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Research Statement

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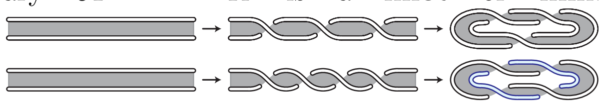
1. Introduction

If you put several half-twists in a thin strip of paper and tape the ends together, you get a surface F whose boundary $\partial F = K$ is a **knot** or **link**:



More generally, given any knot or link K in 3-space, there is an embedded surface F with $\partial F = K$, and in fact there are many such **spanning surfaces** for K. I am a knot theorist who approaches many problems through the lens of spanning surfaces. Next, I will describe why knot theory is useful mathematics and then how my perspective connects me to, and distinguishes me from, other mathematicians and their interests. Then, §2 will introduce some central ideas in my research by describing two Master's theses that I recently co-advised and two undergraduate theses that I am currently advising and co-advising. Finally, §§3–5 will describe some of my other papers, and §6 will offer concluding remarks.

1.1. **Why knot theory?** Two main reasons: (1) radical human accessibility, and (2) surprisingly broad mathematical and scientific significance. On a human level (1), knot theory is uniquely welcoming for at least four reasons. First, the mathematical barrier to entry is low: lots of open problems can be stated in a way that anyone who was taken a proofs course can grasp, and many techniques are similarly accessible. Second, there are lots of flavors to choose from: some techniques involve spatial reasoning, while others are purely diagrammatic, or highly combinatorial, or heavily algebraic. Third, many problems invite computational approaches, inviting empiricists and programmers. Fourth, knot theory is… fun. Regarding (2):

- Geometric topology is both richest and most applicable to the physical world in dimensions three and four, where knot theory is foundational (via handle structures and surgery).
- While the frameworks of general relativity and quantum mechanics remain unreconciled, their mathematical structures—hyperbolic geometry, which is projective relativistic geometry, and operator algebras—have deep and still-mysterious connections via the Jones polynomial, a knot theory gadget that also has deep generalizations (webs and Khovanov homology) which carry information about representation theory and properly embedded surfaces in the 4-ball. It remains a profound mystery as to why any of this math should be related. What does the Jones polynomial *mean*?
- Knot theory has applications to cell biology, synthetic chemistry, and particle physics.

1.2. **What distinguishes my research?** To me "coloring in" a knot with a spanning surface tends both to simplify matters (less knotting, just a little twisting) and to enrich them (every knot has *many* spanning surfaces). This perspective has led to insights along a few avenues:

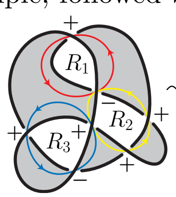
(1) Problems "internal" to the broadly construed topic of spanning surfaces—see [K1, K2, K4, K5, K13, K15];
(2) Fresh takes on foundational theorems and key gadgets—see §§2.4,2.5 and [K3, K8, K9, K10, K11, K12];
(3) Purely linear algebraic problems motivated by spanning surfaces–see §2.2 and [K14]; and
(4) Adaptation to dimension four–see §6.

I also enjoy research problems unrelated to knots, especially ones inviting visual/spatial or combinatorial approaches. So far, I have only one paper like this [K7], but I have specific plans for more–see §6.

2. Student research projects advised

2.1. **Knots, diagrams, surfaces, and matrices.** Every knot in 3-space has spanning surfaces, lots of them actually. Given a knot $K \subset \mathbb{R}^3$, take a generic projection of K to $\mathbb{R}^2$, and record over-under information at the self-intersections, or *crossings*, to get a **diagram** D of K. It is always possible to color the regions of $\mathbb{R}^2 - D$ light and dark in "checkerboard fashion" (see below), so that like-shaded regions abut only at crossings. If we do this so that the unbounded region is light, then there is a spanning surface F for K that lies almost entirely in the dark regions of $\mathbb{R}^2 - D$, except near crossings, where it twists; F is called a **checkerboard surface**.

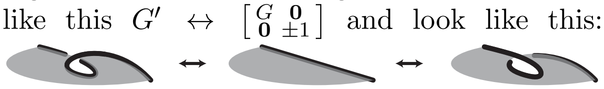
The first homology group of F, denoted $H_1(F)$, is a free abelian group whose elements are represented by oriented circles embedded in F. It has a basis $\mathcal{A} = (a_1, \dots, a_n)$ represented by circles α_i, each of which goes counterclockwise around one bounded light region R_i, and so its rank, $\beta_1(F) = n$, counts "how many holes" are in F. There is a symmetric bilinear pairing $\langle \cdot, \cdot \rangle : H_1(F) \times H_1(F) \to \mathbb{Z}$, first described by Gordon and Litherland [7],[1] which is represented (with respect to $\mathcal{A}$) by a **Goeritz matrix** $G = (g_{ij})$,[2] where each g_{ii} counts (with sign) the number of crossings incident to R_i and each g_{ij} counts (with opposite sign) the number of crossings incident to both R_i and R_j. Here is an example, followed by two key properties:



$$\rightsquigarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

- If $\alpha \subset F$ is a circle representing a homology class a, then $2\langle a, a\rangle$ is the *framing* of α in F, which measures how much F "twists" along α.
- Since $G \in \mathbb{Z}^{n\times n}$ represents a *bilinear* mapping, the change-of-basis formula is $G \to PGP^T$, where $P \in \mathbb{Z}^{n\times n}$ is unimodular, ie $\det P = \pm 1$.

In a chapter in the Concise Encyclopedia of Knot Theory [K6], I proved that any two checkerboard surfaces F_1 and F_2 from any diagrams of a given knot are related by a sequence of isotopy and "kinking" moves, which change Goeritz matrices like this $G' \leftrightarrow \left[\begin{smallmatrix} G & \mathbf{0} \\ \mathbf{0} & \pm 1 \end{smallmatrix}\right]$ and look like this:



This raises the following practical question: given F_1 and F_2, how to find such a sequence of moves between them? Maybe, I thought, a good approach would be to forget the surfaces altogether and just focus on the linear algebra: if G_1 and G_2 are the Goeritz matrices for F_1 and F_2, then any sequence of kinking moves between F_1 and F_2 gives a sequence of moves like $G \leftrightarrow PGP^T$ and $G \leftrightarrow \left[\begin{smallmatrix} G & \mathbf{0} \\ \mathbf{0} & \pm 1 \end{smallmatrix}\right]$ from G_1 to G_2. Call two symmetric integer matrices **kink-equivalent** if they are related by such a sequence. What can we say about this notion?

2.2. Kink-equivalence of matrices, surfaces, 4-manifolds, and quadratic forms.

Two springs ago, I posed this question to a student, John, in my topology topics course as we walked together from our classroom back to the math building. The question quickly grew into a Master's thesis project, which I co-advised with Hugh Howards and Frank Moore. We soon discovered that not every move like $\left[\begin{smallmatrix} G & \mathbf{0} \\ \mathbf{0} & \pm 1 \end{smallmatrix}\right] \to G$ can be realized geometrically, ie there are "fake unkinking moves." Stepping away from topology into pure linear algebra, John asked:

Question 2.1. Is every positive-definite $G_+ \in \mathbb{Z}^{n\times n}$ kink-equivalent to a negative-definite G_-?

(Were it not for fake unkinking moves, there would be no chance of this, because of a theorem from 2017 about definite Goeritz matrices and alternating knots [8].) John proved that if there is an integer matrix C (not necessarily square) such that $G = I + CC^T$, then G is kink-equivalent to the negative-definite matrix $-I - C^TC$. But does such C always exist? In fact, no. John discovered a 6×6 counterexample, and for a while, we held out this matrix as a likely candidate for a negative answer to Question 2.1. Then, as John was getting ready to finish his thesis, he announced that he had figured out how to transform his 6×6 matrix into a negative-definite matrix, and that his method worked in general. This gives the following theorem, which has implications for 4-manifolds, quadratic forms, and indeed anywhere one finds symmetric integer pairings [K14]:

Theorem 2.2. *Any nonsingular symmetric $G \in \mathbb{Z}^{n\times n}$ is kink-equivalent to $\pm$-definite matrices.*

Theorem 2.3. *Every simply connected, closed, topological 4-manifold with nonsingular intersection form has a positive blow-up homeomorphic to a negative blow-up of a positive-definite, simply connected, closed, topological 4-manifold.*

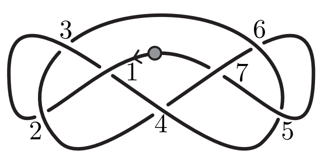
[1] This is my all-time favorite paper, partly because of its content, but especially because they prove their results twice: first, they take a sophisticated approach, involving double-branched covers of the 4-ball (in some sense, this is the best way to understand the deep significance of their work), and then, restarting from scratch, they take a down-to-earth approach that uses only elementary tools and thus makes this profound paper remarkably accessible.

[2] Given $x = \sum_{i=1}^n x_i a_i$ and $y = \sum_{i=1}^n y_i a_i$ in $H_1(F)$ and denoting $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, we have $\langle x, y\rangle = \vec{x}^T G \vec{y}$.

Theorem 2.4. *Let $q : \mathbb{Z}^n \to \mathbb{Q}$ be a nonsingular quadratic form with n_+ positive eigenvalues and n_- negative eigenvalues, and write $q_0 : x \mapsto x^2$. There are $\pm$-definite quadratic forms $q_\pm$ that satisfy these unimodular congruences: $q \oplus (q_0)^{5n_-} \cong q_+ \oplus (-q_0)^{n_-}$ and $q \oplus (-q_0)^{5n_+} \cong q_- \oplus (q_0)^{n_+}$.*

The overall approach of this project—translating an insight about spanning surfaces into an inquiry about matrices and then pursuing the linear algebra—offers a promising blueprint for much more research to come. There is ample intellectual space here for undergraduates to pursue natural questions about the interplay between linear algebra, knots, and surfaces—and perhaps, like John, discover something with unanticipated consequences.

2.3. **Gauss codes and crosscap numbers.** The first knot theorist, Gauss, discovered a way of recording a knot diagram as a tuple of integers. Here is what you do. Pick a basepoint and orient the knot; then walk along the knot, recording each new crossing with the next unused integer and each repeated crossing with the corresponding integer.[3] For example, using the indicated basepoint and orientation, the Gauss code of the following knot diagram is $(1, 2, 3, 1, 4, 5, 6, 3, 2, 4, 7, 6, 5, 7)$.



Gauss codes offer computational approaches to lots of problems in knot theory: if you know how to do something diagrammatically, can you figure out how to do it using only the tuple? If so, and if you know a little Python, you can write a computer program to compute thousands of examples for you. For example, given an alternating diagram D of a knot K, Colin Adams and I figured out how to compute the crosscap number of K from D (this is the smallest $\beta_1(F)$ achieved by any nonorientable spanning surface F for K). During my first postdoc, having no prior experience with Python, I wrote a program that computed the crosscap numbers of all alternating knots through 13 crossings (of which there are several thousand) [K5]. The approach I used for that project was distinct from my approach with Colin Adams, although it relied on our results. This new approach worked for knots but not for links.

I am currently co-advising (with Jason Parsley) an undergraduate thesis project about Gauss codes for alternating knots and links, in which one of our main goals is to compute the crosscap numbers of all alternating knots and links through at least 14 crossings. I met our student, Isaias, when he took my knot theory course last semester. This year, he is applying to graduate programs in computer science.

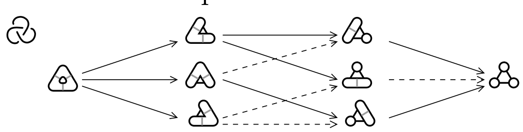
2.4. **An application to statistical physics.** I met my second undergraduate thesis student this year, Tristan, when she took my discrete math/intro to proofs course last fall. Tristan is working on a problem that comes out of an application of knot theory to statistical physics. The Ising model uses statistics of quantum mechanics to capture phase changes or spin behaviors in lattices of molecules. The model features a *partition function* which takes as input a planar graph G whose edges are labeled with signs and maps it to a Gaussian integer $P(G)$. The same planar graph G also encodes—although not quite uniquely—a diagram D of a knot or link L. One would like $P(G)$ to be an invariant of L, in the sense that if two graphs G and G' both encode diagrams of L then $P(G) = P(G')$, but this is not quite the case. The first research question is this: Can we choose a correction factor $f(G)$, so that $f(G) \cdot P(G)$ *is* an invariant of L? Indeed, we will find that we can, but when we do, we will find that it is easier to define $f(G)$ using G *and* D. Yet, G does not encode D uniquely, so buried within this topic are several levels for inquiry and discovery. I am excited to see where Tristan takes this project.

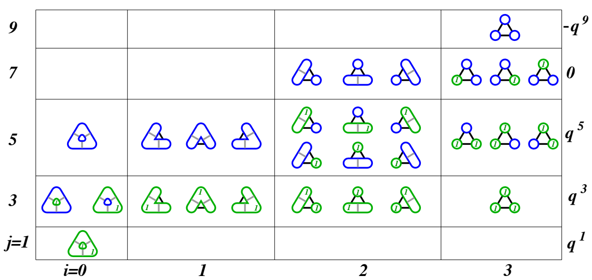
2.5. **Essential states in Khovanov homology.** My first two years at Wake, Hugh Howards and I advised a Master's thesis about Khovanov homology. In general, for a Master's or undergraduate thesis, I would recommend almost any other area among my research interests above this topic above Khovanov homology. But I do have a couple specific projects in mind, and so I included the topic in a list of four or five possibilities when Hugh and I first met with our new advisee, Emma. It turns out that Emma's best friend growing up was the daughter of a mathematician specializing in Khovanov homology, and she had a burning desire to know what

[3] In general, one uses the over/under information at the crossings to attach signs to these integers. For alternating diagrams, however, this is unnecessary, subject to the convention that the first crossing is an over-crossing.

this was all about. So we went for it! We spent the first year learning about Khovanov homology works; then we got to a research question.

Earlier, I mentioned the Jones polynomial, but so far I have not said anything about what it is, or how it is computed. Here is one perspective. Given a diagram D of a knot K, "smooth" each crossing in one of two ways,)($\xleftarrow{A}$ ✕ $\xrightarrow{B}$ ⌣⌢. The resulting diagram x is called a (Kauffman) **state** of D and consists of *state circles* joined by A- and B-labeled arcs, one from each crossing. "Enhance" x by assigning each state circle a binary label: ◯ $\xleftarrow{1}$ ◯ $\xrightarrow{0}$ ◯. Then use the number of A- versus B-smoothings leading to each state and the number of blue versus green circles in each enhancement to assign each "enhanced state" X a bigrading (i_X, j_X), and use this bigrading (and the writhe of the diagram) to arrange the enhanced states in a grid. Here is an example:





For each pair (i, j), let $C_{i,j}(D)$ be the free $\mathbb{Z}$-module generated by those enhanced states X with $(i_X, j_X) = (i, j)$. Then the (unnormalized) Jones polynomial of K is

$$V_K(q) = \sum_j q^j \sum_i (-1)^i \mathrm{rank}(C_{i,j}(D)).$$

For example, the Jones polynomial of the knot in the preceding diagram is $q + q^3 + q^5 - q^9$. Amazingly, $V_K(q)$ does not depend on D. Even more amazingly, Khovanov described "(co)-boundary maps" $d : C_{i,j}(D) \to C_{i+1,j}(D)$ such that the resulting (co)homology groups also are independent of D [11, 21]. All of this works over any commutative ring R with $1 \neq 0$; I will focus on $R = \mathbb{Z}$ and $R = \mathbb{Z}/2\mathbb{Z}$.

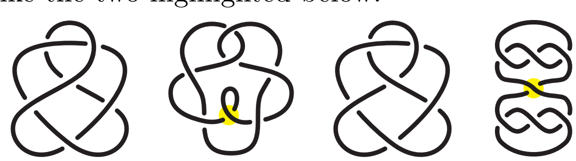
When I first learned about Khovanov homology, I worked a few simple examples, all coming from alternating diagrams, and noticed that many of the nonzero Khovanov homology classes were represented by states that described essential surfaces (in a sense that I will explain in §5.1). In some cases, these essential surfaces were checkerboard surfaces, and in others they were obtained by plumbing checkerboard surfaces. Moreover, any "diagrammatic plumbing" of alternating checkerboard states gives what is called a **homogeneously adequate state**, and these always describe essential surfaces [16]. I proved that such states *always* correspond to nonzero Khovanov homology classes, at least over $\mathbb{Z}/2\mathbb{Z}$ [K3]:

Theorem 2.5. *If x is a homogeneously adequate state, then x gives nonzero Khovanov homology classes over $\mathbb{Z}/2\mathbb{Z}$ in two gradings. If a certain graph G_{x_A} is bipartite*[4]*, then this is also true with integer coefficients.*

The research question for Emma was what happens to such essential states in Khovanov homology when we change the underlying diagram by adding or removing a crossing via a simple kink (called an R1 move). I suspected that such a simple move would preserve all the important features of Khovanov homology classes and their representatives. Surprisingly, though, Emma discovered that sometimes adding a crossing via an R1 move could make things *nicer*. For example, in a 3-crossing diagram of the trefoil, there is a Khovanov homology class that doesn't have a "nice" representative (coming from *a single* state), but after certain R1 moves, this homology class has a nice representative. Emma is now in her second year in the math Ph.D. program at the University of Kentucky.

3. Tait's conjectures, geometrically

In general, it is hard to determine the crossing number of an arbitrary knot, or to determine whether or not a given diagram minimizes crossings, but certainly no diagram that minimizes crossings can have a "nugatory" crossing, like the two highlighted below:



A diagram without nugatory crossings is called *reduced*, whether or not it minimizes crossings. Interestingly, if a knot K has a diagram D that is reduced *and* alternating, then D *always* minimizes crossings: $c(D) = c(K)$. Although this fact was first observed empirically by P.G. Tait

[4] This corresponds to the A-smoothed part of x being a Seifert state.

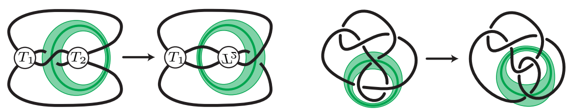
in 1898 [20], it and two related conjectures remained unproven for almost a century:

Tait's conjectures. *Given two reduced alternating diagrams D and D' of a knot $K \subset S^3$:*

(1) D and D' have the same number of crossings, which is minimal: $c(D) = c(D') = c(K)$.

(2) D and D' have equal writhe:[5] *$w(D) = w(D')$.*

(3) If D and D' are prime,[6] *then they are related by a sequence of flype moves:*

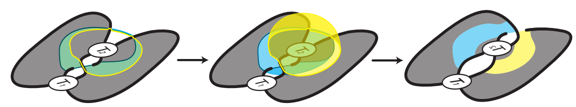


Tait's conjectures remained open until the discovery of the Jones polynomial in the mid-1980's [10], which almost immediately led to three independent proofs of Tait's first conjecture Jones polynomial of a knot K: the degree span, or *breadth*, of $V_K(t)$, denoted $\text{bth}(V_K(t))$, always provides a lower bound for crossing number, $c(K) \leq \text{bth}(V_K(t))$, and any reduced alternating diagram D of K satisfies $c(D) = \text{bth}(V_K(t))$;

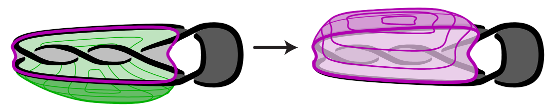
Within a decade, the Jones polynomial had led to proofs of all three of Tait's conjectures. Yet, these proofs were only somewhat satisfying: why did they work? What was the Jones polynomial really measuring? In their 1993 proof of Tait's flyping conjecture, Menasco and Thistlethwaite spotlighted this remaining gap in our understanding: "the question remains open as to whether there exist purely geometric proofs of this and other results that have been obtained with the help of new polynomial invariants." The aim is not just to know, but to understand. I love that the mathematical community values this.

The first partial answer came in 2017 from Greene in a paper where he answered another longstanding question, this one from Ralph Fox: "What [geometrically] is an alternating knot?" First, Greene observed that, if B and W are the checkerboard surfaces from an alternating knot diagram and G_B and G_W are their Goeritz matrices, then G_B is positive-definite and G_W is negative-definite, or vice-versa: "B and W are definite and of opposite signs."[7] Second, Greene proved that alternating knots are the only ones with a two such surfaces. In fact, he proved that if F_+ and F_- are definite spanning surfaces of opposite signs for the same knot K, then K has an alternating diagram whose checkerboard surfaces are "the same as" (are isotopic to) F_+ and F_-. Greene was then able to translate Tait's conjectures into *non-diagrammatic* statements and give the first "purely geometric" proofs of Tait's second conjecture and part of his first: any reduced alternating diagrams D and D' of the same knot satisfy $c(D) = c(D')$ and $w(D) = w(D')$ [8].[8]

Recently, I gave the first purely geometric proof of Tait's "flyping" conjecture [K9]. The first key insight was that flyping a diagram D changes one of its checkerboard surfaces by isotopy and the other, F, by a geometric operation I call *re-plumbing*,



which replaces a disk $U \subset F$ (shown green, left) with another disk V (shown half yellow and half blue, center) that is disjoint from F except along its boundary $\partial V = \partial U$. Here is another replumbing move:



My insight about flyping and re-plumbing translates Tait's diagrammatic conjecture to a geometric statement about spanning surfaces. Then the real work begins. That story is too long for right now though.

Around the time I proved Tait's flyping conjecture, Boden-Karimi extended Greene's insights about definite surfaces in S^3 to the context of thickened surfaces, like $S^1 \times S^1 \times [-1, 1]$ [1]. I used their innovations to extend Tait's flyping conjecture to that context. Many of the arguments adapted directly, but a few technical points required special attention. Some of these subtleties led to fundamental revelations about *virtual knots*, which correspond to knots

[5] Orient D arbitrarily. Then every crossing looks like ⤫ or ⤬. The *writhe* of D is $w(D) = |⤫| - |⤬|$, where bars count components. It is independent of the orientation on D.

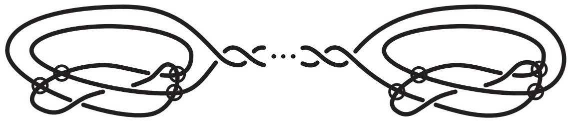
[6] A knot diagram D is *prime* if, for every circle $\gamma \subset S^2$ that intersects D generically in two points, all crossings of D lie on the same side of γ.

[7] $G_B \in \mathbb{Z}^{n \times n}$ is positive-definite if and only if $\vec{x}^T G_B \vec{x} > 0$ for every nonzero $\vec{x} \in \mathbb{Z}^n$.

[8] It remains an open problem to give a purely geometric proof of Tait's full first conjecture, since Greene's insights are less useful regarding the possibilities of non-alternating diagrams of an alternating knot. I have some ideas....

in thickened surfaces. This led to two further spin-off projects.

In one paper [K10], I explained how there are two common, but different, notions of primeness for virtual knots, and likewise for their diagrams; I introduce a new tool called "lassos" and proved how to use it to tell by inspection whether or not the link it represents is prime in either sense. It turns out that the virtual flyping theorem holds for alternating virtual knot diagrams that are prime in the weaker sense, whether or not they are prime in the stronger sense, and as a corollary, it follows that, given any two non-classical alternating virtual knots, there are infinitely many distinct ways to take their connect sum:



The other paper [K11] considers a notion of "essential" surfaces that is more restrictive usual. The initial motivation had to do with implementing a familiar classical technique (Menasco's crossing bubbles) in a thickened surface rather than in S^3, but the results might also prove useful in the study of Seifert solids for knotted surfaces in 4-space, among other possible applications.

4. Genus and crosscap number of an alternating knot

4.1. Seifert matrix and Alexander polynomial of a 2-sided surface. While every spanning surface F has its Gordon-Litherland pairing, when F is a (2-sided) *Seifert surface*, there is another pairing $H_1(F) \times H_1(F) \to \mathbb{Z}$ that carries more information. This *Seifert pairing* is represented by a *Seifert matrix* A, where $A+A^T$ is a Goeritz matrix for F, but with something extra: the polynomial $\Delta_K(t) = \det(A - tA^T)$, called the **Alexander polynomial** of F, depends (up to degree shift) only on K. Moreover, the degree span or *breadth* $\text{bth}(\Delta_K(t))$, which depends only on K, provides a lower bound for $g(K)$:

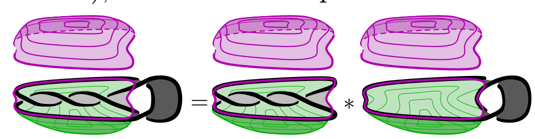
$$\text{bth}(\Delta_K(t)) \leq g(K). \tag{1}$$

4.2. The Crowell–Murasugi theorem. A classical result proven independently in 1958-59 by Crowell and Murasugi asserts conditions under which equality holds in (1):

Theorem 4.1 (Crowell–Murasugi [2, 14])**.** *If F is a surface constructed via Seifert's algorithm from an alternating diagram D of a knot K, then*

$$g(F) = g(K) = \frac{1}{2} bth(K).$$

In part, the Crowell–Murasugi theorem asserts that while, in general a Seifert surface resulting from Seifert's algorithm *need not* have minimal genus, when Seifert's algorithm is applied to an *alternating* diagram, the resulting surface F *always* has minimal genus: $g(F) = g(K)$. Gabai gave a short, elegant, purely geometric, proof of this fact in 1986 [5]. The Crowell–Murasugi theorem, however, had something extra about the Alexander polynomial. When I taught a topics course at UNL in 2021, I wanted to understand, and then share, *why* this *full result* was true. This led me to discover a new, short, extremely satisfying proof of the Crowell-Murasugi theorem [K8]. It involves several of my favorite characters: checkerboard surfaces, linear algebra, and *plumbing* or *Murasugi sum*–a way of gluing two spanning surfaces F_0 and F_1 along a disk U to obtain another spanning surface $F = F_0 * F_1$ (there is one extra condition); here is an example:



Here is a synopsis of my proof. First, to prove that $g(F) = g(K) = \frac{1}{2}\text{bth}(\Delta_K(t))$, I showed that it suffices to prove that the Seifert matrix A from F is invertible. Second, I showed that this is true if F happens to be a checkerboard surface (from an alternating diagram). Third, it turns out that the surfaces obtained via Seifert's algorithm from alternating diagrams are always plumbings of alternating checkerboard surfaces, so it suffices to prove that if F is a plumbing of surfaces F_1 and F_2 which have invertible Seifert matrices A_1 and A_2, then F also has an invertible Seifert matrix. Finally, I showed that, indeed, F has a Seifert matrix of the form $A = \begin{bmatrix} A_1 & 0 \\ B & A_2 \end{bmatrix}$; applying the pigeonhole principle to the formula $\det[a_{ij}]_{i=1}^n = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)}$ thus confirms that $\det(A) = \det(A_1) \cdot \det(A_2) \neq 0$.

4.3. Unoriented genus of an alternating knot. My first research project began in 2005, at Williams' REU, SMALL, where Colin Adams and I generalized Seifert's algorithm in order to construct many spanning surfaces from a given knot diagram. At the time, the construction was new; the surfaces we constructed have since become known as **state surfaces**. We solved the *geography* problem in the alternating case: given a knot K, the problem is to list all pairs $(s(F), \beta_1(F))$ realized by spanning surfaces F for K. The geography problem remains open for most classes of knots and is a fertile area for future research. As an immediate corollary, our classification gives the *unoriented genus* of any alternating knot [K1]. Everything we did also works for alternating links.

4.4. **Average crosscap number of a 2-bridge knot.** A 2-bridge knot is a knot that can be drawn with just two local maxima and two local minima. There are rich correspondences, via continued fractions, between these knots and $\mathbb{Q} \cap (-1, 1) \setminus \{0\}$. For example, the Stevedore (6_1) knot is associated to the rational number $\frac{5}{9}$, which can be represented as a continued fraction in several ways, including

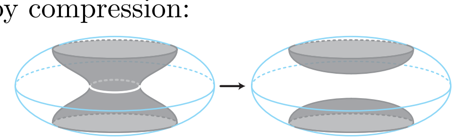
$$\frac{5}{9} = \frac{1}{1 + \frac{1}{1+\frac{1}{4}}}, \quad \frac{5}{9} = \frac{1}{2 - \frac{1}{5}}, \text{ and } \frac{5}{9} = 1 + \frac{1}{-2 - \frac{1}{4}}.$$

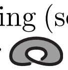
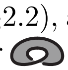
The first (additive) representation describes K via a braid diagram, while the second and third (subtractive) ones describe 1- and 2-sided surfaces that K bounds. Something similar happens with each 2-bridge knot. My four coauthors and I used these three different types of continued fraction representations to obtain a formula for the average crosscap number of all 2-bridge knots with a given crossing number c.

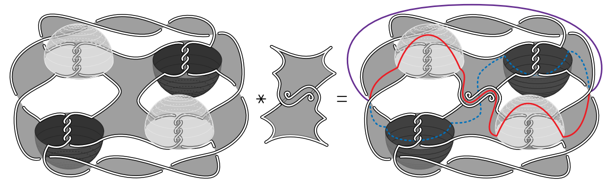
5. Other papers

Here are synopses of my those publications and preprints that I have not yet mentioned, roughly in reverse chronological order:

5.1. **How essential is a spanning surface?** [K13]**.** A spanning surface F for a knot $K \subset S^3$ is *geometrically essential* if it cannot be simplified by compression:



nor by kinking (see §2.2), and F is π_1-*essential* if F is neither  nor  and the inclusion map of the interior of F into the knot complement $S^3 - K$ induces an injective map on fundamental groups. Essential surfaces are fundamental to our understanding of the topology and geometry of 3-manifolds, but the two notions of "essential" were rarely mentioned in the 20$^{\text{th}}$ century, largely because they are equivalent for *orientable* spanning surfaces. For example, Gabai proved in 1983 that any plumbing of essential Seifert surfaces is essential, no qualifier needed [4]. In 2011, however, when Ozawa extended Gabai's result to all spanning surfaces, regardless of orientability, he needed to specify that his surfaces were π_1-essential [16]. The question remained open whether plumbing likewise respects the property of being *geometrically* essential. My first main result answers this question in the negative via this example:



The only hard part of the proof is showing that the surface on the left is indeed geometrically essential; I develop a new technique to do this. The second main result extends Ozawa's result in another direction. I introduce a quantity that measures "how essential" a spanning surface F is. I call it the *essence* of F, denoted $\text{ess}(F)$; F is π_1-essential if and only if $\text{ess}(F) \geq 2$. I extend Ozawa's theorem by proving that if F is a plumbing of π_1-essential surfaces F_1 and F_2, then $\text{ess}(F) \geq \min_{i=1,2} \text{ess}(F_i)$.

5.2. **Efficient multisections of odd-dimensional tori** [K7]**.** This is my one paper, so far, that has nothing to do with knots. Its main result is a construction regarding piecewise-linear, or PL, manifolds, but most of the hard work in the paper is combinatorial. In arbitrary odd dimension $n = 2\ell - 1$, I described a symmetric, efficient **multisection** of the n-torus $T^n = S^1 \times \cdots \times S^1$ (n copies): this is a decomposition into ℓ simple pieces X_i, $i \in \mathbb{Z}/\ell\mathbb{Z}$ with the following intersection property: for each $I \subset \mathbb{Z}_\ell$, the dimension and complexity of $X_I = \bigcap_{i \in I} X_i$ depend only on $|I|$ (namely, $\dim(X_I) = n + 1 - |I|$, and X_I can be built from a ball of that dimension d by gluing on "h-handles" $B^h \times B^{d-h}$ along $S^{h-1} \times B^{d-h}$ for $1 \leq h \leq |I|$).[9] In particular, each X_i is an n-dimensional 1-handlebody, homeomorphic to a "thickened up" wedge of some number, g, of circles;[10] g is called the *genus* of X_i.

[9]Notation: $B^h = \{\vec{x} \in \mathbb{R}^h : |\vec{x}| \leq 1\}$ and $S^{h-1} = \partial B^h = \{\vec{x} \in \mathbb{R}^h : |\vec{x}| = 1\}$.

[10]A wedge of g circles is the space obtained from a disjoint union of g circles by choosing one point on each circle and identifying the chosen points to a single point.

Rubinstein and Tillmann had introduced multisections shortly before this [19], as a natural generalization of an important construction in 3-, and more recently 4-dimensions [6]. They proved that every PL manifold of arbitrary (finite) dimension has a multisection. Yet, their construction tends to produce handlebodies of very large genus, even for simple manifolds. By contrast, the multisections that I constructed for T^n were *efficient* in the sense that each X_i has genus n, which I proved is minimal. Each multisection is also *symmetric* with respect to both the permutation action of S_n on the indices and the $\mathbb{Z}_\ell$ translation action along the main diagonal. I also constructed a related trisection of T^4, lifted all symmetric multisections of tori to certain cubulated manifolds, and obtained two combinatorial identities as corollaries.[11]

5.3. **Alternating links have representativity 2.** [K4]**.** Gromov defined the **distortion** $\delta(\gamma)$ of a rectifiable curve $\gamma \subset \mathbb{R}^3$ by considering how many times farther apart two points are along the curve than in space:

$$\delta(\gamma) = \sup_{p,q\in\gamma} \frac{d_\gamma(p,q)}{d_{\mathbb{R}^3}(p,q)}$$

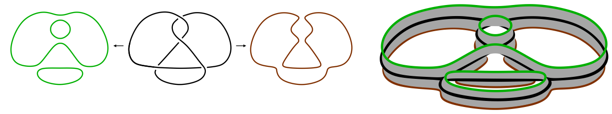
Gromov asked whether knots have arbitrarily large distortion [9]. To show that they do, Pardon established a lower bound for distortion, $160\delta(L) \geq r(L)$, in terms of what is now called the **representativity** of L [18]:

$$r(L) = \max_{F\in\mathcal{F}_L} \min_{X\in\mathcal{X}_F} |\partial X \cap L|.$$

Here, $\mathcal{F}_L$ is the set of positive genus closed surfaces containing L, and $\mathcal{X}_F$ is the set of compressing disks for F. Ozawa computed the representativity of certain pretzel links and all torus and 2-bridge links, and conjectured that alternating links have representativity 2 [17]. I used Menasco's crossing ball structures to confirm Ozawa's conjecture:

Theorem 5.1. *If $L \subset S^3$ is an alternating link and F is a closed surface that contains L (without crossings), then F has a compressing disk whose boundary intersects F in at most two points.*

5.4. **Heegaard diagrams corresponding to Turaev surfaces (with Cody Armond and Nathan Druivenga)** [K2]**.** One builds a **Turaev surface** from a link diagram D on $S^2 \subset S^3$ by pushing the all-A and all-B states of D to opposite sides of S^2, joining them with a cobordism whose saddle points occur precisely at the crossings of D, and capping off each state circle with a disk:



Dasbach, Futer, Kalfagianni, Lee, and Stolzfus showed that the resulting surface Σ is a Heegaard surface for S^3, on which D forms an alternating diagram [3]. Our main theorem proved a converse to this fact, providing a correspondence between Turaev surfaces and certain link-adapted Heegaard diagrams [K2].

6. Conclusion

If you would consider taking a closer look at one or more of my research papers, I recommend [K14], [K8], or [K9]. The first is of broadest interest, the second is most readable, and the third has garnered the most interest.

I have plans for many research projects involving spanning surfaces which I have not described in this statement. Most of these plans would ideally involve students at either the undergraduate or graduate level. I would be delighted to describe any of these specifically if I am so fortunate as to be invited to interview with you, but here is a rough overview of some of my near-term research plans:

Some plans involve translating insights and questions about surfaces into purely linearly algebraic problems, like the approach I took with John and my co-advisors in [K14]. Some involve translating diagrammatic questions into questions about spanning surfaces, like the approach I took in [K9]. Some, like [K9, K8, K13], involve Murasugi sums of spanning surfaces. Some, like my current project with Tristan, apply knot theory to the natural and physical sciences. Some, like [K5] and my current project with Isaias, are computational. Some, like [K3, K12, K10, K11], involve Khovanov homology or virtual knots.

[11] For example, for any $n = 2\ell - 1$, we have

$$\ell^{n-1} = \sum_{i_0=2}^{n} \binom{n}{i_0} \sum_{i_2=4-i_0}^{n-i_0} \binom{n-i_0}{i_1} \sum_{i_3=6-i_0-i_1}^{n-i_0-i_1} \binom{n-i_0-i_1}{i_2} \cdots \sum_{i_{\ell-2}=2\ell-2-\sum_{j=0}^{\ell-3} i_j}^{n-\sum_{j=0}^{\ell-3} i_j} \binom{n-\sum_{j=0}^{\ell-3} i_j}{i_{\ell-1}},$$

which also equals the number of spanning trees of the complete bipartite graph $K_{\ell,\ell}$ [15].

One will require techniques from computational geometry. I would like, in the next few years, to teach a course in this subject and then advise a student on this problem. Hopefully, this experience would open doors to further research in computational geometry, which would not necessarily have anything to do with knots.

All of the plans I just described, however, pertain to knots and surfaces in three dimensions, but my most important aspiration as a researcher is to continue to broaden my mathematical reach. As noted above, computational geometry and applications to the sciences are two of the areas into which I plan to extend my research. I want to conclude by discussing one more such area: higher dimensions, especially dimension four.

After I finished [K7], several questions have lingered with me about the distinctions between the PL and smooth categories. In dimensions four and below, the smooth and PL categories are equivalent, but Milnor's exotic 7-spheres reveal that "by" dimension seven they are distinct. Are they distinct in dimension six? Five? In particular, both PL structures and smooth structures give rise to handle decompositions, for which the PL and smooth types are distinguished by the nature of their attaching maps. It might be interesting to find examples PL handle decompositions that are not smoothable and use them to develop obstructions to smoothability, and then use these obstructions to examine characterize (specifically) how the PL and smooth categories do or do not differ in, say, dimensions five through seven.

Whereas the biggest problem in 3-dimensional topology was solved two decades ago, the biggest problems in 4-dimensional topology—in the PL/smooth category—remain wide open: e.g. the (smooth 4-dimensional) Poincare conjecture, the Schoenflies problem (does every smooth 3-sphere in S^4 bound a smoothly embedded 4-ball?), the slice-ribbon conjecture, does any nontrivially knotted surface $S \subset S^4$ have $\pi_1(S^4 \setminus S) \cong \mathbb{Z}$?

This past summer, after six years of trying, I finally connected with a great 4-manifolds research project. What makes it great, first and foremost, is my collaborators: Maggie Miller, Seungwon Kim, Patrick Naylor, and Homayun Karimi. They all have greater 4-manifolds expertise than I do, but in our meetings they generously encourage my naive questions. The second factor that makes this project a great fit for me is what I bring to it: I suggested the research question—can we adapt Murasugi sum (plumbing) to an operation on spanning solids for knotted surfaces in 4-space?—and my naive questions often lead useful places, because many phenomena, facts, and arguments in dimension 3 translate to dimension 4, but with a caveat or two per sentence. Our whole project aims, in a nutshell, to take my usual research perspective and bump it up a dimension.

A constant concern for the project, then, is to stay sufficiently narrow in our inquiry. To this point, my vantage point has allowed me to anticipate the risk of our inquiry sprawling and to suggest specific delimitations for our conversation. For example, one of our aims is to use our adapted notion of Murasugi sum to construct many new examples of knotted surfaces. This would add value to the field because most knotted surfaces to date have been constructed by "deformed" versions of a certain spinning construction, and this lack of diversity reduces the usefulness of empirical approaches. Yet, "many new examples" is an amorphous goal, so I suggested a refinement: adapt the classical notion of "arborescent" knots to knotted surfaces. This narrowing of the problem has helped focus our exploration. I hope that it also helps us draw a circle around a cohesive set of ideas for a first paper, sooner rather than later, because the overall ambitions of our exploration are vast.

I have found that almost every aspect of classical knot theory can be understood from the perspective of spanning surfaces, and this perspective has already, several times in my young career, led to fundamental insights regarding well-trodden material. My hope is that studying knotted surfaces in 4-space from the perspective of their spanning solids might prove similarly fruitful. In any case, it seems like a promising long-term strategy in my research program.

My main strategy, though, is much more basic. Stay curious. Ask questions, especially naive ones. Listen closely and connect with other humans. Pursue interesting questions, knowing that there are two good ways to recognize them—one comes from the inside and is borne from experience, while the other is more fundamental: a question is interesting if someone else thinks about it deeply and finds it captivating.

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