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Research Statement

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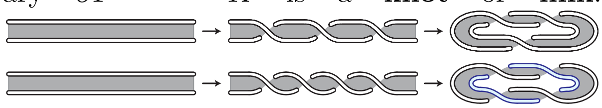
1. Introduction

If you put several half-twists in a thin strip of paper and tape the ends together, you get a surface F whose boundary $\partial F = K$ is a **knot** or **link**:



More generally, given any knot or link K in 3-space, there is an embedded surface F with $\partial F = K$, called a **spanning surface** for K. I am a knot theorist who approaches many problems through the lens of spanning surfaces. To me, "coloring in" a knot with a spanning surface tends both to simplify matters (less knotting, just a little twisting) and to enrich them (every knot has *many* spanning surfaces). My approach gives me fresh takes on topics across knot theory, and this change of perspective often leads to significant new insights [K3, K8, K10, K11, K9, K12]. There are also plenty of interesting and accessible research questions one can ask about (broadly construed) spanning surfaces themselves [K1, K2, K4, K5, K13, K15]. Recently, in work with a Master's student, one such inquiry motivated purely linear algebraic questions which ended up yielding insights regarding 4-manifolds and quadratic forms [K14]. I am currently engaged in joint work which seeks to adapt my approach to dimension four–see §6. I also enjoy research problems unrelated to knots, especially ones inviting visual/spatial or combinatorial approaches. So far, I have only one paper like this [K7], but I have specific plans for more–see §6.

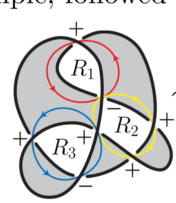
If you would consider taking a closer look at one of my research papers, I recommend [K14], which is my paper of broadest interest. It is also co-authored with a Master's advisee and two co-advisors. I describe it next because this will also allow me to introduce some central ideas in my research. After that, I will two undergraduate theses that I advised and co-advised last year, and two undergraduate/post-baccalaureate research projects I am currently advising. Then I will briefly describe just a few of my other papers and offer concluding remarks.

2. One paper that encapsulates my research

2.1. Knots, diagrams, surfaces, and matrices.

Every knot in 3-space has spanning surfaces, lots of them actually. Given a knot $K \subset \mathbb{R}^3$, take a generic projection of K to $\mathbb{R}^2$, and record over-under information at the self-intersections, or *crossings*, to get a **diagram** D of K. It is always possible to color the regions of $\mathbb{R}^2 - D$ light and dark in "checkerboard fashion" (see below), so that like-shaded regions abut only at crossings. If we do this so that the unbounded region is light, then there is a spanning surface F for K that lies almost entirely in the dark regions of $\mathbb{R}^2 - D$, except near crossings, where it twists; F is called a **checkerboard surface**.

The first homology group of F, denoted $H_1(F)$, is a free abelian group whose elements are represented by oriented circles embedded in F. It has a basis $\mathcal{A} = (a_1, \dots, a_n)$ represented by circles α_i, each of which goes counterclockwise around one bounded light region R_i, and so its rank, $\beta_1(F) = n$, counts "how many holes" are in F. There is a symmetric bilinear pairing $\langle \cdot, \cdot, \rangle : H_1(F) \times H_1(F) \to \mathbb{Z}$, first described by Gordon and Litherland [2],[1] which is represented (with respect to $\mathcal{A}$) by a **Goeritz matrix** $G = (g_{ij})$,[2] where each g_{ii} counts (with sign) the number of crossings incident to R_i and each g_{ij} counts (with opposite sign) the number of crossings incident to both R_i and R_j. Here is an example, followed by two key properties:



$$\rightsquigarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

[1]This is my all-time favorite paper, partly because of its content, but especially because they prove their results twice: first, they take a sophisticated approach, involving double-branched covers of the 4-ball (in some sense, this is the best way to understand the deep significance of their work), and then, restarting from scratch, they take a down-to-earth approach that uses only elementary tools and thus makes this profound paper remarkably accessible.

[2]Given $x = \sum_{i=1}^n x_i a_i$ and $y = \sum_{i=1}^n y_i a_i$ in $H_1(F)$ and denoting $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, we have $\langle x, y \rangle = \vec{x}^T G \vec{y}$.

- If a circle $\alpha \subset F$ represents a homology class a, then $2\langle a, a\rangle$ is the *framing* of α in F, which measures how much F "twists" along α.
- Since $G \in \mathbb{Z}^{n\times n}$ represents a *bilinear* mapping, the change-of-basis formula is $G \to PGP^T$, where $P \in \mathbb{Z}^{n\times n}$ with $\det P = \pm 1$.

In a chapter in the Concise Encyclopedia of Knot Theory [K6], I proved that any two checkerboard surfaces F_1 and F_2 from any diagrams of a given knot are related by a sequence of isotopy and "kinking" moves, which change Goeritz matrices like this $G' \leftrightarrow \left[\begin{smallmatrix} G & \mathbf{0} \\ \mathbf{0} & \pm 1\end{smallmatrix}\right]$ and look like this:

This raises the following practical question: given F_1 and F_2, how to find such a sequence of moves between them? Maybe, I thought, a good approach would be to forget the surfaces altogether and just focus on the linear algebra: if G_1 and G_2 are the Goeritz matrices for F_1 and F_2, then any sequence of kinking moves between F_1 and F_2 gives a sequence of moves like $G \leftrightarrow PGP^T$ and $G \leftrightarrow \left[\begin{smallmatrix} G & \mathbf{0} \\ \mathbf{0} & \pm 1\end{smallmatrix}\right]$ from G_1 to G_2. Call two symmetric integer matrices **kink-equivalent** if they are related by such a sequence. What can we say about this notion?

2.2. **Kink-equivalence of matrices, surfaces, 4-manifolds, and quadratic forms.** Three springs ago, I posed this question to a student, John, in my topology topics course as we walked together from our classroom back to the math building. The question quickly grew into a Master's thesis project, which I co-advised with Hugh Howards and Frank Moore. We soon discovered that not every move like $\left[\begin{smallmatrix} G & \mathbf{0} \\ \mathbf{0} & \pm 1\end{smallmatrix}\right] \to G$ can be realized geometrically, ie there are "fake unkinking moves." Stepping away from topology into pure linear algebra, John asked:

Question 2.1. Is every positive-definite $G_+ \in \mathbb{Z}^{n\times n}$ kink-equivalent to a negative-definite G_-?

(Were it not for fake unkinking moves, there would be no chance of this, because of a theorem from 2017 about definite Goeritz matrices and alternating links [3].) John proved that if there is an integer matrix C (not necessarily square) such that $G = I + CC^T$, then G is kink-equivalent to the negative-definite matrix $-I - C^TC$. But does such C always exist? In fact, no. John discovered a 6×6 counterexample, and for a while, we held out this matrix as a likely candidate for a negative answer to Question 2.1. Then, as John was getting ready to finish his thesis, he announced that he had figured out how to transform his 6×6 matrix into a negative-definite matrix, and that his method worked in general. This gives the following theorem, which has implications for 4-manifolds, quadratic forms, and indeed anywhere one finds symmetric integer pairings [K14]:

Theorem 2.2. *Any nonsingular symmetric $G \in \mathbb{Z}^{n\times n}$ is kink-equivalent to positive- and negative-definite matrices.*

Theorem 2.3. *Every simply connected, closed, topological 4-manifold with nonsingular intersection form has a positive blow-up homeomorphic to a negative blow-up of a positive-definite, simply connected, closed, topological 4-manifold.*

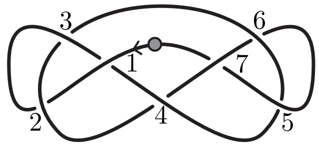
Theorem 2.4. *Let $q : \mathbb{Z}^n \to \mathbb{Q}$ be a nonsingular quadratic form with n_+ positive eigenvalues and n_- negative eigenvalues, and write $q_0 : x \mapsto x^2$. There are positive- and negative-definite quadratic forms $q_\pm$ that satisfy these unimodular congruences: $q \oplus (q_0)^{5n_-} \cong q_+ \oplus (-q_0)^{n_-}$ and $q \oplus (-q_0)^{5n_+} \cong q_- \oplus (q_0)^{n_+}$.*

The overall approach of this project—translating an insight about spanning surfaces into an inquiry about matrices and then pursuing the linear algebra—offers a promising blueprint for much more research to come. There is ample intellectual space here for undergraduates to pursue natural questions about the interplay between linear algebra, knots, and surfaces—and perhaps, like John, discover something with unanticipated consequences.

3. Other student research projects advised

3.1. **Gauss codes and crosscap numbers.** The first knot theorist, Gauss, discovered a way of recording a knot diagram as a tuple of integers. Here is what you do. Pick a basepoint and orient the knot; then walk along the knot, recording each new crossing with the next unused integer and each repeated crossing with the corresponding integer.[3] For example, using the indicated basepoint and orientation, the Gauss code of the following knot diagram is $(1, 2, 3, 1, 4, 5, 6, 3, 2, 4, 7, 6, 5, 7)$.

[3] In general, one uses the over/under information and orientation of the knot, to attach two signs to each of these integers. For alternating knot diagrams, however, this is unnecessary, provided that one does not need to distinguish between a knot and its mirror image.



Gauss codes offer computational approaches to lots of problems in knot theory: if you know how to do something diagrammatically, can you figure out how to do it using only the tuple? If so, and if you know a little Python, you can write a computer program to compute thousands of examples for you. For example, given an alternating diagram D of a knot K, Colin Adams and I figured out how to compute the crosscap number of K from D–this is the smallest $\beta_1(F)$ achieved by any nonorientable spanning surface F for K. During my first postdoc, having no prior experience with Python, I wrote a program that computed the crosscap numbers of all alternating knots through 13 crossings (of which there are several thousand) [K5]. The approach I used for that project was distinct from my approach with Colin Adams, although it relied on our results. This new approach worked for knots but not for links.

Last year, Jason Parsley and I co-advised an undergraduate thesis project about Gauss codes for alternating knots and links, in which our student, Isaias, computed the crosscap numbers of all alternating knots through 18 crossings (there are more than 10 million of them). Along the way, we developed methods that will be helpful in future projects that deal with the Jones polynomial and/or Khovanov homology. Isaias is now in graduate school for computer science, and we have adapted his senior thesis into a research paper [K19].

3.2. **An application to statistical physics.** I met my second undergraduate thesis student from last year, Tristan, when she took my discrete math/intro to proofs course the previous fall. Tristan worked on a problem that came out of an application of knot theory to statistical physics. The Ising model uses statistics of quantum mechanics to capture phase changes or spin behaviors in lattices of molecules. The model features a *partition function* which takes as input a planar graph G whose edges are labeled with signs and maps it to a Gaussian integer $P(G)$. The same planar graph G also encodes—although not quite uniquely—a diagram D of a knot or link L. One would like $P(G)$ to be an invariant of L, in the sense that if two graphs G and G' both encode diagrams of L then $P(G) = P(G')$, but this is not always the case. The first research question was this: Can we choose a correction factor $f(G)$, so that $f(G) \cdot P(G)$ *is* an invariant of L? Indeed, we found that we can, but when we did, we found that it was easier to define $f(G)$ using G *and* D. Yet, G does not encode D uniquely, so buried within this topic were several levels for inquiry and discovery. Tristan and I are currently working on adapting her thesis into a research paper [K18].

3.3. **Essential states in Khovanov homology.** My first two years at Wake, Hugh Howards and I advised a Master's thesis about Khovanov homology. In general, for a Master's or undergraduate thesis, I would recommend almost any other area among my research interests above this topic above Khovanov homology. This is because there is more prerequisite material to learn, and it is more abstract and technical. But I do have a couple specific projects in mind, and so I included the topic in a list of four or five possibilities when Hugh and I first met with our new advisee, Emma. It turns out that Emma's best friend growing up was the daughter of a mathematician specializing in Khovanov homology, and she had a burning desire to know what this was all about. So we went for it! We spent the first year learning about Khovanov homology works; then we got to a research question. Emma ended up making a surprising new discovery. Emma is now in her third year in the math Ph.D. program at the University of Kentucky, and the three of us plan to adapt her thesis into a research paper this year.

3.4. **One-circle Kauffman states.** Given a diagram D of a knot K, one constructs a *Kauffman state* x by "smoothing" each crossing in one of two ways,)($\xleftarrow{A}$ ✕ $\xrightarrow{B}$ ✕. This state x consists of *state circles* joined by A- and B-labeled arcs, one from each crossing. There is an associated *state surface* F_x that spans K, which one constructs by capping off each state circle of x with a disk and attaching a half-twist band at each crossing. If D has n crossings, then it admits 2^n states, some of which have a single state circle.

If x is a one-circle Kauffman state from an n-crossing knot diagram D, then its state surface F_x has an $n \times n$ Gordon-Litherland matrix G, and when D is alternating G can be taken to be of the form $\begin{bmatrix} I & A \\ A^T & -I \end{bmatrix}$, where A is a matrix of 1's and 0's. My five advisees–four post-baccalaureate students and an undergraduate–begin with two main questions:

Question 3.1. Which matrices A of 1's and 0's arise in this way?

Question 3.2. Given a matrix $\begin{bmatrix} I & A \\ A^T & -I \end{bmatrix}$ where A is admissible by Question 3.1, how uniquely does it determine a diagram D?

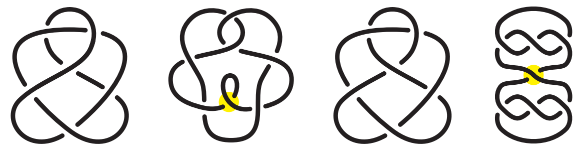
In Question 3.1, we aim for some analogue of Kuratowski's Theorem, where A is admissible if and only if it has no submatrix belonging to, say, either of two prohibited families. My intuition is that the answer to Question 3.2 will be "Up to mutation." If that is correct, then it will follow that it is possible to compute the Jones polynomial from our nice matrix, and following this lead will be ripe material for future projects.

3.5. **Triangle number of a spanning surface.** The *stick number* of a knot K is the smallest number of straight line segments needed to construct (a knot isotopic to) K in Euclidean 3-space $\mathbb{R}^3$. In this project, my undergraduate advisee Libby and I are exploring a new invariant, which we call the *triangle number* of a spanning surface F: it is the smallest number of triangles needed to construct F in $\mathbb{R}^3$.

Computing stick numbers is notoriously difficult in general, due to the difficulty of obtaining effective lower bounds. Anticipating similar challenges here, we have begun with a narrow scope, aiming to determine the triangle numbers of a handful of spanning surfaces for the three links with three crossings or fewer. I anticipate Libby's project being just the first of many on this topic. In this and future projects, as the knots and surfaces under consideration increase in complexity, I expect that we will seek out increasingly sophisticated tools, many of them computational-geometric.

4. Tait's conjectures, geometrically

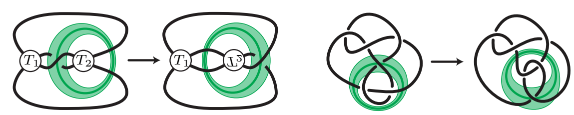
In general, it is hard to determine the crossing number of an arbitrary knot, or to determine whether or not a given diagram minimizes crossings, but certainly no diagram that minimizes crossings can have a "nugatory" crossing, like the two highlighted below:



A diagram without nugatory crossings is called *reduced*, whether or not it minimizes crossings. Interestingly, if a knot K has a diagram D that is reduced *and* alternating, then D *always* minimizes crossings: $c(D) = c(K)$. Although this fact was first observed empirically by P.G. Tait in 1898 [7], it and two related conjectures remained unproven for almost a century:

Tait's conjectures. *Given two reduced alternating diagrams D and D' of a knot $K \subset S^3$:*

(1) D, D' have the same number of crossings, which is minimal: $c(D) = c(D') = c(K)$.
(2) D, D' have equal writhe:[4] *$w(D) = w(D')$.*
(3) If D, D' are prime,[5] *then they are related by a sequence of flype moves:*



Tait's conjectures remained open until the discovery of the Jones polynomial in the mid-1980's [4], which almost immediately led to three independent proofs of Tait's first conjecture Jones polynomial of a knot K: the degree span, or *breadth*, of $V_K(t)$, denoted $\mathrm{bth}(V_K(t))$, always provides a lower bound for crossing number, $c(K) \leq \mathrm{bth}(V_K(t))$, and any reduced alternating diagram D of K satisfies $c(D) = \mathrm{bth}(V_K(t))$;

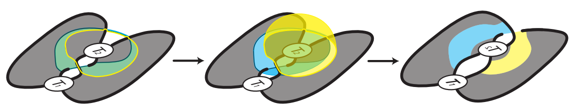
Within a decade, the Jones polynomial had led to proofs of all three of Tait's conjectures. Yet, these proofs were only somewhat satisfying: why did they work? What was the Jones polynomial really measuring? In their 1993 proof of Tait's flyping conjecture [6], Menasco and Thistlethwaite spotlighted this remaining gap in our understanding: "the question remains open as to whether there exist purely geometric proofs of this and other results that have been obtained with the help of new polynomial invariants." The aim is not just to know, but to understand. I love that the mathematical community values this.

[4]Orient D arbitrarily. Then every crossing looks like ⤧ or ⤧. The *writhe* of D is $w(D) = |⤧| - |⤧|$, where bars count components. It is independent of the orientation on D.

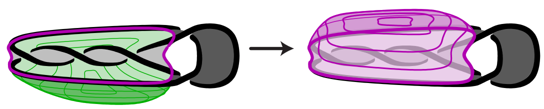
[5]A knot diagram D is *prime* if, for every circle $\gamma \subset S^2$ that intersects D generically in two points, all crossings of D lie on the same side of γ.

The first partial answer came in 2017 from Greene in a paper where he answered another longstanding question, this one from Ralph Fox: "What [geometrically] is an alternating knot?" First, Greene observed that, if B and W are the checkerboard surfaces from an alternating knot diagram and G_B and G_W are their Goeritz matrices, then G_B is positive-definite and G_W is negative-definite, or vice-versa: "B and W are definite and of opposite signs."[6] Second, Greene proved that alternating knots are the only ones with a two such surfaces. In fact, he proved that if F_+ and F_- are definite spanning surfaces of opposite signs for the same knot K, then K has an alternating diagram whose checkerboard surfaces are "the same as" (are isotopic to) F_+ and F_-. Greene was then able to translate Tait's conjectures into *non-diagrammatic* statements and give the first "purely geometric" proofs of Tait's second conjecture and part of his first: any reduced alternating diagrams D and D' of the same knot satisfy $c(D) = c(D')$ and $w(D) = w(D')$ [3].[7]

Recently, I gave the first purely geometric proof of Tait's "flyping" conjecture [K10]. The first key insight was that flyping a diagram D changes one of its checkerboard surfaces by isotopy and the other, F, by a geometric operation I call *re-plumbing*,



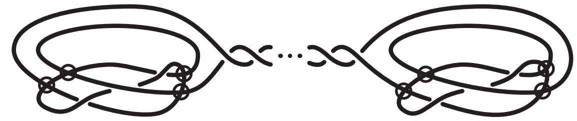
which replaces a disk $U \subset F$ (shown green, left) with another disk V (shown half yellow and half blue, center and right) that is disjoint from F except along its boundary $\partial V = \partial U$. Here is another replumbing move:



My insight about flyping and re-plumbing translates Tait's diagrammatic conjecture to a geometric statement about spanning surfaces. Then the real work begins. That story is too long for right now though.

Around the time I proved Tait's flyping conjecture, Boden-Karimi extended Greene's insights about definite surfaces in S^3 to the context of thickened surfaces, like $S^1 \times S^1 \times [-1, 1]$ [1]. I used their innovations to extend Tait's flyping conjecture to that context. Many of the arguments adapted directly, but a few technical points required special attention. Some of these subtleties led to fundamental revelations about *virtual knots*, which correspond to knots in thickened surfaces. This led to two further spin-off projects.

In one paper [K11], I explained how there are two common, but different, notions of primeness for virtual knots, and likewise for their diagrams; I introduced a new tool called "lassos" and proved how to use it to tell by inspection whether or not the link it represents is prime in either sense. As a corollary, it follows that, given any two non-classical alternating virtual knots, there are infinitely many distinct ways to take their connect sum:



The other paper [K9] considers a notion of "essential" surfaces that is more restrictive than usual. The initial motivation had to do with implementing a familiar classical technique (Menasco's crossing bubbles) in a thickened surface rather than in S^3, but the results might also prove useful in the study of Seifert solids for knotted surfaces in 4-space, among other possible applications.

5. Continued fractions and 2-bridge links

A 2-bridge knot or link is one that can be drawn with just two local maxima and two local minima. There are rich correspondences, via continued fractions, between these links and $\mathbb{Q} \cap (0, 1)$. For example, the Stevedore (6_1) knot is associated to the rational number $\frac{5}{9}$, which can be represented as a continued fraction in several ways, including

$$\frac{5}{9} = \frac{1}{1 + \frac{1}{1+\frac{1}{4}}}, \quad \frac{5}{9} = \frac{1}{2 - \frac{1}{5}}, \text{ and } \frac{5}{9} = 1 + \frac{1}{-2 - \frac{1}{4}}.$$

The first (additive) representation describes K via a braid diagram, while the second and third (subtractive) ones describe 1- and 2-sided surfaces that K bounds. Something similar happens with each 2-bridge knot. My four coauthors and I used these three different types of continued fraction representations to obtain a formula for the average crosscap number of all

[6] $G_B \in \mathbb{Z}^{n \times n}$ is positive-definite if and only if $\vec{x}^T G_B \vec{x} > 0$ for every nonzero $\vec{x} \in \mathbb{Z}^n$.

[7] It remains an open problem to give a purely geometric proof of Tait's full first conjecture, since Greene's insights are less useful regarding the possibilities of non-alternating diagrams of an alternating knot. I have some ideas....

2-bridge knots with a given crossing number c [K15].

Two-bridge knots and links (and their spanning surfaces) carry a wealth of data that I am interested in parsing further, in part because I would like to broaden my research program to involve statistical methods and data analytics, and this seems like the perfect venue. That is just one of several possible follow-up project here. My co-authors and I are currently pursuing another such project, where we consider how many distinct minimal length continued fraction representations a given rational number has [K16]. Another would be to delve into possible number theoretic implications of these surfaces. A good preliminary step would be to deepen my understanding of the topology of numbers by teaching a course, or advising a reading group, on that subject.

6. Conclusion

I have plans for many research projects involving spanning surfaces which I have not described in this statement. Many of these plans would ideally involve students at the undergraduate level. I would be delighted to describe any of these specifically if I am so fortunate as to be invited to interview with you, but here is a rough overview of some of my near-term research plans (in addition to those described in the previous paragraph).

Some plans involve translating insights and questions about surfaces into purely linearly algebraic problems, like the approach I took with John and my co-advisors in [K14]. Some involve translating diagrammatic questions into questions about spanning surfaces, like the approach I took in [K10]. Some, like [K10, K8, K13], involve Murasugi sums of spanning surfaces. Some, like my recent project with Tristan, apply knot theory to the natural and physical sciences. Some, like [K5] and my recent project with Isaias, are computational. Some, like my current project with Libby, involved computational geometry. Some, like [K3, K12, K11, K9], involve Khovanov homology or virtual knots.

All of the plans I just described pertain to knots and surfaces in three dimensions, but my most important aspiration as a researcher is to continue to broaden my mathematical reach. As noted above, computational geometry and applications to the sciences are two of the areas into which I plan to extend my research. I want to conclude by discussing one more such area: dimension four.

Two summers ago, after six years of trying, I finally connected with a great 4-manifolds research project. What makes it great, first and foremost, is my collaborators: Maggie Miller, Seungwon Kim, Patrick Naylor, and Homayun Karimi. They all have greater 4-manifolds expertise than I do, but in our meetings they generously encourage my naive questions. The second factor that makes this project a great fit for me is what I bring to it: I suggested the research question—can we adapt Murasugi sum (plumbing) to an operation on spanning solids for knotted surfaces in 4-space?—and my naive questions often lead useful places, because many phenomena, facts, and arguments in dimension 3 translate to dimension 4, but with a caveat or two per sentence. Our whole project aims, in a nutshell, to take my usual research perspective and bump it up a dimension.

A constant concern for the project, then, is to stay sufficiently narrow in our inquiry. To this point, my vantage point has allowed me to anticipate the risk of our inquiry sprawling and to suggest specific delimitations for our conversation. For example, one of our aims is to use our adapted notion of Murasugi sum to construct many new examples of knotted surfaces. This would add value to the field because most knotted surfaces to date have been constructed by "deformed" versions of a certain spinning construction, and this lack of diversity reduces the usefulness of empirical approaches. Yet, "many new examples" is an amorphous goal, so I suggested a refinement: adapt the classical notion of "arborescent" knots to knotted surfaces. This narrowing of the problem has helped focus our exploration. I hope that it also helps us draw a circle around a cohesive set of ideas for a first paper, sooner rather than later, because the overall ambitions of our exploration are vast.

I have found that almost every aspect of classical knot theory can be understood from the perspective of spanning surfaces, and this perspective has already, several times in my young career, led to fundamental insights regarding well-trodden material. My hope is that studying knotted surfaces in 4-space from the perspective of their spanning solids might prove similarly fruitful. In any case, it seems like a promising long-term strategy in my research program.

My main strategy, though, is much more basic. Stay curious. Ask questions, especially naive ones. Listen closely and connect with other humans. Pursue interesting questions, knowing that there are two good ways to recognize them—one comes from the inside and is borne from experience, while the other is more fundamental: a question is interesting if someone else thinks about it deeply and finds it captivating.

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