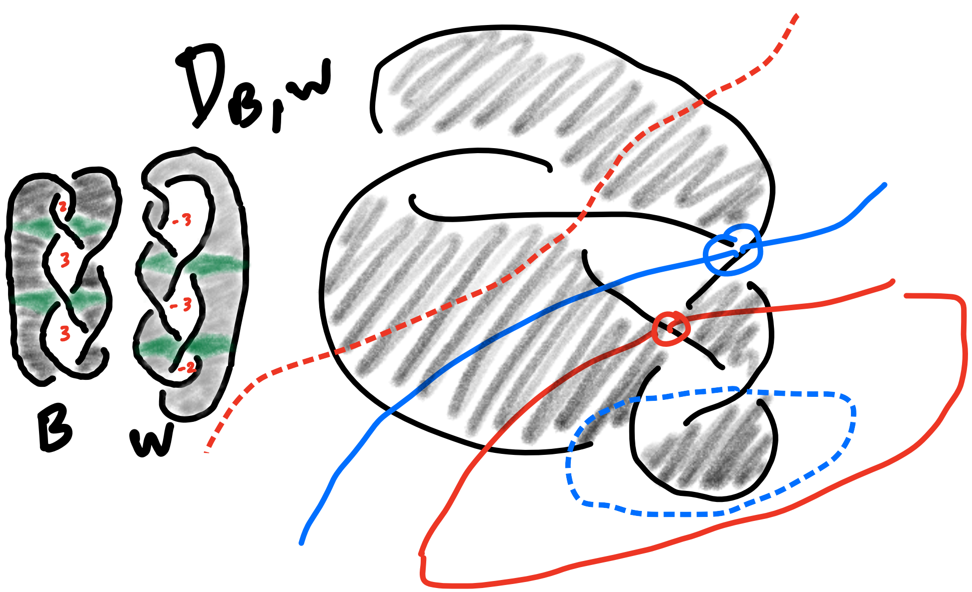
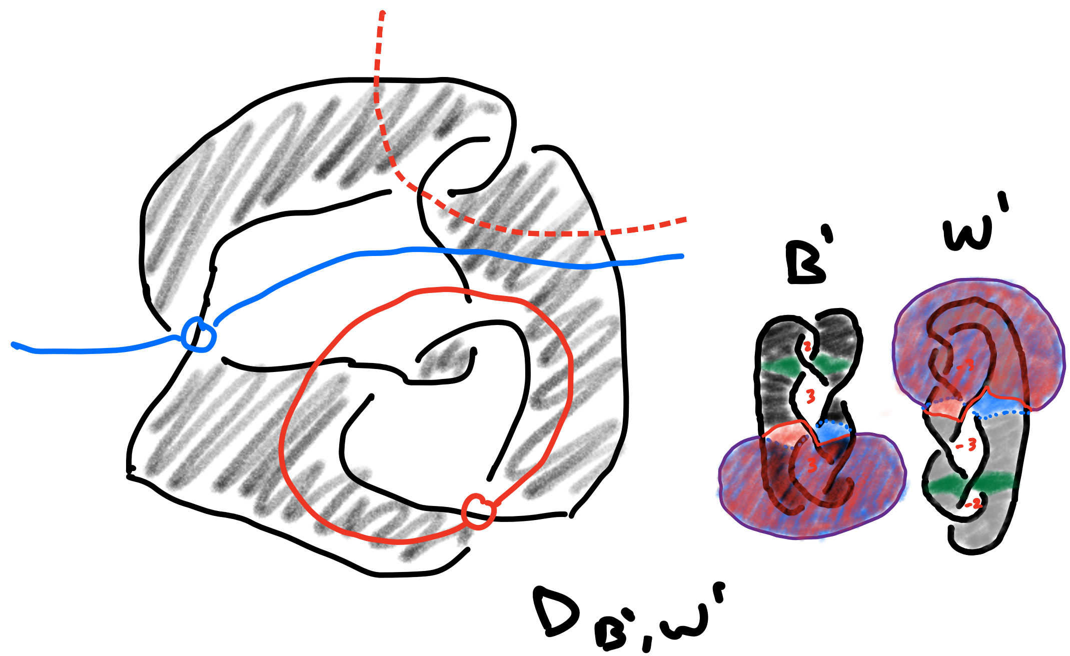
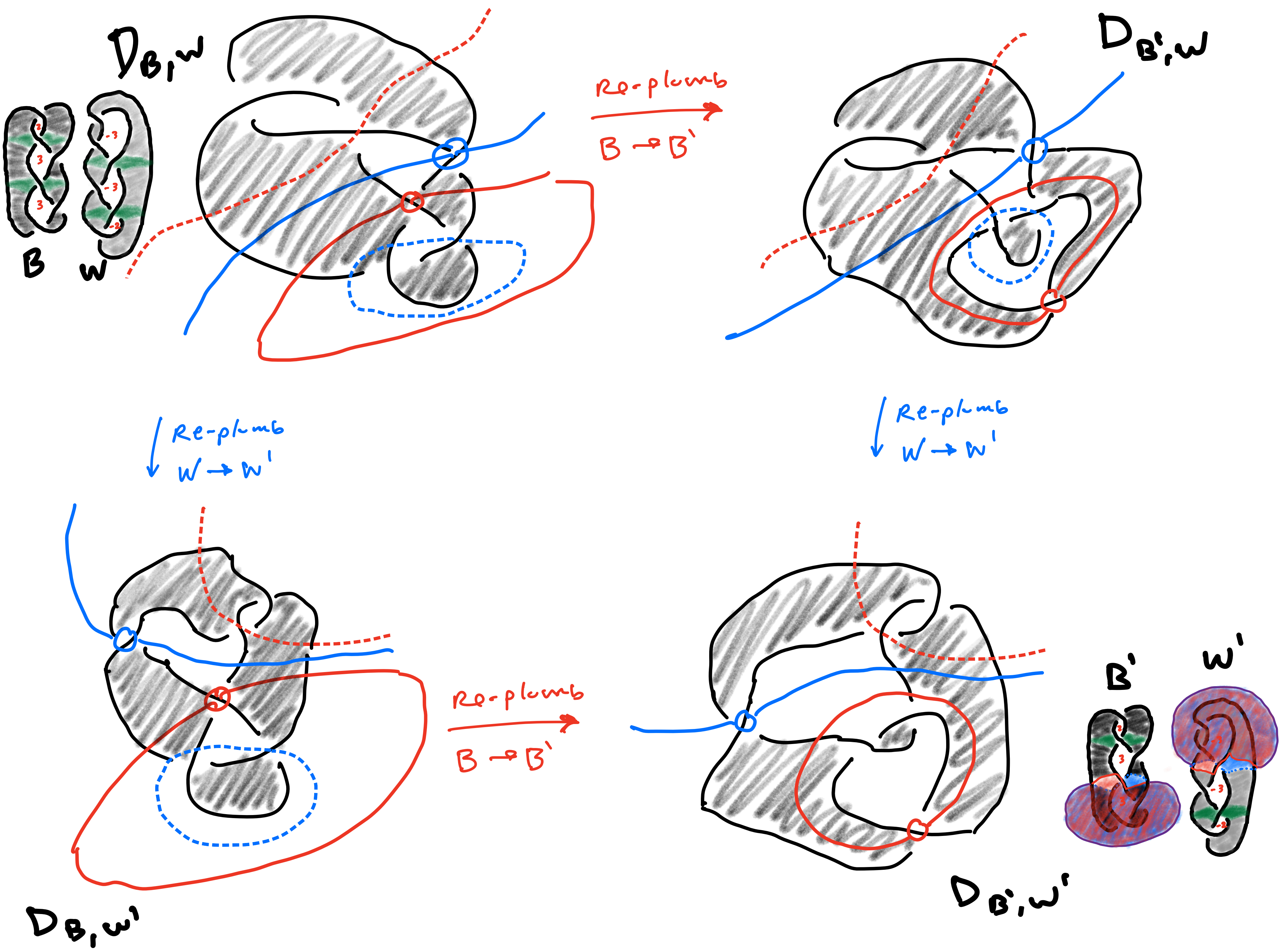
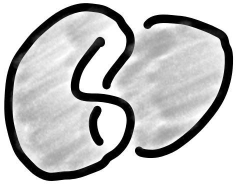

D_{B,W}
B
W


B'
W'
D_{B',W'}



$D_{B,W}$
B
W
Re-plumb
B → B'
$D_{B',W}$
Re-plumb
W → W'
Re-plumb
W → W'
Re-plumb
B → B'
$D_{B,W'}$
$D_{B',W'}$
B'
W'

NOTE: In general, pairs B, W of spanning surfaces for a given knot need not determine a diagram, let alone uniquely.

EX:  AND 

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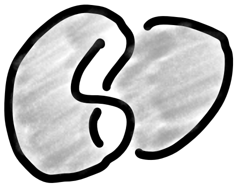
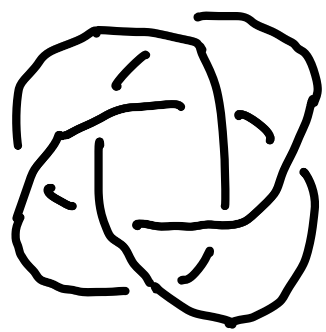
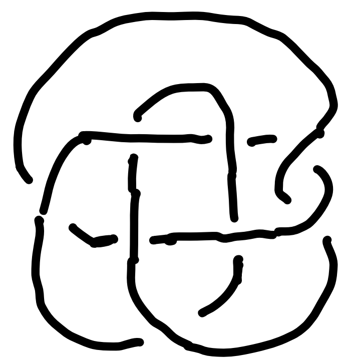
EX:  AND 

EX: TWO COPIES OF  $\rightsquigarrow$  AND 

The GOERITZ MATRIX G of a checkerboard surface F measures how much F twists:

IDEA: Circle $\gamma \subset F \rightsquigarrow$ Vector $\vec{x} \rightsquigarrow \vec{x}^T \cdot G \cdot \vec{x} = |\gamma|$

EX:

F: 

$$G: \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

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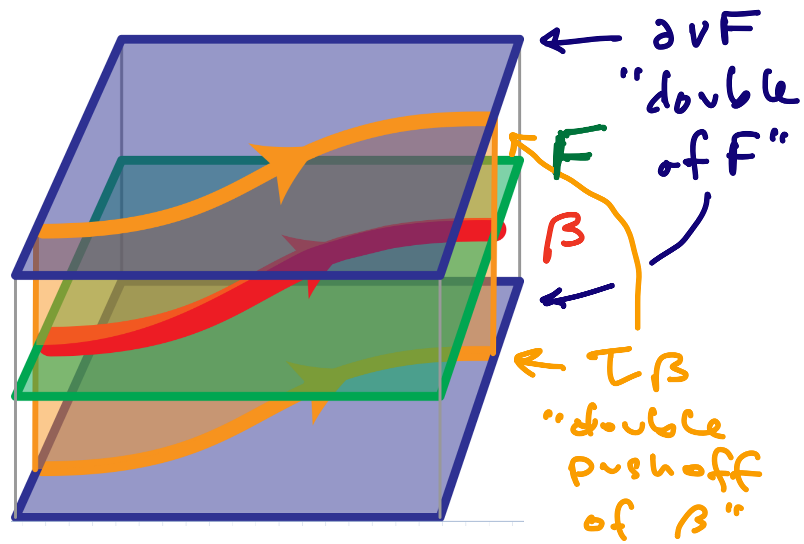
G REPRESENTS THE GORDON-LITHERLAND PAIRING

$$\langle \cdot , \cdot \rangle : H_1(F) \otimes H_1(F) \to \mathbb{Z}$$

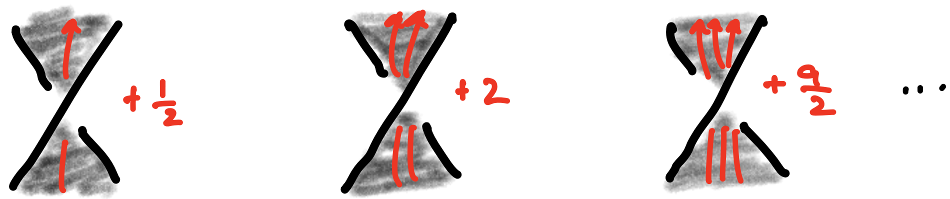
$$\langle \alpha, \beta \rangle = lk(\alpha, \tau\beta)$$



DEFN: F is $\pm$ definite if its GL pairing is $\pm$ definite.

THEOREM (GREENE '17): A knot $K \subset S^3$ is alternating if and only if it has positive & negative definite spanning surfaces $F_\pm$. In that case, K has an alternating diagram whose checkerboard surfaces are isotopic rel boundary to $F_\pm$.

PARTIAL INTUITION:



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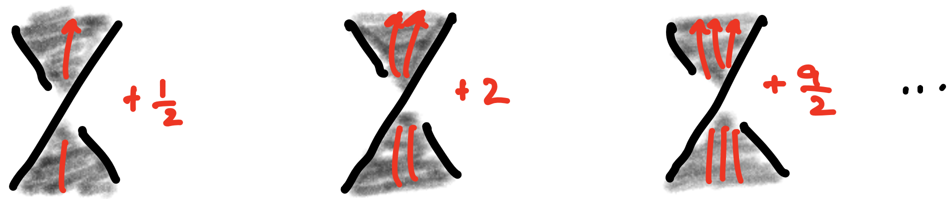

SURPRISING OBSERVATION: THE (G-L PAIRING ON THE) DOUBLE OF A CHECKERBOARD SURFACE FOR A REDUCED ALTERNATING DIAGRAM NEED NOT BE DEFINITE...

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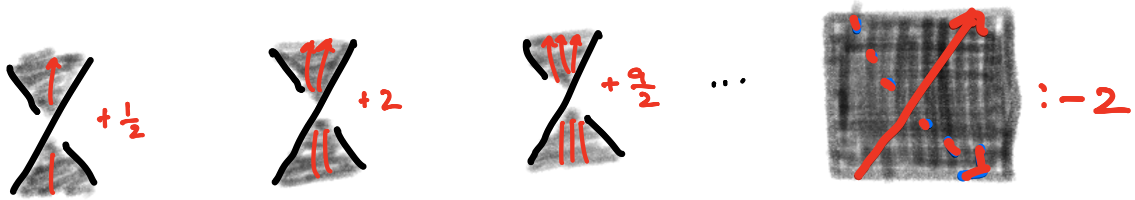
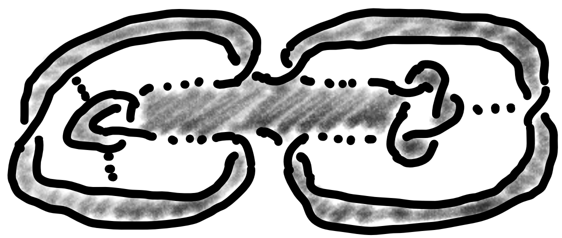
- 

- 

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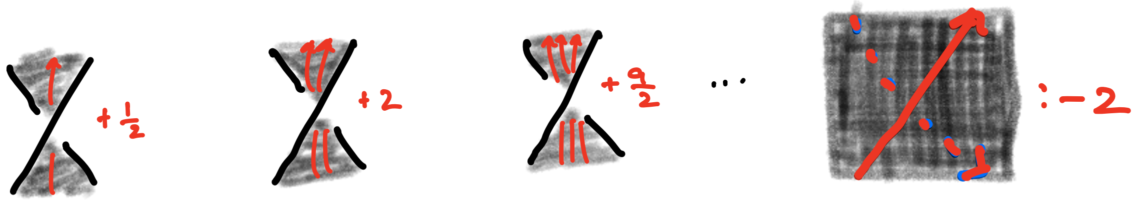
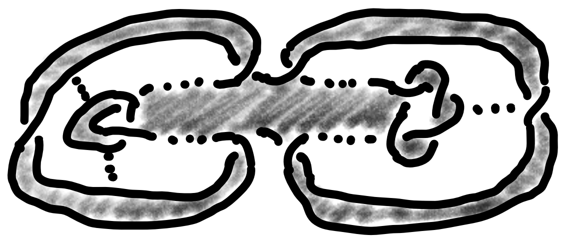
- 

- 
- SIMPLER EXAMPLES (JOINT W/ H. HOWARDS, F. MOORE, J. TOLBERT*)

OPEN QUESTIONS: CAN THE DOUBLE OF A CHECKERBOARD SURFACE CONTAIN AN ESSENTIAL UNKNOTTED SIMPLE CLOSED CURVE W/ zero framing? WITH NEGATIVE FRAMING?

GOERITZ MATRIX v. GORDON-LITHERLAND PAIRING

TRALDI:

TWO KNOTS HAVE DIAGRAMS w/ THE SAME GOERITZ MATRICES

$\longleftrightarrow$ THE KNOTS ARE MUTANTS.

* ANY MUTATION-INVARIANT KNOT INVARIANT IS DETERMINED BY GOERITZ MATRICES.

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K: All checkerboard surfaces are related by kinking/unkinking moves:

Hence, all "G-L MATRICES" FOR ALL SUCH SURFACES FOR A GIVEN KNOT ARE RELATED BY

$G \longleftrightarrow P^T G P$ (P UNIMODULAR) ; $G \longleftrightarrow \begin{bmatrix} G & 0 \\ 0 & \pm 1 \end{bmatrix}$

(change of basis) ($\pm$ (un)kinking)

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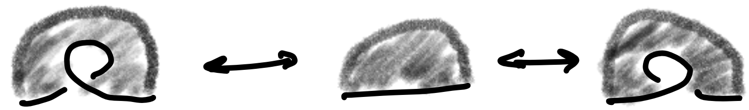

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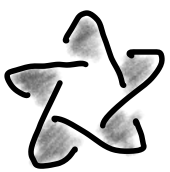
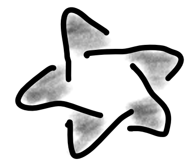
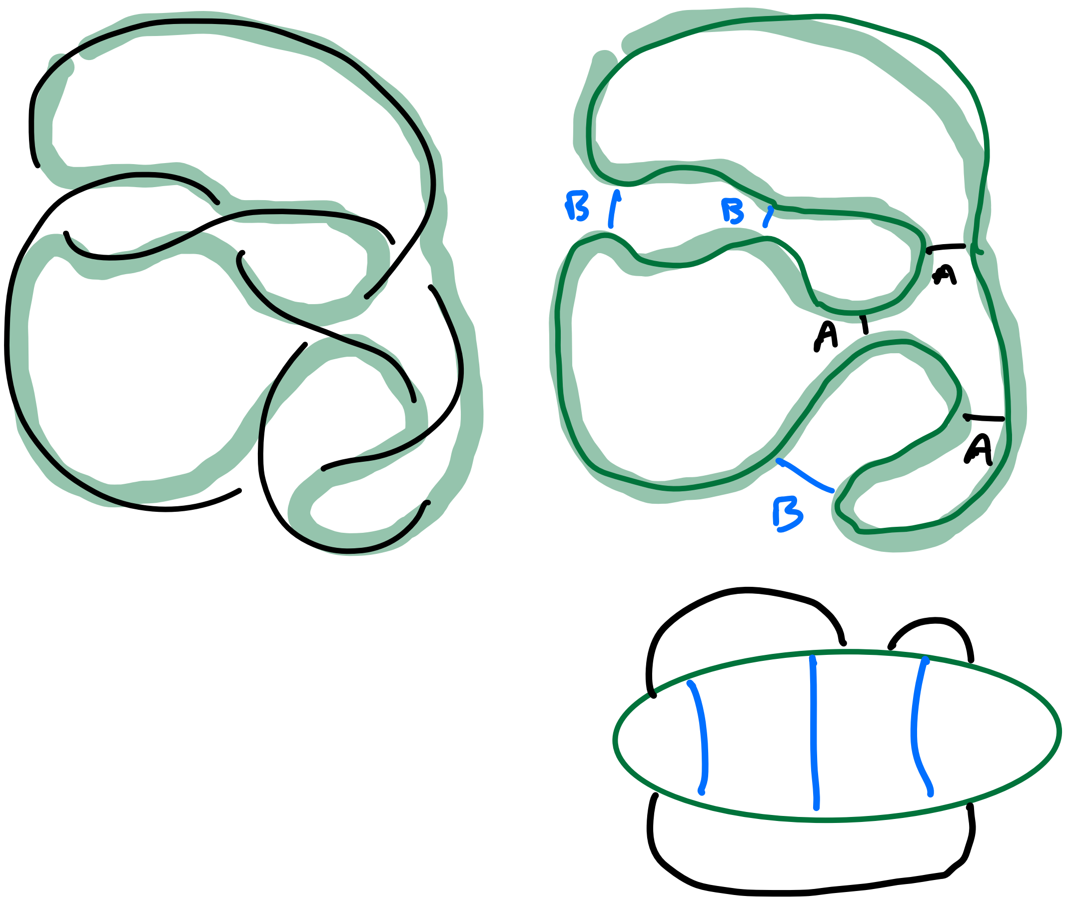
Ex  $\longrightarrow [5] \longrightarrow \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \xrightarrow[\text{change of basis}]{} \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow [-5] \longrightarrow$ 

PROBLEM: FIND OBSTRUCTIONS FOR MOVES ON G-L MATRICES VIS-A-VIS MOVES ON GOERITZ MATRICES

INSIGHT: EVERY KNOT DIAGRAM HAS A KAUFFMAN STATE WITH A SINGLE STATE CIRCLE:

EX:



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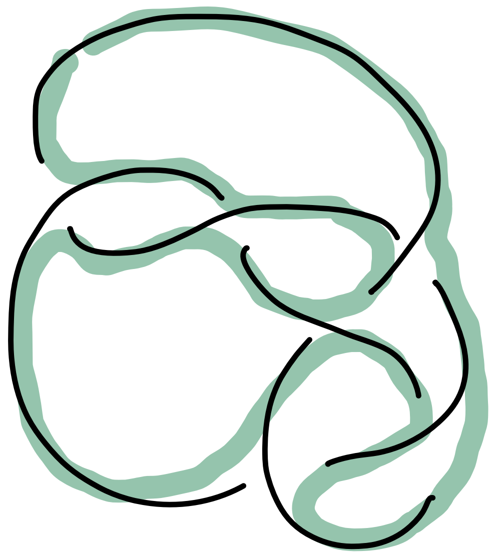
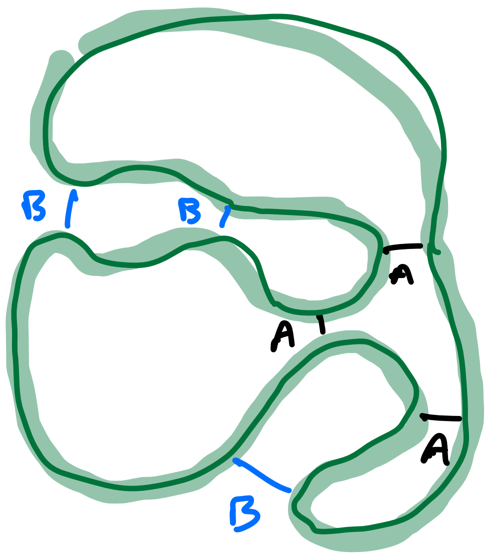
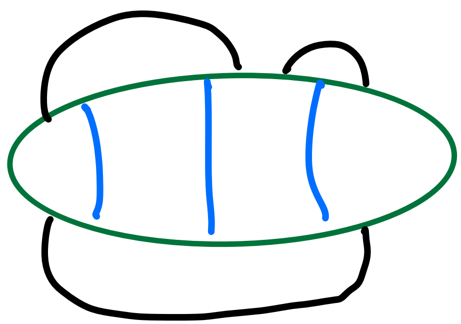




$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

OBS: THE SIZE OF THIS MATRIX PROVIDES A LOWER BOUND FOR CROSSING #...



$$\left[\begin{array}{c|c} \begin{matrix} I & 0 \\ 0 & -I \end{matrix} & A \\ \hline A^T & \begin{matrix} I & 0 \\ 0 & -I \end{matrix} \end{array}\right]$$

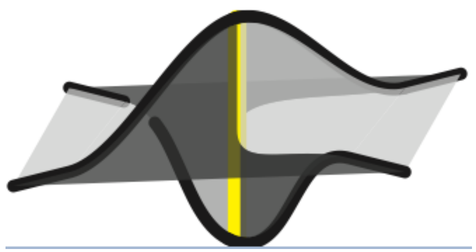
NOTE: THE STATE SURFACE F_x IS A PLUMBING $F_x = F_1 * F_2$ WHERE

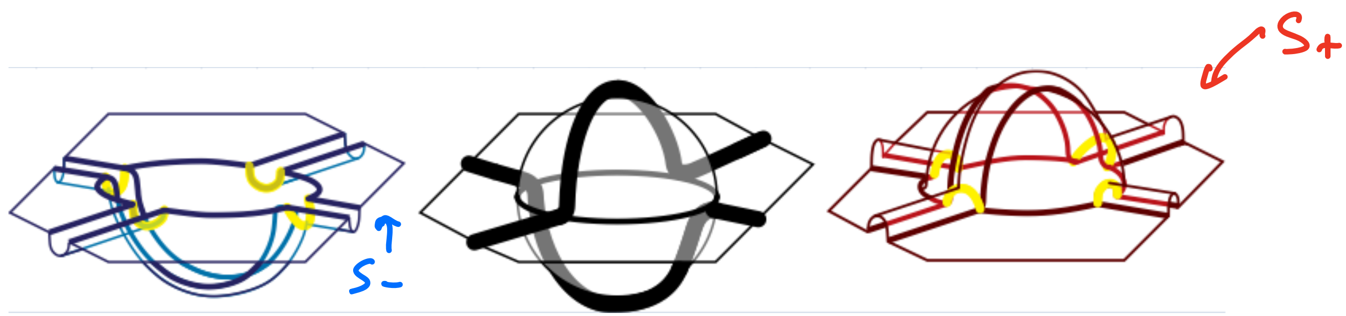
both F_i ARE $\natural$'s of 's AND 

POSSIBLE APPLICATION TO "TAIT 1"?

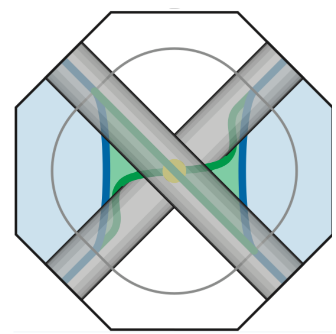
BACK TO THE GEOMETRIC PROOF:

- Given diagrams $D_{B,W}$ and $D_{B',W'}$ consider how B' sits relative to B, W





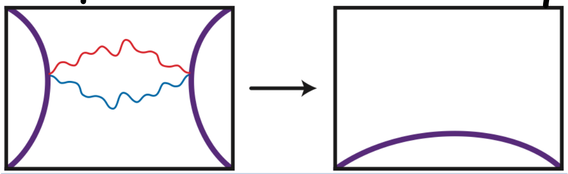
- Insist that near any crossing band, B' lies <u>BELOW</u> $B \cup W$:



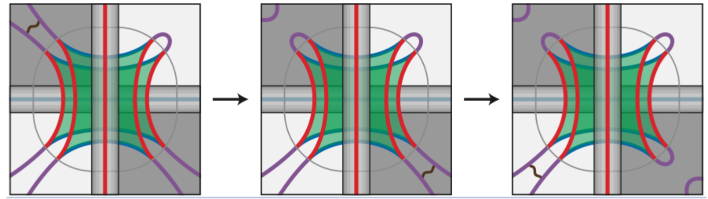
- Now, circles of $B' \cap S_+$ will suggest helpful simplifying moves.

Hierarchy of isotopy moves 1-9:

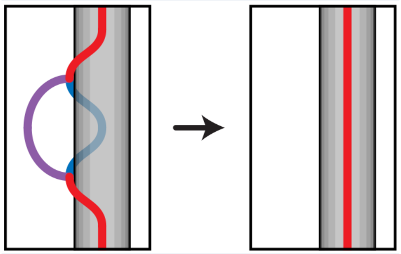
Move 1:



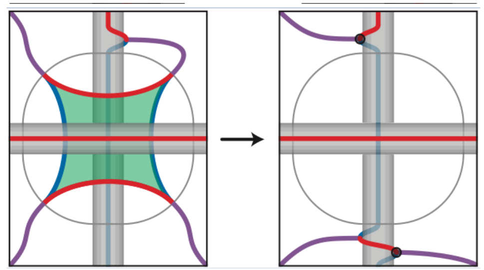
Move 2:



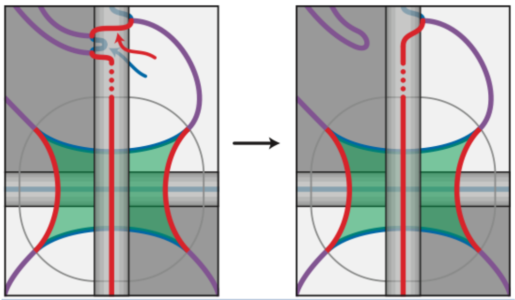
Move 3:



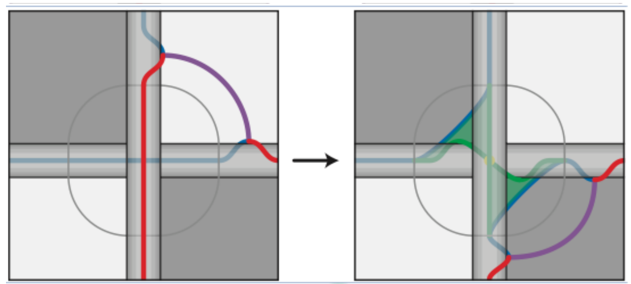
Move 4:

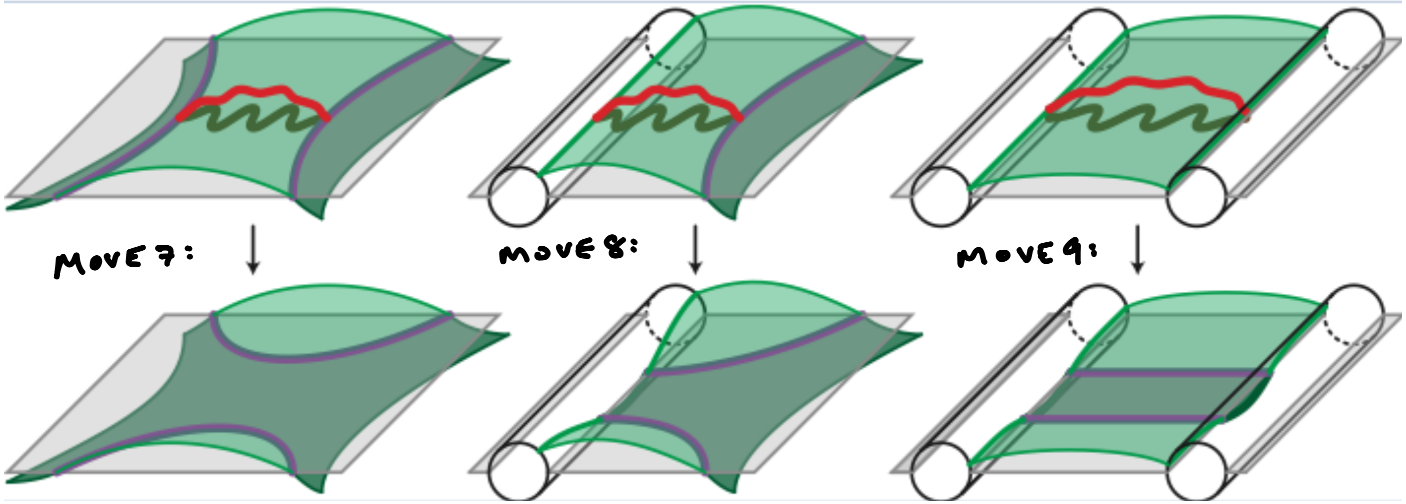


Move 5:



Move 6:





Move 7: Move 8: Move 9:

Upshot: After exhausting moves 1-9, it is possible to perform a certain re-plumbing (move 10). After exhausting moves 1-10, B' is isotopic to B.

UTILIZING THE HIERARCHY:

1. Define a notion of complexity.
   Use it to argue that sequences terminate.

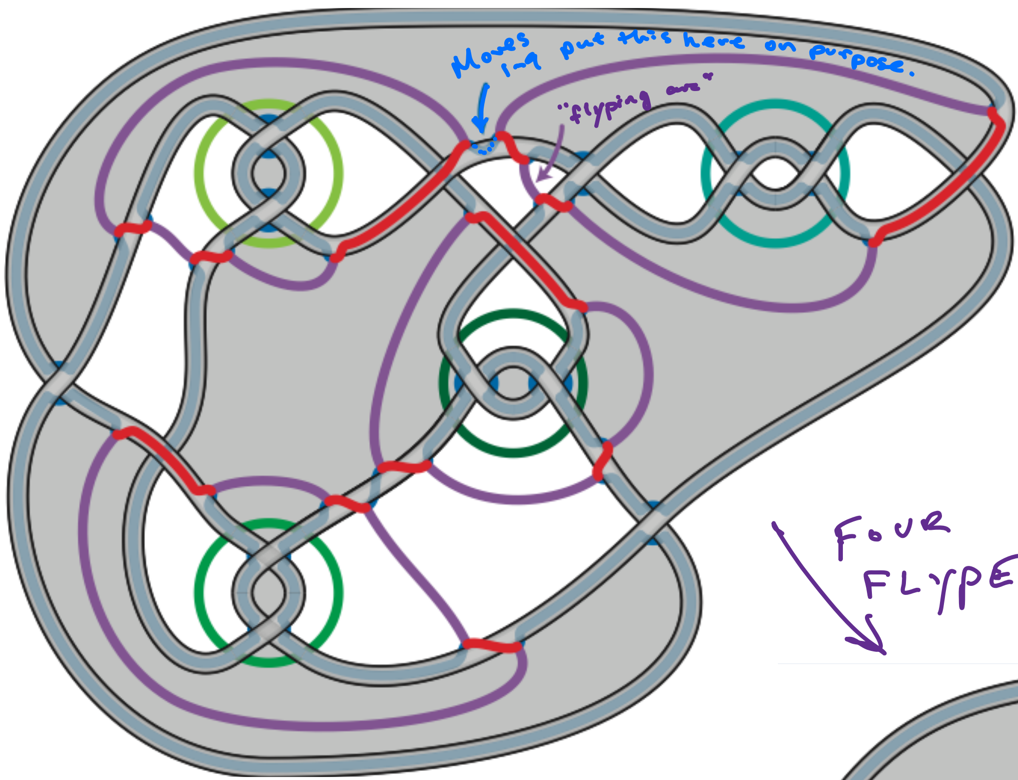
2. Argue that the resulting picture is nice.

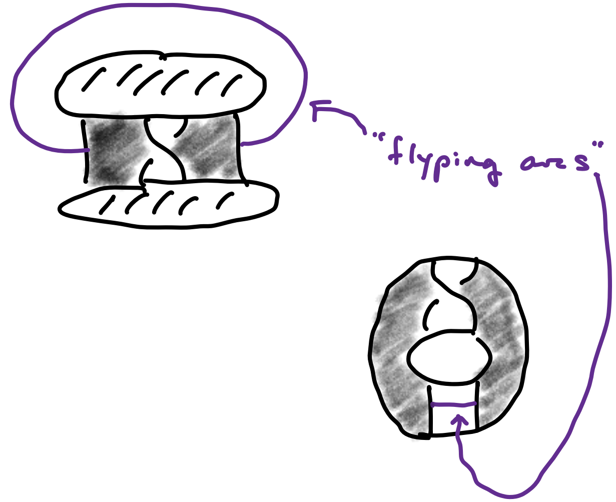
3. Press the rewind button. What happened last?

<u>THM</u>: AFTER ONE HAS EXHAUSTED ISOTOPY MOVES 1-9, EVERY CIRCLE OF $B' \cap S_+$ IS a "flyping circle"

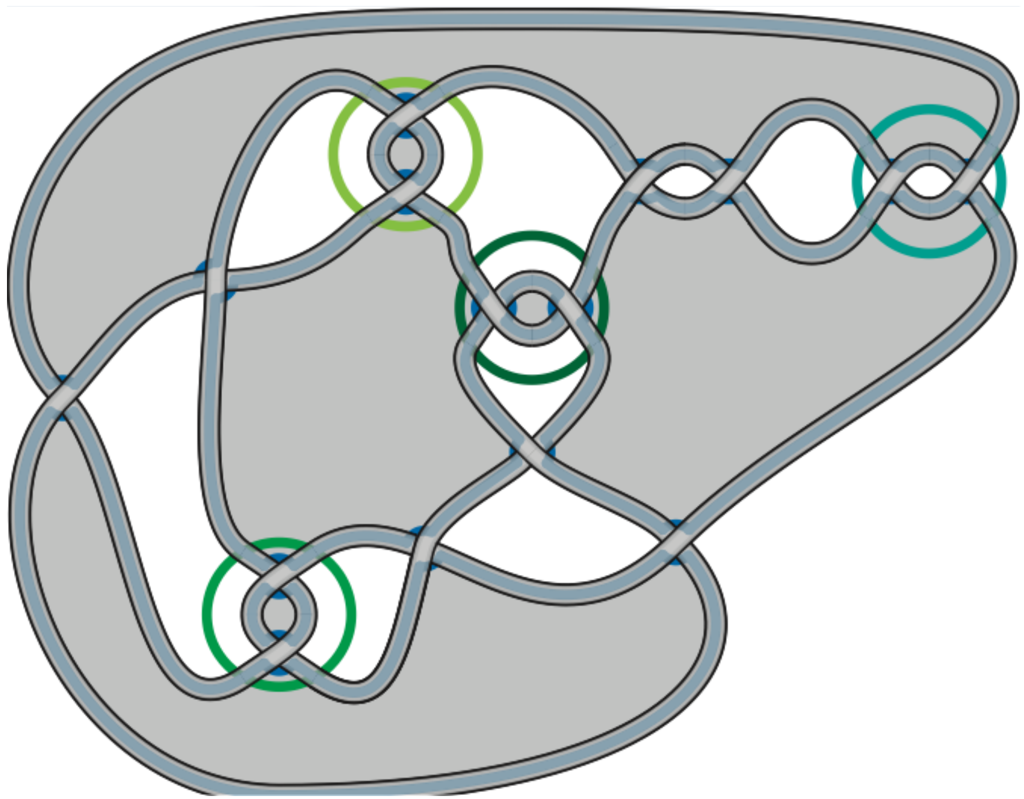
* Now we can read the entire flyping sequence $D_{B,W} \to D_{B',W}$ straight off of $B' \cap S_+$. (THEN DO THE SAME FOR W' TO FLYPE $D_{B',W} \to D_{B',W'}$.

Ex:





Four flypes.



Each circle of $B' \cap S_+$ here (purple-red) is a flyping circle.

AND EVERYTHING WORKS FOR VIRTUAL KNOTS TOO...

IF ONE IS CAREFUL:

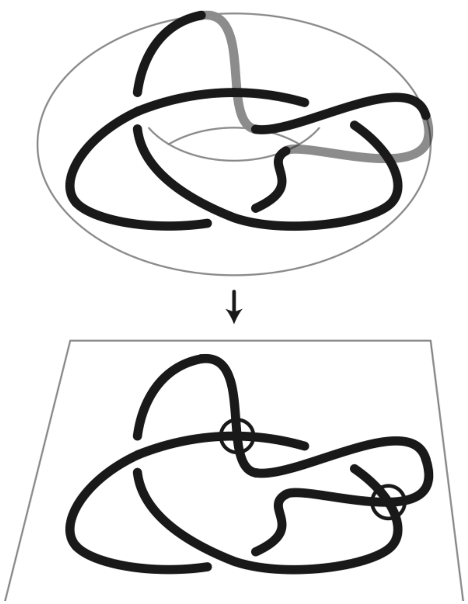
Q: IS THIS VIRTUAL KNOT PRIME?

AND EVERYTHING WORKS FOR VIRTUAL KNOTS TOO...

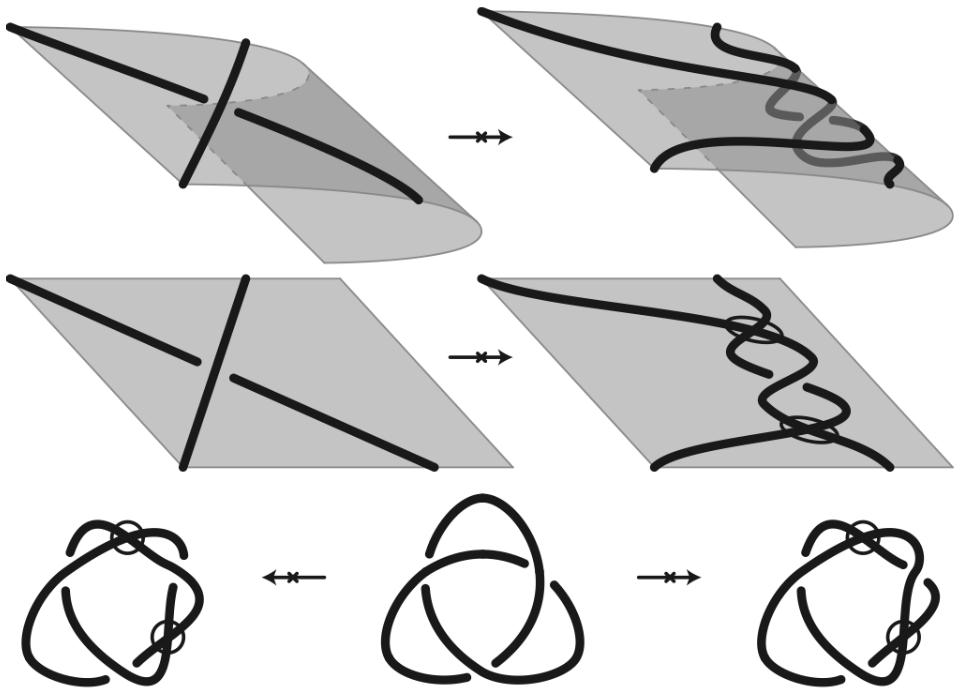
IF ONE IS CAREFUL:

Q: IS THIS VIRTUAL KNOT PRIME?

REGARDING THIS CORRESPONDENCE:



THERE'S A CAVEAT:



THANK YOU!

HAPPY BIRTHDAY LOU!